Imperial College London

MSc EXAMINATION June 2010

This paper is also taken for the relevant Examination for the Associateship

ADVANCED FIELD THEORY

For Students in Quantum Fields and Fundamental Forces

Friday, 7 May 2010: 14:00 to 17:00

Answer 3 out of the following 4 questions.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the THREE answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

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Go to the next page for questions

You may use the following results without proof:

- Loop integral in d dimensions (Minkowskian):

$$I_n(m^2) \equiv \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2 - m^2)^n} = (-1)^n \frac{i m^{d-2n}}{(4\pi)^{d/2}} \frac{\Gamma(n - d/2)}{\Gamma(n)}$$

- Gamma functions: $z\Gamma(z) = \Gamma(z+1)$ and

$$\Gamma\left(-m + \frac{\epsilon}{2}\right) = \frac{(-1)^m}{m!}\left(\frac{2}{\epsilon} + \sum_{p=1}^{m} \frac{1}{p} - \gamma + O(\epsilon)\right) \quad \text{for integer } \; m \geq 0$$

- Feynman parameters:

$$\frac{1}{ab} = \int_0^1 \frac{dx}{[xa + (1-x)b]^2}$$

- Gaussian two-point function

$$\frac{\int d^N q \, q_i q_j e^{-\frac{1}{2} q^T M q}}{\int d^N q e^{-\frac{1}{2} q^T M q}} = \left(M^{-1}\right)_{ij} .$$

- Gaussian Grassmann integral

$$\int \left(\prod_i d\theta_i^* d\theta_i\right) \exp\left(-\theta_i^* B_{ij} \theta_j\right) = \det \mathbf{B}.$$

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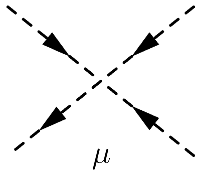
1. The Lagrangian of scalar electrodynamics (theory of an electrically charged scalar field) is

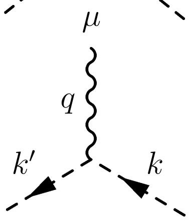
$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + (D_\mu\phi)^*(D^\mu\phi) - m^2\phi^*\phi - \frac{\lambda}{4}(\phi^*\phi)^2 \, ,$$

where ϕ is a complex scalar, A_μ is an Abelian gauge field, $D_\mu = \partial_\mu + ieA_\mu$ is a covariant derivative, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor.

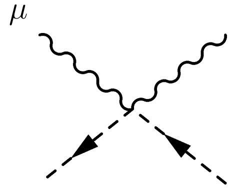
(i) Show that the Feynman rules for the interaction vertices are



$\leftrightarrow \quad -i\lambda,$



$\leftrightarrow \quad -ie(k_\mu + k'_\mu),$

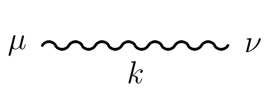


$\leftrightarrow \quad 2ie^2 g_{\mu\nu}.$

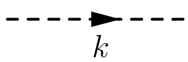
[6 marks]

(ii) Draw the 1PI diagrams that contribute to the photon two-point function at one-loop level. [3 marks]

(iii) Use the propagators

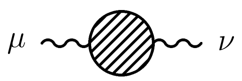


$\leftrightarrow \quad \frac{-i}{k^2}\left[g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right],$



$\leftrightarrow \quad \frac{i}{k^2 - m^2},$

to write down the integrals corresponding to the 1PI diagrams in (ii). Show that the whole one-loop correction is



$$= e^2 \int \frac{d^dp}{(2\pi)^d} \frac{(2p+k)^\mu(2p+k)^\nu - 2g^{\mu\nu}[(p+k)^2 - m^2]}{(p^2 - m^2)((p+k)^2 - m^2)}.$$

[7 marks]

(iv) Show that this integral vanishes for $k = 0$ in arbitrary number of dimensions d. (Hint:

$$m^2 I_2(m^2) = \frac{d-2}{2} I_1(m^2).)$$

[4 marks]

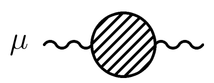
[Total 20 marks]

2. The bare Lagrangian of Quantum Electrodynamics is

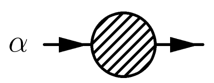
$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \bar{\psi}(i\partial\!\!\!/ - m)\psi - e\bar{\psi}\gamma^\mu\psi A_\mu,$$

where ψ is the electron field, A_μ is the U(1) gauge field, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor, γ^μ are the Dirac gamma matrices and $\partial\!\!\!/ = \gamma^\mu\partial_\mu$.

