Imperial College London

MSc EXAMINATION May 2012

This paper is also taken for the relevant Examination for the Associateship

ADVANCED QUANTUM FIELD THEORY

For Students in Quantum Fields and Fundamental Forces

Friday, 25 May 2012: 14:00 to 17:00

Answer 3 out of the following 4 questions.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the THREE answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

Go to the next page for questions

You may use the following results without proof:

- Loop integral in d dimensions (Minkowskian):

$$I_n(m^2) \equiv \int \frac{d^dp}{(2\pi)^d} \frac{1}{(p^2 - m^2)^n} = (-1)^n \frac{i m^{d-2n}}{(4\pi)^{d/2}} \frac{\Gamma(n - d/2)}{\Gamma(n)}$$

- Gamma functions: $z\Gamma(z) = \Gamma(z+1)$ and

$$\Gamma\left(-m + \frac{\epsilon}{2}\right) = \frac{(-1)^m}{m!} \left(\frac{2}{\epsilon} + \sum_{p=1}^{m} \frac{1}{p} - \gamma + O(\epsilon) \right) \quad \text{for integer} \;\; m \geq 0$$

- Feynman parameters:

$$\frac{1}{ab} = \int_0^1 \frac{dx}{[xa + (1-x)b]^2}$$

- Gaussian integral
  - Normal

$$\frac{\int d^N\mathbf{q}\, q_i q_j e^{-\frac{1}{2}\mathbf{q}^T\mathbf{M}\mathbf{q}}}{\int d^N\mathbf{q}\, e^{-\frac{1}{2}\mathbf{q}^T\mathbf{M}\mathbf{q}}} = \left(\mathbf{M}^{-1}\right)_{ij},$$

    where $\mathbf{q} = (q_1, \ldots, q_N)$ and $\mathbf{M}$ is an $N \times N$ matrix.
  - Grassmannian

$$\frac{\int d^N\mathbf{c}^* d^N\mathbf{c}\, c_i c_j^* e^{-\mathbf{c}^\dagger\mathbf{M}\mathbf{c}}}{\int d^N\mathbf{c}^* d^N\mathbf{c}\, e^{-\mathbf{c}^\dagger\mathbf{M}\mathbf{c}}} = \left(\mathbf{M}^{-1}\right)_{ij},$$

    where $\mathbf{c} = (c_1, \ldots, c_N)$ and $\mathbf{M}$ is an $N \times N$ matrix.

- Gaussian two-point function
  - Normal

$$\frac{\int \mathcal{D}\phi\, \phi(x)\phi(y) e^{-\frac{1}{2}\int d^4x d^4y \phi(x) M(x,y) \phi(y)}}{\int \mathcal{D}\phi\, e^{-\frac{1}{2}\int d^4x d^4y \phi(x) M(x,y) \phi(y)}} = M^{-1}(x,y).$$

  - Grassmannian

$$\frac{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, \psi_\alpha(x)\bar{\psi}_\beta(y) e^{-\int d^4x d^4y \bar{\psi}_\alpha(x) M_{\alpha\beta}(x,y) \psi_\beta(y)}}{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{-\int d^4x d^4y \bar{\psi}_\alpha(x) M_{\alpha\beta}(x,y) \psi_\beta(y)}} = \left(M^{-1}\right)_{\alpha\beta}(x,y).$$

- Gamma matrices:

$$\mathrm{tr}\gamma^\mu = 0, \quad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu},$$
$$\mathrm{tr}\gamma^5 = 0, \quad (\gamma^5)^\dagger = \gamma^5, \quad \{\gamma^5, \gamma^\mu\} = 0.$$

 Go to the next page for questions

1. (i) Show that in $4-\epsilon$ dimensions

$$\mu^\epsilon \int \frac{d^{4-\epsilon}p}{(2\pi)^{4-\epsilon}} \frac{1}{p^2-m^2} = \frac{im^2}{16\pi^2}\left(\frac{2}{\epsilon} + \ln\frac{4\pi\mu^2}{m^2} + 1 - \gamma + O(\epsilon)\right),$$

and

$$\mu^\epsilon \int \frac{d^{4-\epsilon}p}{(2\pi)^{4-\epsilon}} \frac{1}{(p^2-m^2)^2} = \frac{i}{16\pi^2}\left(\frac{2}{\epsilon} + \ln\frac{4\pi\mu^2}{m^2} - \gamma + O(\epsilon)\right).$$

