Einstein's equations

Derive Einstein's equations from the Einstein-Hilbert action
$I = \gamma \int d^4x \sqrt{-g} R$ where $R = g^{\mu\nu} R_{\mu\nu}(\Gamma(g))$ and $g = \det(g_{\mu\nu})$
$\gamma = \frac{2}{\kappa^2} = \frac{1}{16\pi G}$ where G is Newton's constant

The Ricci tensor is $R_{\mu\nu} = R_{\rho\mu}{}^{\rho}{}_{\nu}$ where the Riemann curvature tensor derives from the covariant derivative for a vector field $V^\nu(x)$
$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma_{\mu\rho}^{\nu} V^\rho$ $\Gamma_{\mu\rho}^{\nu}$: Christoffel connection
through the commutator
$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) V^\rho = R_{\mu\nu}{}^{\rho}{}_{\sigma} V^\sigma$

From this relation, one sees that $R_{\mu\nu}{}^{\rho}{}_{\sigma} = R_{\mu\nu}{}^{\rho}{}_{\sigma}(\Gamma)$ depends directly only on the symbol connection $\Gamma_{\mu\nu}^{\rho}$. Dependence on the metric $g_{\mu\nu}$ comes only via $\Gamma_{\mu\nu}^{\rho}(g, \partial g)$
Explicitly, one has
$R_{\mu\nu}{}^{\rho}{}_{\sigma} = \partial_\mu \Gamma_{\nu\sigma}^{\rho} - \partial_\nu \Gamma_{\mu\sigma}^{\rho} + \Gamma_{\mu\lambda}^{\rho} \Gamma_{\nu\sigma}^{\lambda} - \Gamma_{\nu\lambda}^{\rho} \Gamma_{\mu\sigma}^{\lambda}$

Consider for a moment the Palatini (1st order form) action
$I_{Pal} = \gamma \int d^4x \sqrt{-g}\, g^{\mu\nu} R_{\mu\nu}(\Gamma)$ where $\Gamma_{\mu\nu}^{\rho}$ is treated as an independent set of variables. One has initially $I_{Pal}(g, \Gamma, \partial\Gamma)$ but one can integrate by parts ∂_μ to obtain $I_{Pal}(g, \partial g, \Gamma)$ containing terms with one Γ and terms with two.
Varying $\frac{\delta I}{\delta \Gamma_{\mu\nu}^{\rho}}$ one then obtains an algebraic equation which is solved by $\Gamma_{\mu\nu}^{\rho}(g) = \frac{1}{2} g^{\rho\sigma}(g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma})$
where $g_{\mu\nu,\sigma} = \partial_\sigma g_{\mu\nu}$.
Note that the algebraic equation obtained is equivalent to
$g^{\rho\sigma} \nabla_\sigma g_{\mu\nu} = g^{\rho\sigma}(\partial_\sigma g_{\mu\nu} - \Gamma_{\sigma\mu}^{\lambda} g_{\lambda\nu} - \Gamma_{\sigma\nu}^{\lambda} g_{\mu\lambda}) = 0$
which is solved by the Christoffel $\Gamma_{\mu\nu}^{\rho}(g)$.

Consideration of the Palatini form of the action principal leads to a useful shortcut in deriving Einstein's equations: "1.5 order form".

Consider the action in the form

$I = \gamma \int d^4x \sqrt{-g}\, g^{\mu\nu} R_{\mu\nu}(\Gamma(g))$ where $\Gamma^\rho_{\mu\nu}(g)$ is the Christoffel connection.

In varying $\frac{\delta I}{\delta g_{\mu\nu}}$, one encounters terms involving $\delta \Gamma^\rho_{\mu\nu} = \frac{\delta \Gamma^\rho_{\mu\nu}(g)}{\delta g_{\lambda\sigma}} \delta g_{\lambda\sigma}$.

These become very complicated if they are broken up. The key strategy is not to do so. Keeping such $\delta \Gamma^\rho_{\mu\nu}$ grouped as they arise, the co-factors in δI are just the $\frac{\delta I}{\delta \Gamma^\rho_{\mu\nu}}$ equations. For $\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu}(g)$ Christoffel connections, these equations are solved, $\frac{\delta I}{\delta \Gamma^\rho_{\mu\nu}}\Big|_{\Gamma = \Gamma(g)} = 0$.

