Imperial College London

MSc EXAMINATION May 2013

This paper is also taken for the relevant Examination for the Associateship

ADVANCED QUANTUM FIELD THEORY

For Students in Quantum Fields and Fundamental Forces

Friday, 17 May 2013: 14:00 to 17:00

Answer 3 out of the following 4 questions.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the THREE answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

Go to the next page for questions

You may use the following results without proof:

- Loop integral in d dimensions (Minkowskian):

$$I_n(m^2) \equiv \int \frac{d^dp}{(2\pi)^d} \frac{1}{(p^2 - m^2)^n} = (-1)^n \frac{im^{d-2n}}{(4\pi)^{d/2}} \frac{\Gamma(n - d/2)}{\Gamma(n)}$$

- Gamma functions: $z\Gamma(z) = \Gamma(z+1)$ and

$$\Gamma\left(-m + \frac{\epsilon}{2}\right) = \frac{(-1)^m}{m!}\left(\frac{2}{\epsilon} + \sum_{p=1}^{m} \frac{1}{p} - \gamma + O(\epsilon)\right) \quad \text{for integer } \; m \geq 0$$

- Feynman parameters:

$$\frac{1}{ab} = \int_0^1 \frac{dx}{[xa + (1-x)b]^2}$$

- Gaussian path integrals
  - Real

$$\int \mathcal{D}\phi \, e^{-\frac{1}{2}\int d^dx d^dy \phi(x)\mathbf{M}(x,y)\phi(y)} = \frac{\text{const}}{\sqrt{\det \mathbf{M}}}$$

  - Grassmannian

$$\int \mathcal{D}\theta^* \mathcal{D}\theta \, e^{-\int d^dx d^dy \theta^*(x)\mathbf{M}(x,y)\theta(y)} = \det \mathbf{M}$$

- Gamma matrices:

$$\text{tr}\gamma^\mu = 0, \quad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu},$$
$$\text{tr}\gamma^5 = 0, \quad (\gamma^5)^\dagger = \gamma^5, \quad \{\gamma^5, \gamma^\mu\} = 0.$$

Go to the next page for questions

1. Consider a simple harmonic oscillator with action

$$S = \int_{-\infty}^{\infty} dt \left(\frac{1}{2} m\dot{x}^2 - \frac{1}{2} m\omega_0^2 x^2 \right).$$

The amplitude for the oscillator to move from point x_a at time t_a to point x_b at time t_b is given by the path integral

$$U(x_a, x_b; t_b - t_a) \equiv \langle x_b; t_b | x_a; t_a \rangle = \int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}x(t) e^{iS},$$

where $|x; t\rangle = \exp(i\hat{H}t)|x\rangle$ is an eigenstate of the coordinate operator $\hat{x}(t)$ with eigenvalue x. Furthermore, we have for any operator $\hat{\mathcal{O}}[\hat{x}]$

$$\langle x_b; t_b | T\hat{\mathcal{O}} | x_a; t_a \rangle = \int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}x(t)\, \mathcal{O}[x(t)] e^{iS}.$$

(i) Show that you can write the ground state $|0\rangle$ as

$$|0\rangle \propto \lim_{T\to\infty} |x_a; -T\rangle,$$

with an appropriate rotation of the time coordinate on the complex plane. [4 marks]

(ii) Show that the ground state two-point correlator $\langle x(0)x(t)\rangle \equiv \langle 0|T\hat{x}(0)\hat{x}(t)|0\rangle$ can be expressed in terms of path integrals as

$$\langle x(0)x(t)\rangle = \frac{\int \mathcal{D}x\, x(0)x(t)\, e^{iS}}{\int \mathcal{D}x\, e^{iS}}.$$

[4 marks]

(iii) How can the two-point correlator $\langle x(0)x(t)\rangle$ be obtained from the the generating functional

$$Z[J] \equiv \int \mathcal{D}x\, e^{iS + i\int dt J(t)x(t)} \,?$$

[4 marks]