[6 marks]

(ii) Consider the integral

$$I_2^{\mu\nu\rho} = \mu^\epsilon \int \frac{d^{4-\epsilon}q}{(2\pi)^{4-\epsilon}} \frac{q^\mu q^\nu q^\rho}{((k+q)^2-m^2)(q^2-m^2)}.$$

(a) Show that the divergent part is

$$I_2^{\mu\nu\rho} = -\frac{i}{64\pi^2}\left[(k^\mu g^{\nu\rho} + k^\nu g^{\mu\rho} + k^\rho g^{\mu\nu})\left(m^2 - \frac{k^2}{6}\right) + k^\mu k^\nu k^\rho\right]\frac{2}{\epsilon} + \text{finite}.$$

[8 marks]

(b) Use the integrals

$$\begin{aligned}
\int_0^1 dx\, x \ln(1-2x)^2 &= -1, \\
\int_0^1 dx\, x^2 \ln(1-2x)^2 &= -\frac{5}{9}, \\
\int_0^1 dx\, x^3 \ln(1-2x)^2 &= -\frac{1}{3},
\end{aligned}$$

to show that for $k^2 = 4m^2$, the full integral, including finite parts, is

$$\begin{aligned}
I_2^{\mu\nu\rho} &= -\frac{im^2}{192\pi^2}(k^\mu g^{\nu\rho} + k^\nu g^{\mu\rho} + k^\rho g^{\mu\nu})\left(\frac{2}{\epsilon} + \ln\frac{4\pi\mu^2}{m^2} - \gamma + \frac{5}{3}\right) \\
&\quad -\frac{im^2}{64\pi^2}(k^\mu k^\nu k^\rho)\left(\frac{2}{\epsilon} + \ln\frac{4\pi\mu^2}{m^2} - \gamma + \frac{4}{3}\right). \qquad (1)
\end{aligned}$$

[6 marks]

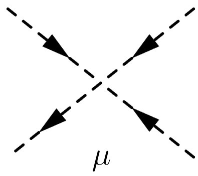
[Total 20 marks]

2. The Lagrangian of scalar electrodynamics (theory of an electrically charged scalar field) is

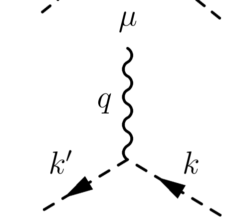
$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + (D_\mu\phi)^*(D^\mu\phi) - m^2\phi^*\phi - \frac{\lambda}{4}(\phi^*\phi)^2\,,$$

where ϕ is a complex scalar, A_μ is an Abelian gauge field, $D_\mu = \partial_\mu + ieA_\mu$ is a covariant derivative, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor.

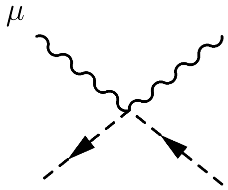
(i) Show that the Feynman rules for the interaction vertices are



$$\leftrightarrow \quad -i\lambda, \tag{1}$$



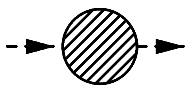
$$\leftrightarrow \quad -ie(k_\mu + k'_\mu), \tag{2}$$



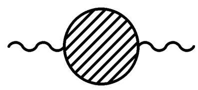
$$\leftrightarrow \quad 2ie^2 g_{\mu\nu}. \tag{3}$$

Indicate also the momenta of the legs and show how momentum is conserved at each vertex. [6 marks]

(ii) Draw the 1PI diagrams that contribute to the scalar and photon two-point functions ,

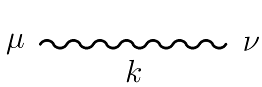


and

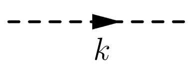
,

at one-loop level. [3 marks]