Consequently, these complicated terms can be ignored in the variation of I. The Einstein equations arise only from the explicit $\delta g_{\mu\nu}$ variations.

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Consider now these explicit variations.

For $g = \det(g_{\mu\nu})$, one has $\delta g = g\, g^{\mu\nu} \delta g_{\mu\nu}$

So $\delta \sqrt{-g} = \frac{1}{2\sqrt{-g}} (-g\, g^{\mu\nu} \delta g_{\mu\nu}) = \frac{1}{2} \sqrt{-g}\, g^{\mu\nu} \delta g_{\mu\nu}$.

Also $g^{\mu\sigma}$ is the inverse of $g_{\sigma\nu}$, ie $g^{\mu\sigma} g_{\sigma\nu} = \delta^\mu_\nu$

Thus $\delta g^{\mu\sigma} g_{\sigma\nu} + g^{\mu\sigma} \delta g_{\sigma\nu} = 0$

and hence $\delta g^{\mu\nu} = - g^{\mu\sigma} \delta g_{\sigma\rho} g^{\rho\nu}$

Accordingly,

$$\delta \int d^4x \sqrt{-g}\, g^{\mu\nu} R_{\mu\nu}(\Gamma) = \int d^4x \sqrt{-g} \left(\tfrac{1}{2} g^{\mu\nu} \delta g_{\mu\nu} R - g^{\sigma\mu} \delta g_{\mu\nu} g^{\nu\rho} R_{\sigma\rho} \right)$$

Consequently, the Einstein equations $\frac{\delta I}{\delta g_{\mu\nu}} = 0$ in the absence of matter are simply (dropping δ coeff)

$$-g^{\mu\sigma} g^{\nu\rho} R_{\sigma\rho} + \tfrac{1}{2} g^{\mu\nu} R = 0 \,, \quad \text{ie.} \quad -G^{\mu\nu} = 0$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein tensor.

In the presence of matter, one needs to include the matter action I_{matter} and vary it also.

From $\gamma \int d^4x \sqrt{-g}\, R + I_{\text{matter}}$ with $\gamma = \frac{2}{\kappa^2} = \frac{1}{16\pi G}$

one then obtains

$$-G^{\mu\nu} + \frac{\kappa^2}{2} \frac{\delta I_{\text{matter}}}{\delta g_{\mu\nu}} = 0$$

Now $\frac{\delta I_{\text{matter}}}{\delta g_{\mu\nu}} = \frac{1}{2} T^{\mu\nu}_{\text{matter}}$, the matter stress tensor.

Accordingly, one has

$$G^{\mu\nu} = \frac{\kappa^2}{4} T^{\mu\nu}_{\text{matter}} \qquad \text{and since } \kappa^2 = 32\pi G$$

this becomes the standard form of Einstein's equation

$$G^{\mu\nu} = 8\pi G\, T^{\mu\nu}_{\text{matter}}$$

Note how this matter-coupled equation agrees consistently with the required conservation of the matter $T^{\mu\nu}$. In flat space, one has $\partial_\mu T^{\mu\nu} = 0$ following from the matter's own field equations. In the presence of gravity, this becomes $\nabla_\mu T^{\mu\nu} = 0$, also following from the gravitationally coupled matter field equations. On the LHS of the Einstein equation, this accords with the Bianchi identity $\nabla_\mu G^{\mu\nu} \equiv 0$.

Some Differential Geometry

- **Manifold:**

An n-dimensional smooth manifold M can be covered with a set of local coordinate charts $\{\mathcal{O}_\alpha, \psi_\alpha\}$ where $\mathcal{O}_\alpha$ are open subsets in M and ψ_α are maps from $\mathcal{O}_\alpha$ to open subsets of $\mathbb{R}^n$ in a 1-1 and onto manner. For any two $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$ that overlap, the composition maps $\psi_\alpha \circ \psi_\beta^{-1}$, mapping an open subset of $\mathbb{R}^n$ to another, are taken to be C^∞ (smooth).