(iv) For the harmonic oscillator, we have

$$Z[J] = C \exp\left[-\frac{1}{2} \int \frac{d\omega}{2\pi} \frac{d\omega'}{2\pi} J(\omega) \frac{i2\pi\delta(\omega+\omega')}{m(\omega^2 - \omega_0^2 + i\epsilon)} J(\omega') \right],$$

where C is a constant and $J(\omega) = \int dt e^{i\omega t} J(t)$. Calculate the two-point function $\langle x(\omega)x(\omega')\rangle$ of the Fourier transformed coordinate variable

$$x(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} x(t).$$

[4 marks]

(v) Take the Fourier transform of your result to show that you obtain the standard result for zero-point fluctuations

$$\langle x(t)^2 \rangle = \frac{1}{2m\omega_0}.$$

[Hint: To do the integral, you may make use of $\Gamma(1/2) = \sqrt{\pi}$.] [4 marks]

[Total 20 marks]

2. The Lagrangian of the SU(N) Yang-Mills field A^a_μ coupled to a complex scalar (Higgs) field ϕ_i in the fundamental representation is

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}) \\ &\quad + \frac{g}{2} f^{abc}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) A^{b\mu} A^{c\nu} - \frac{g^2}{4} f^{abc} f^{ade} A^b_\mu A^c_\nu A^{d\,\mu} A^{e\,\nu} \\ &\quad + \partial_\mu \phi^*_i \partial^\mu \phi_i + i g t^a_{ij} A^a_\mu \left[(\partial^\mu \phi^*_i)\phi_j - \phi^*_i(\partial^\mu \phi_j)\right] + g^2 t^a_{ij} t^b_{jk} g^{\mu\nu} \phi^*_i A^a_\mu A^b_\nu \phi_j,\end{aligned}$$

where g is the gauge coupling constant, f^{abc} are the structure constants and t^a_{ij} are the group generators. The colour indices are $i, j \in \{1, \ldots, N\}$ and $a, b, c, d, e \in \{1, \ldots, N^2 - 1\}$.

(i) Find the dimensionalities of the fields ϕ_i and A^a_μ and the coupling g in d spacetime dimensions. [3 marks]

(ii) Draw the interaction vertices. Find the corresponding Feynman rules in their symmetric form, labelling all indices and momenta carefully. [12 marks]

(iii) Gauge fixing introduces an extra factor

$$\det\left[i\partial^\mu(\delta^{ab}\partial_\mu + g f^{abc} A^c_\mu)\right]$$

in the path integral. Explain how this determinant can be represented by a ghost field c, and describe the properties of this field. Write down the new terms that appear in the Lagrangian and the Feynman rules for the new vertex/vertices. [5 marks]

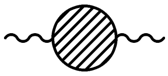
[Total 20 marks]

3. The bare Lagrangian of Quantum Electrodynamics is

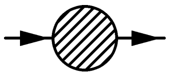
$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \bar{\psi}(i\partial\!\!\!/ - m)\psi - e\bar{\psi}\gamma^\mu\psi A_\mu,$$

where ψ is the electron field, A_μ is the U(1) gauge field, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor, γ^μ are the Dirac gamma matrices and $\partial\!\!\!/ = \gamma^\mu\partial_\mu$.

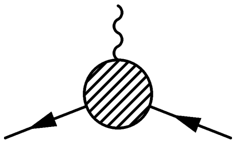
At one-loop level in *naive* perturbation theory, the superficially divergent one-particle irreducible correlation functions are



$$= -\frac{ie^2}{12\pi^2}(k^2 g^{\mu\nu} - k^\mu k^\nu)\left(\frac{2}{\epsilon} + \log\frac{4\pi\mu^2}{m^2} - \gamma\right) + \text{finite.}$$



$$= \frac{ie^2}{16\pi^2}\left(\xi k\!\!\!/ + (3+\xi)m\right)_{\beta\alpha}\left(\frac{2}{\epsilon} + \log\frac{4\pi\mu^2}{m^2} - \gamma\right) + \text{finite.}$$



$$= -ie\gamma^\mu_{\alpha\beta} - \frac{ie^3\xi}{16\pi^2}\gamma^\mu_{\alpha\beta}\left(\frac{2}{\epsilon} + \log\frac{4\pi\mu^2}{m^2} - \gamma\right) + \text{finite.} \qquad (1)$$

