IMPERIAL COLLEGE

MSc EXAMINATION

Black Holes

Time Allowed:	3 hours
Date:	18th May 2007
Time:	1pm
Instructions:	**Answer Question 1 (40%) and TWO out of Questions 2, 3 and 4 (30% each).**
Marks:	The marks shown are indicative of those the examiners intend to assign.

DO NOT TURN TO THE FIRST PAGE OF THE QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY THE INVIGILATOR

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Question 1.

(a) (*10 marks*) Write down the Schwarzschild metric for a black hole of mass M in Schwarzschild coordinates (t, r, θ, ϕ). By transforming to ingoing Eddington-Finkelstein coordinates, (v, r, θ, ϕ), prove that $r = 2M$ is a coordinate singularity. Explain, qualitatively in one sentence, why $r = 0$ is not a coordinate singularity.

(b) (*10 marks*) Draw the Carter-Penrose diagram for the spacetime of a spherically symmetric star that collapses to form a black hole. Label all its features and boundaries.

(c) (*10 marks*) Draw the Carter-Penrose diagram for the maximally extended Schwarzschild-Kruskal vacuum spacetime. Label all its features and boundaries. How is this spacetime related to that in part (b)?

(d) (*5 marks*) Using the Carter Penrose diagram from part (b), explain qualitatively what is seen by an observer who remains at a fixed radial coordinate far from the black hole, as a spaceship falls towards and into the black hole.

(e) (*5 marks*) Two identical spaceships, labelled A and B, both fall into the black hole from rest starting from the same Schwarzschild radial coordinate, $r = R$ where $R > 2M$. A starts falling at Schwarzschild time coordinate $t = t_1$ and B starts falling later at time coordinate $t = t_2$ where $t_2 > t_1$. Draw the two spaceships' worldlines on a Carter-Penrose diagram of the black hole. Does B lose sight of A as A falls across the horizon of the black hole? Justify your answer.

Question 2.

(a) (*9 marks*) State the black hole Area Theorem, also known as the "Second Law of Black Hole Mechanics".

Raychaudhuri's equation for null geodesic congruences is

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \hat{\sigma}^{\mu\nu}\hat{\sigma}_{\mu\nu} + \hat{\omega}^{\mu\nu}\hat{\omega}_{\mu\nu} - R_{\mu\nu}t^\mu t^\nu \tag{1}$$

Explain the terms and parameters in this equation and its qualitative meaning.

(b) (*5 marks*)

Suppose that the tangent vector to the null geodesic congruence is everywhere normal to a family of null hypersurfaces and that the spacetime is a solution of Einstein's equations in which the energy momentum tensor satisfies the Weak Energy Condition. Prove that

$$\frac{d\theta}{d\lambda} \leq -\frac{1}{2}\theta^2 \,. \tag{2}$$

(You may assume Frobenius' Theorem that $\hat{\omega}_{\mu\nu}$ vanishes for null hypersurfaces.)

(c) (*14 marks*)

Sketch a proof of the Area Theorem. You can use results you do not prove if you state them clearly.

(d) (*2 marks*) Suppose a cosmological constant Λ is added to the theory so that the energy momentum tensor is a sum:

$$8\pi T^{(total)}_{\mu\nu} = 8\pi T^{(matter)}_{\mu\nu} - \Lambda g_{\mu\nu} \tag{3}$$

where $T^{(matter)}_{\mu\nu}$ satisfies the Weak Energy Condition. Does this affect the Area Theorem and does it make a difference whether Λ is positive or negative? Justify your answer.

Question 3.

(a) (*10 marks*) Describe how a free massive real scalar field is quantised in a stationary spacetime with a Cauchy surface. Define the vacuum state.

(b) (*12 marks*) Suppose a spacetime (M, g) has the form of a sandwich: there are two non-intersecting Cauchy surfaces, Σ_1 and Σ_2 such that the spacetime is stationary to the past of Σ_1 and to the future of Σ_2, and in between it is time dependent.

Describe how this may lead to particle production. Derive the formula, in terms of Bogoliubov coefficients, for the expectation value of the number of particles in a certain mode as measured by an observer in the far future, in the vacuum state as defined by an observer in the far past?

Explain qualitatively, with no calculation, how this sandwich spacetime is relevant to the derivation of Hawking radiation in the spacetime of gravitational collapse to a black hole.

(c) (*8 marks*) In Planck units where $G = c = \hbar = 1$, Stefan's law implies that

$$\frac{dM}{dt} \approx -\frac{\pi^2}{60} A\, T^4 \qquad (4)$$

where M is the mass of a black hole, A is its surface area and T is its Hawking temperature, and Boltzmann's constant is set to 1.

Calculate the approximate lifetime in seconds of a Schwarzschild black hole of mass 1 gram. You may use these approximate values for the Planck mass, Planck length and Planck time respectively:

$$M_p = 10^{-8}\, kg$$
$$L_p = 10^{-35}\, m$$
$$T_p = 10^{-43}\, s\,.$$

Question 4.

(a) (*10 marks*) The metric for a spherically symmetric black hole with mass M in a spacetime with positive cosmological constant Λ is the Schwarzschild-de Sitter metric, given here in static coordinates:

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega_2^2 \tag{5}$$

where

$$f(r) = 1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}, \tag{6}$$

and $d\Omega_2^2$ is the round metric on the 2-sphere.

$f(r)$ can be expressed, $f(r) = -\frac{\Lambda}{3}r^{-1}(r-r_1)(r-r_2)(r-r_3)$ where $r_1 < r_2 < r_3$. Show that r_1 is negative and r_2 and r_3 are positive. Show also that $f(r)$ is positive for r between r_2 and r_3.

Prove that both $r = r_2$ and $r = r_3$ are coordinate singularities.

(b) (*15 marks*)

Assuming that $r = r_2$ is the black hole horizon and $r = r_3$ is the cosmological horizon, prove that both these horizons are null hypersurfaces. Prove that vector $V = \frac{\partial}{\partial t}$ is a Killing vector. Prove that both the black hole and cosmological horizons are Killing horizons.

Calculate the surface gravity, κ, of the black hole horizon with respect to V.

(c) (*5 marks*)

An observer in the region between the horizons, $r_2 < r < r_3$, measures a temperature for the black hole and also a temperature for the cosmological horizon. The temperature for the black hole is related (by the standard formula) to its surface gravity w.r.t. V and the temperature for the cosmological horizon is related (by the standard formula) to its surface gravity w.r.t. $-V$ (the sign is reversed because the observer is inside the cosmological horizon). Prove that these two temperatures cannot be equal.