Imperial College London

MSc EXAMINATION May 2020

This paper is also taken for the relevant Examination for the Associateship

BLACK HOLES

For Students in Quantum Fields and Fundamental Forces

Friday, June 5th 2020: 10:00 to 14:00

Answer Question 1 and EITHER Question 2 or Question 3.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

QFFF examinations in the Spring 2020 examination session may be taken open-book.

At the top of each page of your answers, write your CID number, module code, question number and page number. Scan and upload your answers to the Turnitin dropboxes as described in the guidance documents in the Blackboard module for this exam. Upload each answer to the dropbox provided for that specific question.

Your uploaded file name should be of the form CID_ModuleCode_QuestionNumber(s).pdf

For each answer you should prepare a coversheet which should be the first page of your scanned answer. The coversheet should contain the following:

- your CID
- module name and code
- the question number
- the number of pages in your answer

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

1. The Schwarzschild metric for a black hole of mass M in Schwarzschild coordinates (t, r, θ, ϕ) is

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

(i) (a) Show that on radial null geodesics one has $dt^2 = (1-\frac{2M}{r})^{-2}dr^2 =: dr_*^2$, which defines the tortoise coordinate r_*. Show that $r_* = r + 2M\ln\left|\frac{r-2M}{2m}\right|$ where $-\infty < r_* < \infty$ for spacetime outside the horizon. Give equations defining ingoing and outgoing radial null coordinates using the coordinates (t, r_*).

(b) Show that this metric can be written in ingoing Eddington-Finkelstein coordinates in the form

$$ds^2 = -\left(1-\frac{2M}{r}\right)dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

(c) Explain why the subsurface at $r = 2M$ is not in fact singular. Explain why $r = 0$ is singular.

(d) Draw the Carter-Penrose diagram for the maximally extended Schwarzschild spacetime, which is also known as the Kruskal spacetime. Label $\mathscr{I}^+$, $\mathscr{I}^-$ (*i.e.* "scri-plus", "scri-minus") and i_0 in one of the asymptotically flat regions. Show the locations of the black hole horizon and singularity in this region.

(e) Label regions and show in which parts of the diagram the ingoing Eddington-Finkelstein coordinates $(v, r.\theta, \phi)$ are defined.

[18 marks]

(ii) (a) Define a causal vector. Define a causal curve. Define a time orientation, Define a future directed causal vector field

(b) Explain why the Killing vector field ∂_t is not acceptable as a time orientation in the Schwarzschild spacetime.

(c) Pick an appropriate time orientation vector field and consider the fate of a future-directed causal curve $x^\mu(\lambda)$ with nonvanishing $dx^\mu/d\lambda$, where $r(\lambda_0) \leq 2M$. Show that $r(\lambda) \leq 2M$ for all $\lambda \geq \lambda_0$.

(d) Define a black hole and show that the Schwarzschild geometry contains a black hole.

[16 marks]

(iii) (a) Consider a point particle falling radially towards the center of the Schwarzschild black hole, and parametrise it's worldline using the proper time, $d\tau^2 = -ds^2$. Show that $\mathscr{E} = -g_{tt}\frac{dt}{d\tau}$ is a constant of the motion for this particle motion.

(b) Show that

$$(\dot{r})^2 = \frac{1}{\mathscr{E}^2}(1-\frac{2M}{r})^2(\frac{2M}{r} - 1 + \mathscr{E}^2); \quad \dot{r} = \frac{dr}{dt}.$$

(c) For a particle starting from rest at $r = r_{\rm max} > 2M$, show that $r_{\rm max} = \frac{2M}{1-\mathscr{E}^2}$. For gravitationally bound motion, show that $0 < \mathscr{E} < 1$.

[9 marks]

(iv) (a) Find an integral expression for the time taken for a particle starting at rest at $r = r_{\rm max} > 2M$ to fall in to $r = 2M$ and show that this time diverges as $r \to 2M$.

(b) For a comoving observer traveling along with the infalling particle, using proper-time parametrisation, show that the proper time evolved in falling from $r = r_{\rm max}$ to $r_{\rm end}$ is

$$\tau(r_{\rm end}) = (1-\mathscr{E}^2)^{-\frac{1}{2}} r_{\rm max}\left((z(1-z))^{\frac{1}{2}} + \arcsin[(1-z)^{\frac{1}{2}}]\right)$$

where $z = r_{\rm end}/r_{\rm max}$.

(c) Hence show that the comoving observer proper time for the infalling particle to reach $r = 2M$ is finite, and that the proper time to fall to the central singularity at $r = 0$ is $\tau(0) = \pi M(1-\mathscr{E}^2)^{-\frac{3}{2}}$.

[17 marks]

[Total 60 marks]

2. (i) (a) The Lie derivative of the metric with respect to a vector k can be written

$$\mathscr{L}_k g_{\mu\nu} = k^\rho \partial_\rho g_{\mu\nu} + g_{\rho\nu}\partial_\mu k^\rho + g_{\mu\rho}\partial_\nu k^\rho \ .$$

A Killing vector satisfies $\mathscr{L}_k g_{\mu\nu} = 0$. Show that $\nabla_{(\mu}k_{\nu)} = 0$.

