Imperial College London

MSc EXAMINATION May 2011

This paper is also taken for the relevant Examination for the Associateship

ADVANCED QUANTUM FIELD THEORY

For Students in Quantum Fields and Fundamental Forces

Friday, 13 May 2011: 14:00 to 17:00

Answer 3 out of the following 4 questions.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the THREE answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

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Go to the next page for questions

You may use the following results without proof:

- Loop integral in d dimensions (Minkowskian):

$$I_n(m^2) \equiv \int \frac{d^dp}{(2\pi)^d} \frac{1}{(p^2-m^2)^n} = (-1)^n \frac{im^{d-2n}}{(4\pi)^{d/2}} \frac{\Gamma(n-d/2)}{\Gamma(n)}$$

- Gamma functions: $z\Gamma(z) = \Gamma(z+1)$ and

$$\Gamma\left(-m+\frac{\epsilon}{2}\right) = \frac{(-1)^m}{m!}\left(\frac{2}{\epsilon}+\sum_{p=1}^{m}\frac{1}{p}-\gamma+O(\epsilon)\right) \quad \text{for integer } \ m \geq 0$$

- Feynman parameters:

$$\frac{1}{ab} = \int_0^1 \frac{dx}{[xa+(1-x)b]^2}$$

- Gaussian two-point function

$$\frac{\int \mathcal{D}\phi\, \phi(x)\phi(y) e^{-\frac{1}{2}\int d^4x d^4y \phi(x) M(x,y)\phi(y)}}{\int \mathcal{D}\phi\, e^{-\frac{1}{2}\int d^4x d^4y \phi(x) M(x,y)\phi(y)}} = M^{-1}(x,y).$$

- Grassmann Gaussian two-point function

$$\frac{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, \psi_\alpha(x)\bar{\psi}_\beta(y) e^{-\int d^4x d^4y \bar{\psi}_\alpha(x) M_{\alpha\beta}(x,y)\psi_\beta(y)}}{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{-\int d^4x d^4y \bar{\psi}_\alpha(x) M_{\alpha\beta}(x,y)\psi_\beta(y)}} = \left(M^{-1}\right)_{\alpha\beta}(x,y).$$

- Gamma matrices:

$$\mathrm{tr}\gamma^\mu = 0, \quad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu},$$
$$\mathrm{tr}\gamma^5 = 0, \quad (\gamma^5)^\dagger = \gamma^5, \quad \{\gamma^5, \gamma^\mu\} = 0.$$

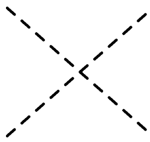
1. The Lagrangian of the pseudoscalar Yukawa theory is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_\phi^2\phi^2 - \frac{1}{4!}\lambda\phi^4 + \bar{\psi}(i\partial\!\!\!/ - m)\psi - ig\bar{\psi}\gamma^5\psi\phi,$$

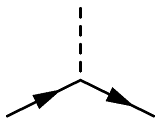
where ϕ is a real (pseudo)scalar field and ψ is a fermion field.

(i) Using the expressions for the Gaussian two-point functions in the rubric, derive the propagators for the two fields in bare perturbation theory. [5 marks]

(ii) Explain why the Feynman rules for the interaction vertices are



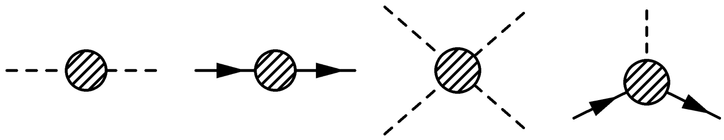
$\leftrightarrow \quad -i\lambda$



$\leftrightarrow \quad g\gamma^5.$

[2 marks]

(iii) The superficially divergent correlators are



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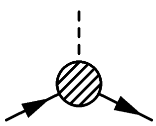
Draw the one-particle irreducible diagrams that contribute to these at one-loop level. [4 marks]

(iv) Use the Feynman rules to write the corresponding integrals. [6 marks]

(v) By considering zero external momenta, show that the divergent part of the scalar-fermion interaction at one-loop level is



$$= -\frac{g^3}{16\pi^2}\gamma^5\frac{2}{\epsilon} + \text{finite terms.}$$

[3 marks]

[Total 20 marks]

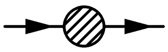
2. As in Question 1, consider the pseudoscalar Yukawa theory,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_\phi^2\phi^2 - \frac{1}{4!}\lambda\phi^4 + \bar{\psi}(i\partial\!\!\!/ - m)\psi - ig\bar{\psi}\gamma^5\psi\phi,$$

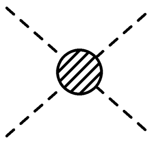
where ϕ is a real (pseudo)scalar field and ψ is a fermion field.