- **Vectors**

A vector is formally defined as a directional derivative. Let $\mathcal{F} = \{f: M \to \mathbb{R}, f \in C^\infty\}$ be the set of functions on M. A vector V at a point $p \in M$ is then defined to be a map $V: \mathcal{F} \to \mathbb{R}$ taking $f \in \mathcal{F}$ to $V(f)$, satisfying

1. Linearity: $V(af + bg) = a\,V(f) + b\,V(g)$ where a & $b \in \mathbb{R}$ and f & $g \in \mathcal{F}$
2. Leibniz rule: $V(fg) = f(p)\,V(g) + g(p)\,V(f)$

– Note that in local coordinates x^μ one can write $V = V^\mu \frac{\partial}{\partial x^\mu}$, where $V^\mu \in \mathbb{R}$ are the components of V and $V(f) = V^\mu \partial_\mu f$. If one denotes a basis set of vectors by $\{\partial/\partial x^\mu\}$, then V^μ are the components of V with respect to this basis.

– Upon making a coordinate transformation $x^\mu \to x^{\mu'}(x)$, one has $V = V^\mu \frac{\partial}{\partial x^\mu} = V^\mu \frac{\partial x^{\nu'}}{\partial x^\mu} \partial_{\nu'} = V^{\nu'} \partial_{\nu'}$

So $V^{\nu'} = V^\mu \frac{\partial x^{\nu'}}{\partial x^\mu}$ gives the usual transformation law for the components of V.

- A <u>vector field</u> on M is a specification of a vector $V(p)$ at each point $p \in M$. A vector field is said to be smooth if $V(f)$ (now a function on M) is smooth. In a local coordinate patch $\mathcal{O}_\alpha$, this is equivalent to the statement that the components $V^\mu(x)$ are smooth functions of x^ν.

  – Consider a smooth curve $\gamma : \mathbb{R} \to M$ taking $\lambda \in \mathbb{R}$ to $\gamma(\lambda) \in M$. For any function $f \in \mathcal{F}$, the composition $f \circ \gamma$ is a map from $\mathbb{R}$ to itself. For each point p lying on the curve γ in M, one can specify a vector $V(f) = \frac{d}{d\lambda}(f \circ \gamma)|_{\gamma^{-1}(p)}$.
In local coordinates, one has $V(f) = \frac{\partial f}{\partial x^\mu}\frac{dx^\mu}{d\lambda}$, so the components of V at the point $\gamma(\lambda)$ are $V^\mu = \frac{dx^\mu}{d\lambda} = \dot{x}^\mu$. This vector V is the <u>tangent vector</u> to the curve γ at the point $\gamma(\lambda)$ in M.

  – Conversely, given a vector field V, one can construct <u>integral curves</u> which have the property that one and only one curve passes through each point $p \in M$, for which the tangent vector to that curve at the point p is $V(p)$.

  – For a given vector field V it is possible to choose local coordinates such that $V = \frac{\partial}{\partial x^1}$, i.e. $V^\mu = (1, 0, \cdots, 0)$.

Vector field
Coordinate transformation $x^\mu \to x^{\mu'}(x) : V^{\mu'}(x') = \frac{\partial x^{\mu'}}{\partial x^\rho} V^\rho(x)$.

- <u>Co-vectors</u>, or <u>one-forms</u>

  The <u>co-vectors</u> at a point $p \in M$ live in the vector space dual to the vector space of vectors at p, i.e. they are linear maps taking vectors to the real numbers. The basis of co-vectors dual to the basis of vectors $\{\partial_\mu\}$ is denoted by $\{dx^\mu\}$. The action of dx^μ on ∂_ν is δ^μ_ν.

A general co-vector at a point $p \in M$ is written $W = W_\mu dx^\mu$, where the W_μ are the components of the co-vector. The action of W on a vector $V = V^\mu \partial_\mu$ is the contraction $V^\mu W_\mu \in \mathbb{R}$. A <u>co-vector field</u> is a specification of a co-vector $W(p)$ at each point $p \in M$.