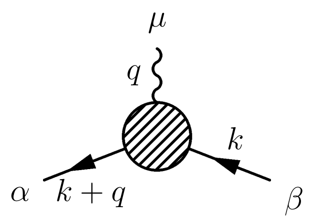
At one-loop level in *naive* perturbation theory, the superficially divergent one-particle irreducible correlation functions are



$$= -\frac{ie^2}{12\pi^2}(k^2 g^{\mu\nu} - k^\mu k^\nu)\left(\frac{2}{\epsilon} + \log\frac{4\pi\mu^2}{m^2} - \gamma\right) + O(k^4).$$



$$= \frac{ie^2}{16\pi^2}\left(\xi k\!\!\!/ + (3+\xi)m\right)_{\beta\alpha}\left(\frac{2}{\epsilon} + \log\frac{4\pi\mu^2}{m^2} - \gamma\right) + O(k^2).$$



$$= -ie\gamma^\mu_{\alpha\beta} - \frac{ie^3\xi}{16\pi^2}\gamma^\mu_{\alpha\beta}\left(\frac{2}{\epsilon} + \log\frac{4\pi\mu^2}{m^2} - \gamma\right) + O(q). \qquad (2.1)$$

(i) Consider the same theory in *renormalised* perturbation theory. The renormalised fields are defined by $\psi = Z_\psi^{1/2}\psi_R$ and $A^\mu = Z_A^{1/2}A_R^\mu$, and renormalised parameters e_R, m_R and ξ_R by

$$\begin{aligned} Z_A^{1/2} Z_\psi e &= e_R + \delta e, \\ Z_\psi m &= m_R + \delta m, \\ Z_A/\xi &= Z_\xi/\xi_R. \end{aligned}$$

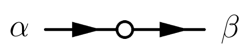
The renormalisation constants are written in terms of counterterms as $Z_A = 1 + \delta Z_A$, $Z_\psi = 1 + \delta Z_\psi$ and $Z_\xi = 1 + \delta Z_\xi$.

Doing these changes of variables explicitly, write the Lagrangian in terms of renormalised fields, renormalised parameters, and counterterms. [3 marks]

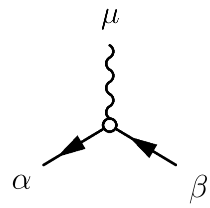
(ii) Identify the terms that are treated as interactions in renormalised perturbation theory. [3 marks]

(iii) The Feynman rules for the counterterm vertices are

$$\leftrightarrow \quad -i\left[\delta Z_A\left(k^2 g^{\mu\nu} - k^\mu k^\nu\right) - \frac{\delta Z_\xi}{\xi_R}k^\mu k^\nu\right]$$



$$\leftrightarrow \quad i\left(\delta Z_\psi k\!\!\!/ - \delta m\right)_{\beta\alpha}$$



$$\leftrightarrow \quad -i\delta e\gamma^\mu_{\alpha\beta}$$

Write the correlation functions (2.1) in renormalised perturbation theory in terms of renormalised parameters and counterterms (i.e., without imposing a renormalisation condition yet). [3 marks]

[This question continues on the next page . . .]

(iv) Determine the values of the counterterms in the $\overline{\text{MS}}$ scheme. Using them, write the correlation functions (2.1) in terms of renormalised parameters only. [8 marks]

(v) Show that, to leading order in e, the beta function of the theory is

$$\beta(e) \equiv M \left. \frac{\partial e_R}{\partial M} \right|_B = \frac{e^3}{12\pi^2}.$$

[3 marks]

[Total 20 marks]

3. Consider the d-dimensional (Minkowskian) integral

$$I_2^\mu(k;m,0) = \int \frac{d^dp}{(2\pi)^d} \frac{p^\mu}{(p^2-m^2)(p+k)^2}.$$

(i) Explain why $I_2^\mu(k;m,0)$ has to be parallel to k^μ. [2 marks]

