Imperial College London

MSc EXAMINATION May 2016

This paper is also taken for the relevant Examination for the Associateship

ADVANCED QUANTUM FIELD THEORY

For Students in Quantum Fields and Fundamental Forces

Friday, 6th May 2016: 14:00 to 17:00

Answer THREE out of the following FOUR questions.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the THREE answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

Go to the next page for questions

You may use the following results without proof:

- Chain rule for functional differentiation:

$$\frac{\delta F(x)}{\delta B(y)} = \int \mathrm{d}^4 z \, \frac{\delta F(x)}{\delta A(z)} \frac{\delta A(z)}{\delta B(y)}.$$

- Minkowski-space loop integral in $d = 4 - \epsilon$ dimensions:

$$I_n(m^2) := \mu^\epsilon \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{1}{(p^2 - m^2 + \mathrm{i}\epsilon')^n} = (-1)^n \frac{\mathrm{i}\mu^\epsilon m^{d-2n}}{(4\pi)^{d/2}} \frac{\Gamma(n - \frac{1}{2}d)}{\Gamma(n)},$$

  where $\Gamma(z+1) = z\Gamma(z)$, such that $\Gamma(m) = (m-1)!$ if m is a positive integer and

$$\Gamma(\delta) = \delta^{-1} - \gamma + O(\delta), \qquad \Gamma(-1+\delta) = -(\delta^{-1} + 1 - \gamma) + O(\delta),$$

  for small δ where $\gamma = 0.577216...$ is the Euler–Mascheroni constant.

- Gaussian path integrals:

$$\text{real:} \qquad \int \mathcal{D}\phi \; \mathrm{e}^{-\frac{1}{2}\int \mathrm{d}^d x\, \mathrm{d}^d y\, \phi(x) M(x,y) \phi(y)} = \frac{\text{const.}}{\sqrt{\det M}},$$

$$\text{Grassmannian:} \qquad \int \mathcal{D}\theta^* \, \mathcal{D}\theta \; \mathrm{e}^{-\int \mathrm{d}^d x\, \mathrm{d}^d y\, \theta^*(x) M(x,y) \theta(y)} = \det M$$

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1. Consider a particle with action

$$S[q;t_a,t_b] = \int_{t_a}^{t_b} \mathrm{d}t\,\left[\tfrac{1}{2}m\dot{q}^2 - V(q)\right]$$

Let $|q,t\rangle$ be the eigenstate of the position operator $\hat{q}(t)|q;t\rangle = q|q;t\rangle$ at time t. The amplitude for the particle to evolve from position q_a at time t_a to position q_b at time t_b is given by the path integral

$$U(q_a,q_b;t_b-t_a) := \langle q_b;t_b|q_a;t_a\rangle = \int_{q(t_a)=q_a}^{q(t_b)=q_b} \mathcal{D}q\ \mathrm{e}^{\mathrm{i}S[q:t_a,t_b]/\hbar}.$$

(i) Discuss briefly the physical interpretation of this path integral. In the limit $\hbar \to 0$, which paths $q(t)$ in the integral give the largest contribution to the amplitude? Why?

By inserting factors of $\mathbb{1} = \int \mathrm{d}q\,|q,t\rangle\langle q,t|$, show that

$$\begin{aligned}\langle q_b;t_b|\hat{q}(t)|q_a;t_a\rangle &= \int \mathrm{d}q\,U(q,q_b;t_b-t)\,qU(q_a,q;t-t_a)\\ &= \int_{q(t_a)=q_a}^{q(t_b)=q_b} \mathcal{D}q\ q(t)\mathrm{e}^{\mathrm{i}S[q;t_a,t_b]/\hbar}.\end{aligned}$$

Hence argue that in general one gets the time-ordered expression

$$\int_{q(t_a)=q_a}^{q(t_b)=q_b} \mathcal{D}q\ q(t_1)\cdots q(t_n)\,\mathrm{e}^{\mathrm{i}S[q;t_a,t_b]/\hbar} = \langle q_b;t_b|T\,\hat{q}(t_1)\cdots\hat{q}(t_n)|q_a;t_a\rangle.$$

[7 marks]

(ii) Setting $\hbar = 1$ and given $|q;t\rangle = \mathrm{e}^{\mathrm{i}\hat{H}t}|q\rangle$ where $\hat{H}$ is the Hamiltonian, show that $\lim_{T\to\infty(1-\mathrm{i}\epsilon)}|q;-T\rangle \propto |\Omega\rangle$ where $|\Omega\rangle$ is the ground state. Hence show that

$$\langle\Omega|T\,\hat{q}(t_1)\cdots\hat{q}(t_n)|\Omega\rangle = \lim_{T\to\infty(1-\mathrm{i}\epsilon)} \frac{\int \mathcal{D}q\ q(t_1)\cdots q(t_n)\,\mathrm{e}^{\mathrm{i}S[q;-T,T]}}{\int \mathcal{D}q\ \mathrm{e}^{\mathrm{i}S[q;-T,T]}}.$$