(i) Consider the same theory in *renormalised* perturbation theory. The renormalised fields are defined by $\psi = Z_\psi^{1/2}\psi_R$ and $A^\mu = Z_A^{1/2}A_R^\mu$, and renormalised parameters e_R, m_R and ξ_R by

$$\begin{aligned} Z_A^{1/2}Z_\psi e &= e_R + \delta e, \\ Z_\psi m &= m_R + \delta m, \\ Z_A/\xi &= Z_\xi/\xi_R. \end{aligned}$$

The renormalisation constants are written in terms of counterterms as $Z_A = 1 + \delta Z_A$, $Z_\psi = 1 + \delta Z_\psi$ and $Z_\xi = 1 + \delta Z_\xi$.

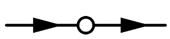
Doing these changes of variables explicitly, write the Lagrangian in terms of renormalised fields, renormalised parameters, and counterterms.

Identify the terms that are treated as interactions in renormalised perturbation theory.

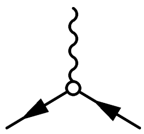
[5 marks]

(ii) Using the counterterm Feynman rules

$$\leftrightarrow \quad -i\left[\delta Z_A\left(k^2 g^{\mu\nu} - k^\mu k^\nu\right) - \frac{\delta Z_\xi}{\xi_R}k^\mu k^\nu\right]$$



$$\leftrightarrow \quad i\left(\delta Z_\psi k\!\!\!/ - \delta m\right)_{\beta\alpha}$$



$$\leftrightarrow \quad -i\delta e\gamma^\mu_{\alpha\beta},$$

write the correlation functions (1) in renormalised perturbation theory in terms of renormalised parameters and counterterms (i.e., without imposing a renormalisation condition yet). Then determine the values of the counterterms in the $\overline{\text{MS}}$ scheme, and use them to write the correlation functions (1) in terms of renormalised parameters only. [7 marks]

[This question continues on the next page ...]

(iii) Show that, to leading order in e, the beta function of the theory is

$$\beta(e) \equiv M \left.\frac{\partial e_R}{\partial M}\right|_B = \frac{e^3}{12\pi^2}.$$

Assuming that at $M = 0.5$ MeV, $e = 0.3$, and that there is only one species of charged particles (i.e. electron/positron), find the Landau pole of the theory (a rough approximate value is enough). Discuss the implications of your answer. [8 marks]

[Total 20 marks]

Please go to the next page

4. Consider the d-dimensional (Minkowskian) integral

$$I_2^\mu(k;m,0) = \int \frac{d^dp}{(2\pi)^d} \frac{p^\mu}{(p^2-m^2)(p+k)^2}.$$

(i) Explain why $I_2^\mu(k;m,0)$ has to be parallel to k^μ. [2 marks]

(ii) Show that

$$I_2^\mu(k;m,0) = \int_0^1 dx \int \frac{d^dp}{(2\pi)^d} \frac{p^\mu}{[(p+kx)^2-(1-x)(m^2-xk^2)]^2}.$$

[5 marks]

(iii) Show that

$$I_2^\mu(k;m,0) = -k^\mu \int_0^1 dx\, x \int \frac{d^dp}{(2\pi)^d} \frac{1}{[p^2-(1-x)(m^2-xk^2)]^2}.$$

[3 marks]

(iv) Carry out the momentum integration to express $I_2^\mu(k;m,0)$ as an integral over x only. [2 marks]

(v) Using

$$\int_0^1 dx\, x \ln(1-\alpha x) = \frac{1}{2}\left(1-\frac{1}{\alpha^2}\right)\ln(1-\alpha) - \frac{1}{4}\left(1+\frac{2}{\alpha}\right),$$

calculate the integral in $d = 4-\epsilon$ dimensions (with $\epsilon \ll 1$). You should find

$$I_2^\mu(k;m,0) = -\frac{ik^\mu}{32\pi^2}\left[\frac{2}{\epsilon} + \log\frac{4\pi}{m^2} - \gamma + 2 + \frac{m^2}{k^2} - \left(1-\frac{m^4}{k^4}\right)\ln\left(1-\frac{k^2}{m^2}\right)\right] + O(\epsilon).$$

[8 marks]

[Total 20 marks]

 End of examination paper