(iii) Use the Feynman gauge propagators



$$\leftrightarrow \quad \frac{-ig^{\mu\nu}}{k^2},$$



$$\leftrightarrow \quad \frac{i}{k^2 - m^2},$$

to write down the integrals corresponding to the 1PI diagrams in (ii) in dimensional regularisation. [7 marks]

(iv) Using the expressions given in Question 1.(i), calculate the photon two-point function at $k = 0$. Explain the significance of your result. [4 marks]

[Total 20 marks]

3. Consider again scalar electrodynamics (see Question 2)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*(D^\mu\phi) - m^2\phi^*\phi - \frac{\lambda}{4}(\phi^*\phi)^2\,,$$

this time in d spacetime dimensions.

(i) Find the dimensionalities of the fields A_μ and ϕ and parameters e, λ and m. [3 marks]

(ii) Show that the dimensionality of the momentum-space 1PI correlator $\tilde{\Gamma}_{E_A,E_\phi}$ with E_A external photon legs and E_ϕ external scalar legs is

$$[\tilde{\Gamma}_{E_A,E_\phi}] = d - (E_A + E_\phi)\frac{d-2}{2}.$$

[4 marks]

(iii) Using momentum conservation or other arguments, show that the number of loops in a diagram with E_A external photon legs and E_ϕ external scalar legs, and V_1, V_2 and V_3 vertices of the types (1), (2) and (3), respectively (see Question 2) is

$$L = V_1 + \frac{1}{2}V_2 + V_3 - \frac{1}{2}(E_A + E_\phi) + 1.$$

[4 marks]

(iv) Using dimensional analysis, show that the the superficial degree of divergence of the diagram is

$$D = 4 - (E_A + E_\phi) + (d-4)L.$$

[4 marks]

(v) Use the previous result to explain how the renormalisability of the theory depends on d? [3 marks]

(vi) What counterterms do you need to introduce to renormalise the theory in $d = 3$ dimensions? Explain. [2 marks]

[Total 20 marks]

4. Consider the Georgi-Glashow model, with an SU(2) gauge field A_μ and a scalar field Φ in the adjoint representation. The Lagrangian is

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} + \mathrm{Tr}\,(D_\mu\Phi)(D^\mu\Phi) - \lambda\left(\mathrm{Tr}\Phi^2 - v^2\right)^2,$$

with field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$ and covariant derivate $D_\mu\Phi = \partial_\mu\Phi + ig[A_\mu, \Phi]$. This Lagrangian is invariant under the SU(2) gauge symmetry

$$A_\mu \to A_\mu^U = UA_\mu U^\dagger - \frac{i}{g}U\partial_\mu U^\dagger, \quad \Phi \to \Phi^U = U\Phi U^\dagger.$$

(i) Express the kinetic terms of the gauge field A_μ in terms of its components A_μ^a defined as $A_\mu = A_\mu^a t^a$, where the 2×2 matrices T^a satisfy $[t^a, t^b] = (i/2)\epsilon^{abc}t^c$ and $\mathrm{Tr}\, t^a t^b = \delta^{ab}/2$. Show that without gauge fixing the tree-level two-point function $\langle A_\mu^a(k)A_\nu^b(q)\rangle$ is not well defined, and explain why. [6 marks]

(ii) Explain why any gauge-invariant expectation value can be calculated using the modified Lagrangian

$$\mathcal{L}_\xi = \mathcal{L} - \frac{1}{2\xi}(\partial^\mu A_\mu^a)^2 + \partial^\mu c^{a*}\partial_\mu c^a - g\epsilon^{abc}\partial^\mu c^{a*}A_\mu^b c^c.$$

What are the properties of the field c, and what type of particle does it describe? How is the value of the parameter ξ determined? [11 marks]

(iii) Show that with Lagrangian $\mathcal{L}_\xi$, the tree-level gauge-field two-point function is

$$\langle A_\mu^a(k)A_\nu^b(q)\rangle = (2\pi)^4\delta(k+q)\frac{-i\delta^{ab}}{k^2}\left[g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right].$$

[3 marks]

[Total 20 marks]

 End of examination paper