[6 marks]

(iii) The generating functional is given by

$$Z[J] = \lim_{T\to\infty(1-\mathrm{i}\epsilon)} \int \mathcal{D}q\,\mathrm{e}^{\mathrm{i}S[q]+\mathrm{i}\int \mathrm{d}t\,q(t)J(t)}.$$

By considering a change of integration variables in $Z[J]$ from $q(t)$ to $q'(t) = q(t)+\alpha(t)$ for arbitrary, small $\alpha(t)$, show that

$$\int \mathrm{d}t\,\alpha(t) \lim_{T\to\infty(1-\mathrm{i}\epsilon)} \int \mathcal{D}q\,\left(\frac{\delta S[q]}{\delta q(t)} + J(t)\right)\,\mathrm{e}^{\mathrm{i}S[q]+\mathrm{i}\int \mathrm{d}t\,q(t)J(t)} = 0.$$

(You may assume $\mathcal{D}q = \mathcal{D}q'$.) [3 marks]

(iv) Hence show that

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}\langle\Omega|T\,\hat{q}(t)\hat{q}(t')|\Omega\rangle + \langle\Omega|T\,\widehat{\frac{\mathrm{d}V(t)}{\mathrm{d}q}}\,\hat{q}(t')|\Omega\rangle = -\mathrm{i}\delta(t-t').$$

[4 marks]

[Total 20 marks]

2. The generating functionals $Z[J]$, $E[J]$ and $\Gamma[\phi_{\text{cl}}]$ for a scalar theory are related by

$$\mathrm{i}\ln Z[J] = E[J] = -\Gamma[\phi_{\text{cl}}] - \int \mathrm{d}^4x\,\phi_{\text{cl}}(x)J(x),$$

where $\phi_{\text{cl}}(x) = -\delta E[J]/\delta J(x)$, and the corresponding correlation functions are given by

$$G_n(x_1,\ldots,x_n) = \frac{(-\mathrm{i})^n}{Z[0]}\,\frac{\delta^n Z[J]}{\delta J(x_1)\cdots\delta J(x_n)}\bigg|_{J=0},$$

$$\mathcal{G}_n(x_1,\ldots,x_n) = (-\mathrm{i})^{n+1}\,\frac{\delta^n E[J]}{\delta J(x_1)\cdots\delta J(x_n)}\bigg|_{J=0},$$

$$\Gamma_n(x_1,\ldots,x_n) = \mathrm{i}\,\frac{\delta^n \Gamma[\phi_{\text{cl}}]}{\delta\phi_{\text{cl}}(x_1)\cdots\delta\phi_{\text{cl}}(x_n)}\bigg|_{\phi_{\text{cl}}=0}.$$

(i) In a perturbation expansion, what kind of Feynman diagrams contribute to G_n? What about $\mathcal{G}_n$ and Γ_n?
Consider $\lambda\phi^4$ theory. Argue that $G_2(x_1,x_2) = \mathcal{G}_2(x_1,x_2)$ and draw the tree-level and one-loop Feynman diagrams that contribute to the functions G_4, $\mathcal{G}_4$ and Γ_4. [6 marks]

(ii) If A is a constant, the effective potential $V_{\text{eff}}(A)$ is defined by

$$\Gamma[\phi_{\text{cl}}]\big|_{\phi_{\text{cl}}=A} = -\int \mathrm{d}^4x\,V_{\text{eff}}(A).$$

Ignoring terms independent of A, show that

$$V_{\text{eff}}(A) = \mathrm{i}\sum_{n=1}^{\infty}\frac{1}{n!}A^n\widetilde{\Gamma}_n(0,\ldots,0),$$

where the Fourier transform of $\Gamma_n(x_1,\ldots,x_n)$ is $(2\pi)^4\delta^{(4)}(\sum_i p_i)\widetilde{\Gamma}_n(p_1,\ldots,p_n)$. [4 marks]

(iii) Draw the Feynman diagrams that give the one-loop contribution to $V_{\text{eff}}(A)$ in $\lambda\phi^4$ theory. Using the standard Feynman rules show that

$$\begin{aligned}V_{\text{eff}}^{\text{1-loop}}(A) &= \sum_{n=1}^{\infty}\frac{\mathrm{i}}{2^{n+1}n}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\left(\frac{\lambda_0}{p^2-m_0^2+\mathrm{i}\epsilon'}\right)^n A^{2n},\\ &= -\tfrac{1}{2}\mathrm{i}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\log\left(\frac{p^2-m_0^2-\frac{1}{2}\lambda_0A^2+\mathrm{i}\epsilon'}{p^2-m_0^2+\mathrm{i}\epsilon'}\right)\end{aligned}$$