• Tensors

<u>Tensors</u> of type (r,s) have components $T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}$. Higher rank tensors can be obtained by taking tensor products. For example, if S and T are two co-vectors, then $W = S \otimes T$ is an $(0,2)$ tensor with components $B_{\mu\nu} = S_\mu T_\nu$.

Tensors can have symmetry properties between indices of the same type: covariant or contravariant (i.e. down or up).

$$T_{(\mu\nu)} := \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu}) \quad \text{(symmetric)} \qquad T_{[\mu\nu]} := \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu}) \quad \text{(antisymmetric)}$$

These definitions are "strength one": if $S^{\mu\nu} = S^{(\mu\nu)}$ and $T_{\mu\nu}$ is a general tensor without symmetry, $S^{\mu\nu} T_{\mu\nu} = S^{\mu\nu} T_{(\mu\nu)}$. Similarly, for $A^{\mu\nu} = A^{[\mu\nu]}$ one has $A^{\mu\nu} T_{\mu\nu} = A^{\mu\nu} T_{[\mu\nu]}$. Moreover, $T_{\mu\nu} = T_{(\mu\nu)} + T_{[\mu\nu]}$.

For 3-index tensors, one has similarly

$$T_{(\mu\nu\rho)} := \frac{1}{3!}(T_{\mu\nu\rho} + T_{\mu\rho\nu} + T_{\rho\mu\nu} + T_{\rho\nu\mu} + T_{\nu\rho\mu} + T_{\nu\mu\rho})$$

$$T_{[\mu\nu\rho]} := \frac{1}{3!}(T_{\mu\nu\rho} - T_{\mu\rho\nu} + T_{\rho\mu\nu} - T_{\rho\nu\mu} + T_{\nu\rho\mu} - T_{\nu\mu\rho})$$

Young tableaux

For the 3-index tensors, however, one can also have mixed symmetry, eg. $T_{[\mu(\nu\rho)]} = \frac{1}{4}(T_{\mu\nu\rho} - T_{\nu\mu\rho} + T_{\nu\rho\mu} - T_{\rho\nu\mu})$.

For more general n-index tensors, one has symmetry types corresponding to all representations of the S_n symmetry group. These are characterised by all the corresponding Young tableaux types.

- **Lie derivative**

For a given vector field V, one can define a Lie derivative which acts on all different types of tensors. If $T^{\mu_1\dots\mu_r}{}_{\nu_1\dots\nu_s}$ is a tensor of type (r,s), then $\mathcal{L}_V T$ is also a tensor of type (r,s).

- Acting on a function f, one has $\mathcal{L}_V(f) = V(f)$
- Acting on a vector field W, one has $\mathcal{L}_V W = [V,W]$, where $[V,W]f = V(W(f)) - W(V(f))$.

In components, this becomes

$$(\mathcal{L}_V W)^\mu = V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu$$

The Lie derivative $\mathcal{L}_V$ commutes with contraction and also satisfies the Leibniz rule:

$$\mathcal{L}_V(S\otimes T) = (\mathcal{L}_V S)\otimes T + S\otimes(\mathcal{L}_V T).$$

From these properties, one obtains the action of $\mathcal{L}_V$ on any type of tensor, e.g. for T_ν $(0,1)$-type one has

$$(\mathcal{L}_V T)_\nu = V^\mu \partial_\mu T_\nu + T_\mu \partial_\nu V^\mu$$

while for $B_{\nu\rho}$ $(0,2)$-type one has

$$(\mathcal{L}_V B)_{\nu\rho} = V^\mu \partial_\mu B_{\nu\rho} + B_{\mu\rho}\partial_\nu V^\mu + B_{\nu\mu}\partial_\rho V^\mu, \text{ etc.}$$

