Imperial College London

MSc EXAMINATION May 2014

This paper is also taken for the relevant Examination for the Associateship

ADVANCED QUANTUM FIELD THEORY

For Students in Quantum Fields and Fundamental Forces

Tuesday, 13 May 2014: 14:00 to 17:00

Answer 3 out of the following 4 questions.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the THREE answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

Go to the next page for questions

You may use the following results without proof:

- Loop integral in d dimensions (Minkowskian):

$$I_n(m^2) \equiv \mu^\epsilon \int \frac{d^dp}{(2\pi)^d} \frac{1}{(p^2 - m^2)^n} = (-1)^n \frac{i\mu^\epsilon m^{d-2n}}{(4\pi)^{d/2}} \frac{\Gamma(n - d/2)}{\Gamma(n)}$$

- Gamma functions: $z\Gamma(z) = \Gamma(z+1)$ and

$$\Gamma\left(-m + \frac{\epsilon}{2}\right) = \frac{(-1)^m}{m!}\left(\frac{2}{\epsilon} + \sum_{p=1}^{m} \frac{1}{p} - \gamma + O(\epsilon)\right) \quad \text{for integer } \; m \geq 0$$

- Feynman parameters:

$$\frac{1}{ab} = \int_0^1 \frac{dx}{[xa + (1-x)b]^2}$$

- Gaussian path integrals
  - Real

$$\int \mathcal{D}\phi \, e^{-\frac{1}{2}\int d^dx d^dy \phi(x)\mathbf{M}(x,y)\phi(y)} = \frac{\text{const}}{\sqrt{\det \mathbf{M}}}$$

  - Grassmannian

$$\int \mathcal{D}\theta^* \mathcal{D}\theta \, e^{-\int d^dx d^dy \theta^*(x)\mathbf{M}(x,y)\theta(y)} = \det \mathbf{M}$$

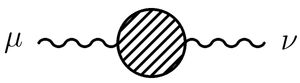
- Gamma matrices:

$$\text{tr}\gamma^\mu = 0, \quad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu},$$
$$\text{tr}\gamma^5 = 0, \quad (\gamma^5)^\dagger = \gamma^5, \quad \{\gamma^5, \gamma^\mu\} = 0.$$

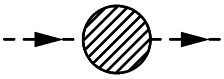
1. Scalar electrodynamics is a theory of an electrically charged (complex) scalar field ϕ. It has the bare Lagrangian

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}(\partial_\mu A_{B\nu} - \partial_\nu A_{B\mu})(\partial^\mu A_B^\nu - \partial^\nu A_B^\mu) \\ &\quad +(\partial_\mu \phi_B + ie_B A_{B\mu}\phi_B)^*(\partial^\mu \phi_B + ie_B A_B^\mu \phi_B) - m_B^2 \phi_B^* \phi_B - \frac{1}{4}\lambda_B(\phi_B^*\phi_B)^2.\end{aligned}$$

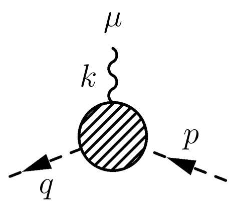
The superficially divergent 1PI correlation functions at one loop in *bare* perturbation theory (in the Lorentz gauge) are



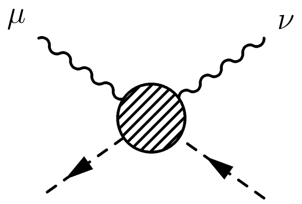
$$= -\frac{ie_B^2}{48\pi^2}(k^2 g^{\mu\nu} - k^\mu k^\nu)\left(\frac{2}{\epsilon} + \log\frac{4\pi\mu^2}{m_B^2} - \gamma + \text{finite}\right),$$



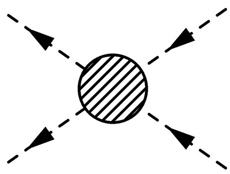
$$\begin{aligned}&= -\frac{3ie_B^2}{16\pi^2}k^2\left(\frac{2}{\epsilon} + \log\frac{4\pi\mu^2}{m_B^2} - \gamma + \text{finite}\right) \\ &\quad +\frac{i\lambda_B}{16\pi^2}m_B^2\left(\frac{2}{\epsilon} + \log\frac{4\pi\mu^2}{m_B^2} - \gamma + \text{finite}\right),\end{aligned}$$



$$= ie_B(p^\mu + q^\mu) - \frac{3ie_B^3}{16\pi^2}(p^\mu + q^\mu)\left(\frac{2}{\epsilon} + \log\frac{4\pi\mu^2}{m_B^2} - \gamma + \text{finite}\right),$$



$$= 2ie_B^2 g^{\mu\nu} - \frac{6ie_B^4}{16\pi^2}g^{\mu\nu}\left(\frac{2}{\epsilon} + \log\frac{4\pi\mu^2}{m_B^2} - \gamma + \text{finite}\right),$$



$$= -i\lambda_B + \left(\frac{3ie_B^4}{4\pi^2} + \frac{5i\lambda_B^2}{32\pi^2}\right)\left(\frac{2}{\epsilon} + \log\frac{4\pi\mu^2}{m_B^2} - \gamma + \text{finite}\right),$$

where γ is the Euler-Mascheroni constant.