At one-loop level in *bare* perturbation theory, the superficially divergent correlation functions are

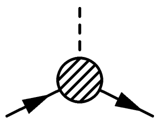
$$= \frac{i}{16\pi^2}\left(\lambda m_\phi^2 + 2g^2k^2\right)\frac{2}{\epsilon} + \text{finite terms.}$$



$$= \frac{i}{16\pi^2}\frac{g^2}{2}k\!\!\!/\frac{2}{\epsilon} + \text{finite terms.}$$



$$= -i\lambda + \frac{i}{16\pi^2}\left(\frac{3}{2}\lambda^2 - 24g^4\right)\frac{2}{\epsilon} + \text{finite terms.}$$



$$= g\gamma^5 - \frac{g^3}{16\pi^2}\gamma^5\frac{2}{\epsilon} + \text{finite terms.}$$

(i) Write the Lagrangian in terms of renormalised fields, renormalised parameters and counterterms, and explain how they are related to the bare parameters. [6 marks]

(ii) Which terms are treated as interactions in *renormalised* perturbation theory. [2 marks]

(iii) What is the contribution from the counterterms to each of the correlation functions? [5 marks]

(iv) What are the values of the counterterms in the $\overline{\text{MS}}$ scheme? [3 marks]

(v) Show that the beta functions of the theory are

$$\beta_g \equiv \left.\frac{\partial g_R}{\partial \ln M}\right|_B = \frac{5g^3}{16\pi^2},$$

$$\beta_\lambda \equiv \left.\frac{\partial \lambda_R}{\partial \ln M}\right|_B = \frac{1}{16\pi^2}\left(3\lambda^2 + 8\lambda g^2 - 48g^4\right).$$

[4 marks]

[Total 20 marks]

3. Consider a simplified version of the Standard Model of particle physics, which consists of the SU(3) gauge field, the top quark and the Higgs. The model has three coupling constants: the gauge coupling g_s, the Higgs self-coupling λ and the top Yukawa coupling g_t. The couplings run as a function of the renormalisation scale M as

$$\frac{dg_s^2}{d\ln M} = -\frac{14}{16\pi^2} g_s^4, \tag{1}$$

$$\frac{dg_t^2}{d\ln M} = \frac{g_t^2}{16\pi^2}\left(9g_t^2 - 16g_s^2\right), \tag{2}$$

$$\frac{d\lambda}{d\ln M} = \frac{1}{16\pi^2}\left(24\lambda^2 + 12g_t^2\lambda - 6g_t^4\right). \tag{3}$$

The gauge coupling measured at $M = v$ is $g_s(v) \approx 1.2$. The mass of top quark is

$$m_t = \frac{g_t(v)v}{\sqrt{2}},$$

and the mass of the Higgs is

$$m_H = \sqrt{2\lambda(v)}v,$$

where $v \approx 246$ GeV is the Higgs expectation value.

(i) Explain what is meant by triviality and asymptotic freedom. [2 marks]

(ii) Consider first the gauge coupling g_s. Is it relevant, marginally relevant, marginally irrelevant or irrelevant? Solve the renormalisation group equation, and find the location of the Landau pole in GeV. What can you say about physics below and above this energy? [5 marks]

(iii) For the set of three coupled equations (1), (2) and (3), find a solution of the form $g_t^2(M) = Ag_s^2(M)$, $\lambda(M) = Bg_s^2(M)$ where $A > 0$ and $B > 0$ are constants. If the couplings satisfy this relation, will the theory have a continuum limit? Given the experimental values of g_s and v, what would the predicted top and Higgs masses be? [7 marks]

(iv) The top mass has been measured to be $m_t \approx 171$ GeV. How do the couplings g_s and g_t behave qualitatively at high M? Depending on the Higgs mass, what are the different possibilities for λ at high M? (A qualitative answer with no numbers is enough.) [3 marks]

(v) Does the theory with realistic masses have a continuum limit? Explain how the existence of a non-trivial fixed point would affect this. [3 marks]

[Total 20 marks]

4. Consider the d-dimensional (Minkowskian) integral

$$I_2^{\mu\nu}(k;0,m) = \int \frac{d^dp}{(2\pi)^d} \frac{p^\mu p^\nu}{(p+k)^2(p^2-m^2)}.$$

(i) Show that

$$I_2^{\mu\nu}(k;0,m) = \int_0^1 dx \int \frac{d^dp}{(2\pi)^d} \frac{p^\mu p^\nu + x^2 k^\mu k^\nu}{[p^2-(1-x)(m^2-xk^2)]^2}.$$

[6 marks]

(ii) Show further that

$$I_2^{\mu\nu}(k;0,m) = \int_0^1 dx \left[\frac{g^{\mu\nu}}{d}\left(I_1(\Delta)+\Delta I_2(\Delta)\right) + x^2 k^\mu k^\nu I_2(\Delta)\right],$$

where the functions $I_1(z)$ and $I_2(z)$ are defined in the rubric, and $\Delta = (1-x)(m^2-xk^2)$.

[6 marks]

(iii) Using the properties of the Gamma function and the results given in the rubric, show that in dimensional regularisation

$$I_2^{\mu\nu}(k;0,m) = \int_0^1 dx \left(\frac{g^{\mu\nu}}{d-2}\Delta + x^2 k^\mu k^\nu\right) I_2(\Delta).$$

[4 marks]

(iv) For $k^2 = m^2$, calculate the integral over x. You should find

$$I_2^{\mu\nu}(k;0,m) = \frac{im^{d-4}}{(4\pi)^{d/2}} \frac{\Gamma\left(\frac{4-d}{2}\right)}{(d-2)(d-1)}\left[m^2 g^{\mu\nu} + \frac{2}{d-3}k^\mu k^\nu\right].$$

[4 marks]

[Total 20 marks]

End of examination paper