(You do not have justify the symmetry factors.) Which terms in the power series expression are divergent? [5 marks]

(iv) Show that, using dimensional regularisation,

$$\frac{\partial V_{\text{eff}}^{\text{1-loop}}}{\partial A^2} = -\frac{\lambda_0}{64\pi^2}\left(m_0^2+\tfrac{1}{2}\lambda_0A^2\right)\left[\frac{2}{\epsilon}+\log\left(\frac{4\pi\mu^2}{m_0^2+\frac{1}{2}\lambda_0A^2}\right)-\gamma+1\right].$$

[5 marks]

[Total 20 marks]

3. The action for electromagnetism is

$$S[A_\mu] = -\frac{1}{4}\int \mathrm{d}^4x\, F_{\mu\nu}F^{\mu\nu}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. It is invariant under gauge transformations

$$A_\mu \mapsto A_\mu^{(\alpha)} = A_\mu + \tfrac{1}{e}\partial_\mu \alpha.$$

(i) Show that in momentum space

$$S[\tilde{A}_\mu] = \tfrac{1}{2}\mathrm{i}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{d}^4q}{(2\pi)^4}\tilde{A}^*_\mu(p)\widetilde{M}^{\mu\nu}(p,q)\tilde{A}_\nu(q),$$

where $\widetilde{M}^{\mu\nu}(p,q) = \mathrm{i}(2\pi)^4\delta^{(4)}(p-q)\,(p^2\eta^{\mu\nu} - p^\mu p^\nu)$ and $\tilde{A}_\mu(p) = \tilde{A}^*_\mu(-p)$ is the Fourier transform of $A_\mu(x)$.
Show that $\widetilde{M}^{\mu\nu}(p,q)$ is not invertible and hence argue that the standard procedure for defining the propagator $\langle 0|TA_\mu(x)A_\nu(y)|0\rangle$ fails. [5 marks]

(ii) Let $G[A_\mu](x) = w(x)$ be some (local) gauge-fixing condition. By inserting

$$1 = \int \mathcal{D}\alpha \det\left(\frac{\delta G[A_\mu^{(\alpha)}]}{\delta\alpha}\right)\delta(G[A_\mu^{(\alpha)}] - w)$$

show that the partition function of $A_\mu(x)$ can be written as

$$Z[J^\mu] = \int \mathcal{D}A_\mu \det\left(\frac{\delta G[A_\mu^{(\alpha)}]}{\delta\alpha}\right)\mathrm{e}^{\mathrm{i}S_\xi[A_\mu]+\mathrm{i}\int \mathrm{d}^4x J^\mu(x)A_\mu(x)},$$

where $S_\xi[A_\mu] = S[A_\mu] - \frac{1}{2\xi}\int \mathrm{d}^4x\, G[A_\mu]^2$ for some constant ξ.
Do you expect correlation functions of A_μ to depend on $G[A_\mu]$ and ξ? What about correlation functions of $F_{\mu\nu}$? [4 marks]

(iii) Locality implies that $\delta G[A_\mu^{(\alpha)}](x)/\delta\alpha(y) = \delta^{(4)}(x-y)\Delta(x)$ for some operator $\Delta(x)$. Show that $Z[J^\mu]$ can be rewritten as

$$Z[J^\mu] = \int \mathcal{D}A_\mu \mathcal{D}c^*\mathcal{D}c\, \mathrm{e}^{\mathrm{i}S[A_\mu,c,c^*]+\mathrm{i}\int \mathrm{d}^4x J^\mu(x)A_\mu(x)}$$

where $S[A_\mu, c, c^*] = S_\xi[A_\mu] - \int \mathrm{d}^4x\, c^*\Delta c$. What are the properties of the new fields $c(x)$ and $c^*(x)$? What type of particle do they describe? [3 marks]

(iv) Show that

$$Z[J^\mu] \propto \int \mathcal{D}A_\mu \mathcal{D}c^*\mathcal{D}c\,\mathcal{D}B\, \mathrm{e}^{\mathrm{i}S[A_\mu,c,c^*,B]+\mathrm{i}\int \mathrm{d}^4x J^\mu(x)A_\mu(x)}$$

where

$$S[A_\mu, c, c^*, B] = \int \mathrm{d}^4x\left(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - c^*\Delta c + \tfrac{1}{2}\xi B^2 + BG[A_\mu]\right).$$

What are the properties of the new field $B(x)$? [3 marks]

[This question continues on the next page . . .]