(i) In renormalised perturbation theory, one defines

$$\begin{aligned} A_B^\mu &= Z_A^{1/2}A^\mu, & Z_\phi^2\lambda_B &= \lambda + \delta\lambda, \\ \phi_B &= Z_\phi^{1/2}\phi, & Z_\phi Z_A^{1/2} e_B &= Z_1 e, \\ Z_\phi m_B^2 &= m_R^2 + \delta m^2, & Z_\phi Z_A e_B^2 &= Z_2 e^2. \end{aligned}$$

and $Z_X = 1 + \delta Z_X$ for any X. Write the Lagrangian in terms of renormalised fields, renormalised couplings and counterterms. [4 marks]

(ii) Write down the Feynman rules for all vertices in renormalised perturbation theory. (Hint: For some of these you can make use of the correlators given above.) [6 marks]

(iii) Write down the same 1PI correlators as above in renormalised perturbation theory (Hint: Again, make use of the correlators given above!), and show that the counterterms in the

[This question continues on the next page . . .]

$\overline{MS}$ scheme are

$$\begin{aligned}
\delta Z_A &= -\frac{e^2}{48\pi^2}\left(\frac{2}{\epsilon}+\log\frac{4\pi\mu^2}{M^2}-\gamma\right),\\
\delta Z_\phi &= \frac{3e^2}{16\pi^2}\left(\frac{2}{\epsilon}+\log\frac{4\pi\mu^2}{M^2}-\gamma\right),\\
\delta m^2 &= \frac{\lambda}{16\pi^2}m^2\left(\frac{2}{\epsilon}+\log\frac{4\pi\mu^2}{M^2}-\gamma\right),\\
\delta\lambda &= \left(\frac{3e^4}{4\pi^2}+\frac{5\lambda^2}{32\pi^2}\right)\left(\frac{2}{\epsilon}+\log\frac{4\pi\mu^2}{M^2}-\gamma\right),\\
\delta Z_1 &= \frac{3e^2}{16\pi^2}\left(\frac{2}{\epsilon}+\log\frac{4\pi\mu^2}{M^2}-\gamma\right),\\
\delta Z_2 &= \frac{3e^2}{16\pi^2}\left(\frac{2}{\epsilon}+\log\frac{4\pi\mu^2}{M^2}-\gamma\right).
\end{aligned}$$

[5 marks]

(iv) Calculate the leading non-trivial terms in the beta functions of the two couplings

$$\beta_\lambda = \frac{d\lambda}{d\log M}, \quad \beta_e = \frac{de}{d\log M},$$

where M is the renormalisation scale. [5 marks]

[Total 20 marks]

2. The Lagrangian of the pseudoscalar Yukawa theory is

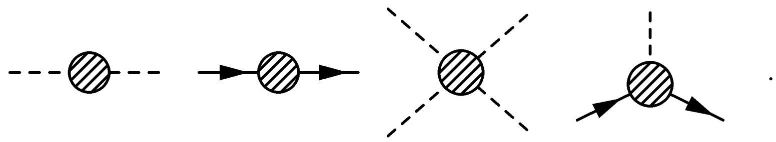
$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_\phi^2\phi^2 - \frac{1}{4!}\lambda\phi^4 + \bar{\psi}(i\partial\!\!\!/ - m)\psi - ig\bar{\psi}\gamma^5\psi\phi,$$

where ϕ is a real (pseudo)scalar field and ψ is a fermion field.

(i) Using the expressions for the Gaussian two-point functions in the rubric, derive the propagators for the two fields in bare perturbation theory. [5 marks]

(ii) Find the Feynman rules for the interaction vertices. [4 marks]

(iii) The superficially divergent correlators are



Draw the one-particle irreducible diagrams that contribute to these at one-loop level. [4 marks]

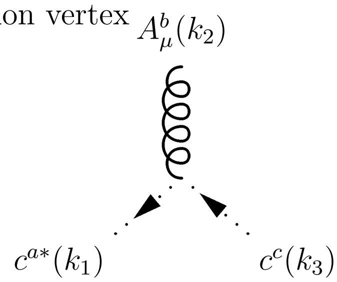
(iv) Use the Feynman rules to write the corresponding integrals. [6 marks]

[Total 19 marks]

3. The gauge-fixed Lagrangian of the Yang-Mills theory is

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}) \\ &\quad +\frac{g}{2} f^{abc}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) A^{b\mu} A^{c\nu} - \frac{g^2}{4} f^{abc} f^{ade} A^b_\mu A^c_\nu A^{d\,\mu} A^{e\,\nu} \\ &\quad -\frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 + \partial^\mu c^{a*}\left(\partial_\mu c^a - g f^{abc} A^b_\mu c^c\right). \end{aligned} \tag{1}$$