- **Metric**

Now consider that the manifold $\mathcal{M}$ is endowed with a metric $g_{\mu\nu}(x) = g_{\nu\mu}(x)$, which is non-degenerate everywhere on $\mathcal{M}$. Using $g_{\mu\nu}$, one can lower indices: $V_\mu = g_{\mu\nu}V^\nu$. Using the inverse metric $g^{\mu\nu}$, defined by $g^{\rho\nu}g_{\nu\mu} = \delta^\rho_\mu$, one may raise indices: $W^\mu = g^{\mu\rho}W_\rho$. The metric is also used to define the Christoffel symbols (Christoffel connection)

$$\Gamma^\mu_{\nu\rho} = \tfrac{1}{2}g^{\mu\sigma}(g_{\sigma\nu,\rho} + g_{\sigma\rho,\nu} - g_{\nu\rho,\sigma}) = \Gamma^\mu_{\rho\nu}$$

from which one constructs the Levi-Civita covariant derivative

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\rho}V^\rho \;;\; \nabla_\mu W_\nu = \partial_\mu W_\nu - \Gamma^\rho_{\mu\nu}W_\rho.$$

By construction, $\nabla_\mu g_{\rho\sigma} = \partial_\mu g_{\rho\sigma} - \Gamma^\tau_{\mu\sigma}g_{\tau\rho} - \Gamma^\tau_{\mu\rho}g_{\tau\sigma} = 0$.

The Riemann curvature tensor satisfies $(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)V^\rho = R_{\mu\nu}{}^\rho{}_\sigma V^\sigma$

— Note that the Lie derivative does not depend upon the metric, but it sometimes may be useful to express Lie derivatives using ∇_μ, e.g.

$$\mathcal{L}_V(f) = V^\mu \nabla_\mu f \qquad \text{scalar function } f(x)$$
$$(\mathcal{L}_V Y)^\mu = V^\nu \nabla_\nu Y^\mu - Y^\nu \nabla_\nu V^\mu \qquad \text{vector field } Y^\mu(x)$$

- Killing vectors

A Killing vector field V has the special property that $\mathcal{L}_V g = 0$, or, in components

$$(\mathcal{L}_V g)_{\mu\nu} = V^\rho \nabla_\rho g_{\mu\nu} + g_{\rho\nu}\nabla_\mu V^\rho + g_{\mu\rho}\nabla_\nu V^\rho$$
$$= \nabla_\mu V_\nu + \nabla_\nu V_\mu = 2\nabla_{(\mu} V_{\nu)}$$

In general, the condition $\nabla_{(\mu} V_{\nu)} = 0$ is highly overdetermined and has no nonzero solution for V_μ. So the existence of a Killing vector field for a given metric $g_{\mu\nu}(x)$ is a special property of that metric.

— Consider making an infinitesimal general coordinate transformation $x^{\mu'} = x^\mu - \xi^\mu(x)$ on the metric:

$$g_{\mu\nu}(x) = \frac{\partial x^{\rho'}}{\partial x^\mu}\frac{\partial x^{\sigma'}}{\partial x^\nu} g_{\rho'\sigma'}(x') \qquad \text{and expand to 1st order in } \xi\text{: } \frac{\partial x^{\rho'}}{\partial x^\mu} = \delta^\rho_\mu - \partial_\mu \xi^\rho$$

$$g_{\mu\nu}(x) = g_{\mu'\nu'}(x') - \partial_\mu \xi^\rho g_{\rho\nu} - \partial_\nu \xi^\sigma g_{\mu\sigma}$$

but also $g_{\mu'\nu'}(x') = g_{\mu'\nu'}(x) - \xi^\rho \partial_\rho g_{\mu\nu}(x)$

so, comparing the effect of an infinitesimal general coordinate transformation on $g_{\mu\nu}(x)$ at the <u>same coordinate point</u> x^μ, one has

$$g_{\mu'\nu'}(x) - g_{\mu\nu}(x) = \delta g_{\mu\nu}(x) = \partial_\mu \xi^\rho g_{\rho\nu} + \partial_\nu \xi^\sigma g_{\mu\sigma} + \xi^\rho \partial_\rho g_{\mu\nu}$$

then rewrite in terms of $\xi_\mu = g_{\mu\rho}\xi^\rho$:

$$\delta g_{\mu\nu}(x) = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = \mathcal{L}_\xi g_{\mu\nu}$$

Hence a Killing vector field ξ generates a <u>symmetry</u> of the metric, under which $\delta g_{\mu\nu} = 0$.