(v) Take $G[A_\mu] = \partial^\mu A_\mu$ and show that $S[A_\mu, c, c^*, B]$ is invariant under the infinitesimal BRST transformations, parameterised by a constant Grassmannian variable ϵ,

$$\delta_\epsilon A_\mu = \tfrac{1}{e}\epsilon\partial_\mu c, \qquad \delta_\epsilon c = 0, \qquad \delta_\epsilon c^* = \epsilon B, \qquad \delta_\epsilon B = 0.$$

Defining an operator Q such that acting on fields $\epsilon Q \cdot A_\mu = \delta_\epsilon A_\mu$, and $\epsilon Q \cdot c = \delta_\epsilon c$ etc., show that $Q^2 = 0$. Outline how the physical states of the theory are defined using Q. [5 marks]

[Total 20 marks]

4. The Lagrangian density for $\lambda\phi^4$ theory is given by

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}m_0^2\phi^2 - \tfrac{1}{4!}\lambda_0\phi^4.$$

(i) In renormalised perturbation theory one defines

$$\phi = Z^{1/2}\phi_R, \qquad Zm_0^2 = m_R^2 + \delta m^2, \qquad Z^2\lambda_0 = \lambda_R + \delta\lambda,$$

where $Z = 1+\delta Z$. Rewrite the Lagrangian density in terms of the renormalised field, renormalised couplings, and counter terms. [2 marks]

(ii) Let $G_2^R(x,y) = \langle\Omega|T\phi_R(x)\phi_R(y)|\Omega\rangle$ be the renormalised two-point function. Using the standard Feynman rules, show that the Fourier transform can be written as

$$\begin{aligned}\widetilde{G}_2^R(p,q) &= (2\pi)^4\delta^{(4)}(p+q)\frac{\mathrm{i}}{p^2 - m_R^2 + \mathrm{i}\epsilon'}\sum_{k=0}^{\infty}\left(\frac{\mathrm{i}\widetilde{\Gamma}_2(p)}{p^2 - m_R^2 + \mathrm{i}\epsilon'}\right)^k \\ &= (2\pi)^4\delta^{(4)}(p+q)\frac{\mathrm{i}}{p^2 - m_R^2 - \mathrm{i}\widetilde{\Gamma}_2(p) + \mathrm{i}\epsilon'}\end{aligned}$$

where $\widetilde{\Gamma}_2(p)$ is the one-particle irreducible (1PI) two-point function.

[5 marks]

(iii) By considering one-loop and counter-term diagrams, show that

$$\widetilde{\Gamma}_2(p) = \tfrac{1}{2}\lambda_R I_1(m_R^2) + \mathrm{i}\left(p^2\delta Z - \delta m^2\right) + O(\lambda_R^2).$$

(where $I_1(m^2)$ is defined on page 2). Hence show that in the $\overline{\mathrm{MS}}$ scheme

$$\delta Z = 0 + O(\lambda_R^2) \qquad \delta m^2 = \frac{\lambda_R m_R^2}{32\pi^2}\left(\frac{2}{\epsilon} + \log\frac{4\pi\mu^2}{M^2} - \gamma\right) + O(\lambda_R^2).$$

[4 marks]

(iv) Now consider the renormalised four-point vertex function $\widetilde{\Gamma}_4^R(p_1,p_2,p_3,p_4)$. By considering one-loop and counter-term diagrams, show that

$$\widetilde{\Gamma}_4^R(0,0,0,0) = -\mathrm{i}\lambda_R + \tfrac{3}{2}\lambda_R^2 I_2(m_R^2) - \mathrm{i}\delta\lambda + O(\lambda_R^3).$$

(where $I_2(m^2)$ is defined on page 2). Hence show that in the $\overline{\mathrm{MS}}$ scheme

$$\delta\lambda = \frac{3\lambda_R^2}{32\pi^2}\left(\frac{2}{\epsilon} + \log\frac{4\pi\mu^2}{M^2} - \gamma\right) + O(\lambda_R^3).$$

[4 marks]

(v) Using the relation between bare and renormalised quantities given in part (i), show that $\gamma_2 = (M/m_R^2)(\partial m_R^2/\partial M)$ and $\beta = M(\partial\lambda_R/\partial M)$ are given by

$$\begin{aligned}\gamma_2 &= -\frac{M}{m_R^2}\frac{\partial\,\delta m_R^2}{\partial M} = \frac{\lambda_R}{16\pi^2} + O(\lambda_R^2), \\ \beta &= -M\frac{\partial\,\delta\lambda}{\partial M} = \frac{3\lambda_R^2}{16\pi^2} + O(\lambda_R^3),\end{aligned}$$

[This question continues on the next page . . .]

where the derivatives are defined holding the bare quantities fixed. Hence argue that λ_R is a marginally irrelevant coupling.

These equations also imply that the mass m_R^2 is a marginal coupling. Is this correct? Would we have found the same result if we had used a simple cut-off to regulate the theory? [5 marks]

[Total 20 marks]

End of examination paper