(i) Using the ghost-gluon vertex

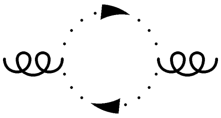


$\leftrightarrow \quad g f^{abc} {k_1}^\mu.$

and the ghost propagator

$c^{b*}(k)$  $c^a(k) \quad \leftrightarrow \quad \dfrac{i\delta^{ab}}{k^2},$

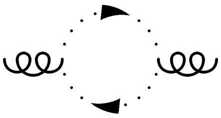
show that the ghost one loop correction to the gluon two-point correlator is given by the integral



$$= -g^2 f^{acd} f^{bcd} \mu^\epsilon \int \frac{d^d p}{(2\pi)^d} \frac{(p+k)^\mu p^\nu}{(p+k)^2 p^2}.$$

[5 marks]

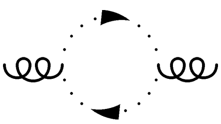
(ii) Show that this can be written as



$$= -g^2 f^{acd} f^{bcd} \int_0^1 dx \left[\mu^\epsilon \int \frac{d^d p}{(2\pi)^d} \frac{p^\mu p^\nu}{(p^2 - \Delta(x))^2} - x(1-x) k^\mu k^\nu \mu^\epsilon \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2 - \Delta(x))^2}\right],$$

where $\Delta(x) = -x(1-x)k^2$. [5 marks]

(iii) Show further that



$$= -g^2 f^{acd} f^{bcd} \int_0^1 dx \left[\frac{g^{\mu\nu}}{d-2}\Delta - x(1-x) k^\mu k^\nu\right] \mu^\epsilon I_2(\Delta), \tag{2}$$

where the function $I_2(\Delta)$ is given in the rubric. [5 marks]

(iv) Using the integral

$$\int_0^1 dx\, x(1-x) \log x = -\frac{5}{36},$$

calculate this diagram in $d = 4 - \epsilon$ dimensions (including the finite piece). [5 marks]

[Total 20 marks]

4. (i) Considering scalar electrodynamics with the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*(D^\mu\phi) - m^2\phi^*\phi - \frac{\lambda}{4}(\phi^*\phi)^2,$$

explain how you would usually calculate the propagator $D_F^{\mu\nu}(k)$ defined by

$$\langle A^\mu(k)A^\nu(q)\rangle = (2\pi)^4\delta(k+q)D_F^{\mu\nu}(k),$$

and show that this procedure fails. [5 marks]

(ii) Show that for a given gauge condition $G[A_\mu, \phi] = 0$, the path integral

$$I_\mathcal{O} = \int \mathcal{D}A_\mu\mathcal{D}\phi\, \mathcal{O}[A_\mu, \phi]e^{iS[A_\mu,\phi]},$$

where $\mathcal{O}[A_\mu, \phi]$ is a gauge-invariant observable, can be written as

$$I_\mathcal{O} \propto \int \mathcal{D}A_\mu\mathcal{D}\phi\, \delta(G[A_\mu, \phi]) \det\left(\frac{\delta G[A_\mu^\alpha, \phi^\alpha]}{\delta\alpha}\right) \mathcal{O}[A_\mu, \phi]e^{iS[A_\mu,\phi]},$$

where A_μ^α is the gauge transform of A_μ,

$$A_\mu^\alpha = A_\mu + \frac{1}{e}\partial_\mu\alpha, \quad \phi^\alpha = e^{i\alpha}\phi.$$

[6 marks]

(iii) Choosing $G[A_\mu, \phi] = \tilde{G}[A_\mu, \phi] - \omega(x)$, where $\omega(x)$ is an arbitrary function in spacetime and $\tilde{G}$ is independent of ω, show that you can write

$$I_\mathcal{O} \propto \int \mathcal{D}A_\mu\mathcal{D}\phi\, \det\left(\frac{\delta\tilde{G}[A_\mu^\alpha, \phi]}{\delta\alpha}\right) \mathcal{O}[A_\mu, \phi]e^{iS_\xi[A_\mu,\phi]},$$

where

$$S_\xi[A_\mu, \phi] = S[A_\mu, \phi] - \int d^4x \frac{1}{2\xi}\tilde{G}^2.$$

Discuss the ξ-dependence of individual Feynman diagrams, renormalisation counterterms and scattering amplitudes. [5 marks]

(iv) Show that you can express $I_\mathcal{O}$ as a path integral

$$I_\mathcal{O} \propto \int \mathcal{D}A_\mu\mathcal{D}\phi\mathcal{D}c^*\mathcal{D}c\mathcal{O}[A_\mu, \phi]e^{iS_{\text{gf}}[A_\mu,\phi,c]},$$

where the full gauge-fixed action is

$$S_{\text{gf}}[A_\mu, \phi, c] = S[A_\mu, \phi] - \int d^4x \left[\frac{1}{2\xi}\tilde{G}^2 + c^*\left(\frac{\delta\tilde{G}}{\delta\alpha}\right)c\right].$$

What are the properties of the new field c? What type of particle does it describe?

[4 marks]

[Total 20 marks]

 End of examination paper