IMPERIAL COLLEGE

MSc EXAMINATION

Black Holes

Time Allowed: 3 hours

Date: 19th May 2006

Time: 1pm

Instructions: **Answer Question 1 (40%) and TWO out of Questions 2,3 and 4 (30% each).**

DO NOT TURN TO THE FIRST PAGE OF THE QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY THE INVIGILATOR

 Version 1.1

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Question 1.

(a) Write down the Schwarzschild metric for a black hole of mass M in Schwarzschild coordinates (t, r, θ, ϕ). Derive the ingoing and outgoing null Eddington-Finkelstein (EF) coordinates, v and u respectively and calculate the Schwarzschild metric in terms of (v, r, θ, ϕ) and (u, r, θ, ϕ). Prove that $r = 2M$ is a coordinate singularity. How do we know that $r = 0$ is not a coordinate singularity? Sketch two spacetime diagrams for the spacetime described in terms of the two types of EF coordinates. Sketch ingoing and outgoing radial null geodesics on each and sketch a few representative lightcones. Show that the region $r < 2M$ has very different physical properties in the two cases. Why is this not a contradiction?

(b) Draw the Carter-Penrose diagram for $3 + 1$ dimensional Minkowski spacetime. Label and explain all its features. Draw ingoing and outgoing null geodesics.

(c) Draw the Carter-Penrose diagram for the maximally extended Schwarzschild-Kruskal vacuum spacetime. Label all its features. Relate this Penrose diagram to the spacetime diagrams you drew for part (a). Point out one unphysical feature of this spacetime.

Draw the Carter-Penrose diagram for the maximally extended Reissner-Nordstrom spacetime of a charged black hole. Label all its features. Point out one unphysical feature of this spacetime.

Question 2.

A spherically symmetric ball of pressure free dust collapses to form a black hole of mass 10^{30} kilogrammes. Initially the ball has radius 10^9 metres and is at rest.

An observer A with a stopwatch sits on one of the dust particles at the outer boundary of the collapsing ball and sets the watch going at the beginning of the collapse. What time approximately does the watch show, in seconds, at the moment the observer A crosses the horizon of the black hole that forms?

An observer B watches the collapse from infinity. What does B observe?

You may assume these approximate values for the Planck mass, Planck length and Planck time respectively:

$$M_p = 10^{-8}\,kg$$
$$L_p = 10^{-35}\,m$$
$$T_p = 10^{-43}\,s\,.$$

Recall that the Planck mass, length and time are those formed from the fundamental constants G (the gravitational constant), c (the speed of light in vacuum) and $\hbar$ (Planck's constant).

Question 3.

(a) Write down the equation for a Killing vector.

Let ξ be a Killing vector. The action of a particle of mass m is

$$S[x,e] = \frac{1}{2}\int_{\lambda_1}^{\lambda_2} d\lambda [e(\lambda)^{-1}\dot{x}^\mu \dot{x}^\nu g_{\mu\nu} - m^2 e(\lambda)]$$

where $\dot{x}^\mu$ denotes $\frac{dx^\mu}{d\lambda}$ and $e(\lambda)$ is an independent nonzero function. Show that the action is invariant, to first order in α, under variations of the path $x^\mu \to x^\mu + \alpha\xi^\mu$.

(b) The Kerr metric for a rotating black hole of mass M and angular momentum $J = Ma$ in Boyer-Lindquist coordinates is

$$\begin{aligned} ds^2 = &-\frac{(\Delta - a^2\sin^2\theta)}{\Sigma}dt^2 - 2a\sin^2\theta\frac{(r^2+a^2-\Delta)}{\Sigma}dt\,d\phi \\ &+ \frac{((r^2+a^2)^2 - \Delta a^2\sin^2\theta)}{\Sigma}\sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 \end{aligned}$$

where

$$\begin{aligned} \Sigma &= r^2 + a^2\cos^2\theta \\ \Delta &= r^2 - 2Mr + a^2 \end{aligned}$$

and $M > a$.

Consider the Killing vector $k \equiv \frac{\partial}{\partial t}$. Prove that k is timelike at infinity. Prove that there is a region *outside* the event horizon where k is spacelike. This region is called the ergoregion and its boundary is called the ergosphere. Sketch the ergosphere and ergoregion in a $t =$ constant hypersurface.

The conserved charge arising from the symmetry of the massive particle action associated with Killing vector k is $Q \equiv p_\mu k^\mu$, where $p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}$. What is the physical interpretation of this charge?

(c) A particle which has energy E at infinity falls freely into the ergoregion where it splits into two particles, A and B. Particle A falls into the black hole. Particle B escapes to infinity. Show that it is possible for particle A to have negative energy and thus for particle B to have more energy than the original particle. How is this consistent with conservation of energy? (You may use the fact that, given a spacelike vector, v^μ, there exist timelike vectors whose inner product with v^μ is positive.)

Use the Second Law of Black Hole Mechanics to prove that there is a limit to the amount of energy that can be extracted from a black hole of mass M and angular momentum J in the manner described above. What is this limit?

Question 4.

(a) Describe how a free massive real scalar field is quantised in a stationary spacetime with a Cauchy surface. Define the vacuum state.

(b) Suppose a spacetime (M, g) has the form of a sandwich: there are two non-intersecting Cauchy surfaces, Σ_1 and Σ_2 such that the spacetime is stationary to the past of Σ_1 and to the future of Σ_2, and in between it is time dependent.

Describe how this may lead to particle production. Derive the formula, in terms of Bogoliubov coefficients, for the expectation value of the number of particles in a certain mode as measured by an observer in the far future, in the vacuum state as defined by an observer in the far past?

Why is this sandwich spacetime example relevant to the derivation of Hawking radiation in the spacetime of gravitational collapse to a black hole?

(c) The surface gravity of a black hole of mass M, electric charge Q and magnetic charge P is

$$\kappa = \frac{1}{2r_+^2}(r_+ - r_-) \ , \tag{1}$$

where $r_\pm = M \pm \sqrt{M^2 - Q^2 - P^2}$. What is the Hawking temperature of the black hole? Calculate the rate of change of temperature with mass at fixed charge. For what range of the parameters is this derivative positive/negative? Suppose $M^2 >> Q^2 + P^2$ to begin with. Describe what happens as the black hole radiates assuming that magnetic monopoles do not exist. What is the endpoint of the process.

Why does Hawking evaporation not violate the Second Law of Black Hole Mechanics?