(ii) Show that

$$I_2^\mu(k;m,0) = \int_0^1 dx \int \frac{d^dp}{(2\pi)^d} \frac{p^\mu}{[(p+kx)^2-(1-x)(m^2-xk^2)]^2}.$$

[5 marks]

(iii) Show that

$$I_2^\mu(k;m,0) = -k^\mu \int_0^1 dx\, x \int \frac{d^dp}{(2\pi)^d} \frac{1}{[p^2-(1-x)(m^2-xk^2)]^2}.$$

[3 marks]

(iv) Calculate the momentum integral to express $I_2^\mu(k;m,0)$ as an integral over x only. [2 marks]

(v) Calculate the integral in d dimensions at $k^2=m^2$.
(Hint:

$$\int_0^1 dx\, x(1-x)^a = \frac{1}{(a+1)(a+2)}.)$$

[3 marks]

(vi) In $d=4-\epsilon$ dimensions and at $k^2=m^2$, show that

$$I_2^\mu(k;m,0) = -\frac{ik^\mu}{32\pi^2}\left(\frac{2}{\epsilon}+\log\frac{4\pi}{m^2}-\gamma+3\right)+O(\epsilon).$$

[5 marks]

[Total 20 marks]

4. Consider a Euclidean scalar field theory with action

$$S_E = \int d^4x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4 \right)$$

and a momentum cutoff Λ. The generating functional is given by the path integral

$$Z[J] = \int_{0<k<\Lambda} \mathcal{D}\phi e^{-S_E - \int d^4x J(x)\phi(x)},$$

where the subscript indicates that only the Fourier modes $\phi(k)$ with $0 < k < \Lambda$ are non-zero.

(i) Derive an expression for the two-point correlation function $\langle\phi(x)\phi(y)\rangle$ in terms of the generating functional $Z[J]$. [3 marks]

(ii) In Wilsonian renormalisation, one derives an effective theory that has a lower cutoff $\Lambda' < \Lambda$ but describes the same physics at low energies. Its action $S_{\text{eff}}^{\Lambda'}[\phi]$ is defined by

$$\int_{0<k<\Lambda'} \mathcal{D}\phi e^{-S_{\text{eff}}^{\Lambda'}[\phi] - \int d^4x J(x)\phi(x)} = Z[J],$$

where $J(x)$ is an arbitrary external field with $J(k) = 0$ for $k > \Lambda'$. Show that $S_{\text{eff}}^{\Lambda'}$ is given by the expression

$$S_{\text{eff}}^{\Lambda'}[\phi] = S_E[\phi] - \log \int_{\Lambda'<k<\Lambda} \mathcal{D}\hat{\phi} e^{-\Delta S[\phi,\hat{\phi}]}, \qquad (4.1)$$

where $\hat{\phi}(k) = 0$ for $k < \Lambda'$ and $k > \Lambda$. Write down $\Delta S[\phi, \hat{\phi}]$. Explain why some of the terms vanish in the derivation of S_{eff}. [4 marks]

(iii) Draw the interaction vertices that arise when you calculate Eq. (4.1) perturbatively. Explain the different types of propagator lines that appear in them. (You do not need to derive the Feynman rules.) [3 marks]

(iv) Draw all connected diagrams that contribute to $S_{\text{eff}}^{\Lambda'}$ up to order λ^2, and identify the term each of them contributes to. [3 marks]

(v) Imagine being able to do the integral (4.1) to all orders in λ. What new terms would you expect to obtain? [2 marks]

(vi) Consider now a rescaling of lengths by factor $b = \Lambda'/\Lambda < 1$, i.e.,

$$x' = bx, \quad k' = k/b.$$

How does the kinetic term

$$S_{\text{eff}}^{\Lambda'} = \int d^4x \frac{1 + \Delta Z}{2} \partial_\mu\phi\partial^\mu\phi + \ldots$$

scale? [2 marks]

(vii) Rescaling the field to take the kinetic term back to its canonical form, show that the coefficient G of a general term with n powers of ϕ and m derivatives becomes

$$G' = \frac{G + \Delta G}{(1 + \Delta Z)^{n/2}} \left(\frac{\Lambda'}{\Lambda}\right)^{m+n-4},$$

where ΔG and ΔZ are the loop corrections to G and the kinetic term, respetively, from the integral in Eq. (4.1). [3 marks]

[Total 20 marks]

End of examination paper