(b) For a Killing vector k, show first that $\nabla_\nu\nabla_\mu k_\rho = R_{\rho\mu\nu\sigma}k^\sigma$ and hence $\nabla_\nu\nabla_\mu k^\nu = R_{\mu\nu}k^\nu$. You may note the covariant derivative commutator identity for an arbitrary vector V^μ: $(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)V^\rho = R_{\mu\nu}{}^\rho{}_\sigma V^\sigma$.

[8 marks]

(ii) (a) Give the definitions of a normal vector to a hypersurface and of a null hypersurface and define the generators of a null hypersurface. Define a Killing horizon $\mathscr{N}$ associated with a Killing vector ζ.

(b) Explain why one can write $\partial_\mu(\zeta^2)|_{\mathscr{N}} = -2\kappa\zeta_\mu|_{\mathscr{N}}$, *i.e.* $\zeta^\rho\nabla_\rho\zeta_\mu|_{\mathscr{N}} = \kappa\zeta_\mu|_{\mathscr{N}}$. The quantity κ is called the surface gravity.

(c) Show that the event horizon of the Schwarzschild solution is a Killing horizon for the vector $\zeta\partial_v$ in ingoing Eddington-Finkelstein coordinates and calculate the value of κ, noting that ζ is normalised so that $\zeta^2 \to -1$ as $r \to \infty$. Explain what happens to κ if one choses a Killing vector with a different normalisation, *i.e.* $\tilde{\zeta} = c\zeta$.

[20 marks]

(iii) Assume that, for ζ normal to $\mathscr{N}$ one has $\zeta_{[\mu}\nabla_\nu\zeta_{\rho]}|_{\mathscr{N}} = 0$ (from the Frobenius theorem), where the [ ] brackets indicate total antisymmetry in the enclosed indices μ, ν, ρ.

(a) Multiply by $\nabla^\mu\zeta^\nu$ and show that

$$\zeta_\rho(\nabla^\mu\zeta^\nu)(\nabla_\mu\zeta_\nu)|_{\mathscr{N}} = -2\kappa^2\zeta_\rho|_{\mathscr{N}}$$

and consequently (except at points where $\zeta = 0$) one has $\kappa^2 = -\frac{1}{2}(\nabla^\mu\zeta^\nu)(\nabla_\mu\zeta_\nu)|_{\mathscr{N}}$.

(b) Use the result of part (ib) to show that for a vector t which is tangent to $\mathscr{N}$ one has

$$t^\rho\partial_\rho\kappa^2 = -(\nabla^\mu\zeta^\nu)t^\rho R_{\nu\mu\rho\sigma}\zeta^\sigma \ .$$

(c) Noting that the null vector ζ is tangent to $\mathscr{N}$ as well as being normal to it, choose $t = \zeta$ and show that the surface gravity κ is constant on the orbits of ζ on $\mathscr{N}$.

[12 marks]

[Total 40 marks]

3. The Kerr metric in Boyer Lindquist coordinates for a rotating black hole of mass M and angular momentum $J = Ma$ is given by

$$ds^2 = -\frac{(\Delta - a^2 \sin^2\theta)}{\Sigma} dt^2 - 2a \sin^2\theta \frac{(r^2 + a^2 - \Delta)}{\Sigma} dt d\phi$$
$$+ \frac{[(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta]}{\Sigma} \sin^2\theta \, d\phi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2$$

where

$$\Delta = r^2 - 2Mr + a^2 \equiv (r - r_+)(r - r_-)$$
$$\Sigma = r^2 + a^2 \cos^2\theta$$

(i) (a) Identify the location of the spacetime singularity submanifold in these coordinates

(b) Draw the extended Carter-Penrose diagram for the Kerr solution for the slice $\theta = 0$. For one asymptotically flat region, label the future and past horizons, past and future null infinity, space-like infinity and the Cauchy horizon and show the location of the spacetime singularity.

[8 marks]

(ii) Define r_* and χ by

$$dr_* = \frac{r^2 + a^2}{\Delta} dr \qquad d\chi = d\phi + \frac{a}{\Delta} dr \,.$$

Then the ingoing Kerr coordinates are (v, r, θ, χ) where $v = t + r_*$.

(a) In the region where $r > r_+$, transform the metric into ingoing Kerr coordinates and show that the metric becomes regular at r_+ and r_- in those coordinates.

(b) Show that $k = \partial_v$ and $m = \partial_\chi$ are Killing vectors of this metric.

(c) On the hypersurface $\mathscr{H}_+$ defined by $S_+ = r - r_+ = 0$, find a normal vector ℓ_+, defined up to a multiplicative function f_+. Show that this normal vector is null.

(d) Show that the normal vector ℓ_+ found in part (iic) is proportional to a linear combination ξ_+ of the Killing vectors of part (iib) and show that $\mathscr{H}_+$ is a Killing horizon.

(e) Rewrite ξ_+ in Boyer-Lindquist coordinates and show that there is a linear relation between ϕ and t on the integral curves (orbits) of ξ_+. Find the constant of proportionality.

(f) Explain why the relation found in part (iie) shows that the generators of the horizon $\mathscr{H}_+$ rotate with respect to a stationary observer at infinity.

[26 marks]

(iii) Show that the Killing vector k is timelike for large r but becomes spacelike outside the $\mathscr{H}_+$ horizon in a region called the ergoregion. Derive the equation for the location of the outer boundary of the ergoregion, known as the ergosurface, as a function of θ. When does the ergosurface coincide with $\mathscr{H}_+$?

[6 marks]

[Total 40 marks]