Working in local coordinates such that $V = \frac{\partial}{\partial x^1}$, the Killing vector condition just becomes $\frac{\partial}{\partial x^1} g_{\mu\nu} = 0$. In this case, we speak of <u>adapted coordinates</u> for this symmetry.

– Note that infinitesimal general coordinate transformations of any tensor field, referred to the same coordinate point, are all expressed by Lie derivatives – this is not only true for the metric.

For example, consider such an infinitesimal general coordinate transformation of a co-vector field $\leftrightarrow$ (0,1) tensor field

$x^{\mu'} = x^\mu - \xi^\mu(x)$ and $W_\mu(x) = \frac{\partial x^{\rho'}}{\partial x^\mu} V_{\rho'}(x')$

so $W_\mu(x) = W_{\mu'}(x) - \xi^\rho \partial_\rho W_\mu(x) - \partial_\mu \xi^\rho W_\rho(x) + \mathcal{O}(\xi^2)$

ie. $\delta W_\mu(x) = W_{\mu'}(x) - W_\mu(x) = \xi^\rho \partial_\rho W_\mu(x) + W_\rho \partial_\mu \xi^\rho = (\mathcal{L}_\xi W)_\mu$.

• Geodesic motion of particles

Consider a spacetime $(\mathcal{M}, g)$ where $g_{\mu\nu}$ is a Lorentz-signature metric. We want to describe the motion of a particle of rest mass m moving on a curve $\gamma(\lambda)$ from point A to point B in $\mathcal{M}$. Let the curve $\gamma(\lambda)$ be specified in local coordinates by $x^\mu(\lambda)$.

λ: worldline parameter

Work in units for which $\hbar = c = G = 1$. The action for such a massive test particle (test: small, so its effect on the background metric $g_{\mu\nu}$ can be ignored) is

$$I = m \int_{\gamma_A}^{\gamma_B} d\tau \qquad \text{where } d\tau^2 = -g_{\mu\nu} dx^\mu dx^\nu, \ (\text{proper time})^2$$

Thus, for motion along $x^\mu(\lambda)$ one has

$$I[x^\mu(\lambda)] = m \int_{\lambda_A}^{\lambda_B} d\lambda \left[-g_{\mu\nu}(x(\lambda)) \dot{x}^\mu \dot{x}^\nu \right]^{1/2} .$$

In the variational principle calculation to find the extremising $x^\mu(\lambda)$, the variations are required to vanish at the end points: $\delta x^\mu(\lambda_A) = \delta x^\mu(\lambda_B) = 0$ (ie. Dirichlet type endpoint conditions). The extremisation condition $\frac{\delta I}{\delta x^\mu(\lambda)} = 0$ then determines the motion from A to B.

– Note that the worldline parametrisation λ may be freely changed by a one-dimensional reparametrisation $\lambda \to \lambda'(\lambda)$ in which $x^\mu(\lambda)$ is treated as a set of worldline scalars, $x^\mu(\lambda) \to x^\mu(\lambda')$

Then $I = m\int_{\lambda'_A}^{\lambda'_B} d\lambda' \frac{\partial\lambda}{\partial\lambda'}\left[-g_{\mu\nu}(x(\lambda'))\frac{\partial x^\mu}{\partial\lambda'}\frac{\partial\lambda}{\partial x^\mu}\frac{\partial x^\nu}{\partial\lambda'}\frac{\partial\lambda}{\partial x^\nu}\right]^{1/2}$

$= m\int_{\lambda'_A}^{\lambda'_B} d\lambda' \left[-g_{\mu\nu}(x(\lambda'))\frac{\partial x^\mu}{\partial\lambda'}\frac{\partial x^\nu}{\partial\lambda'}\right]^{1/2}$ since $\frac{\partial\lambda}{\partial\lambda'}$ factors cancel

Einbeins

A useful alternate presentation of the particle action introduces a new object defined along the curve $\gamma(\lambda)$, the "einbein" $e(\lambda) > 0$:

$\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$

$$\hat{I}[x^\mu(\lambda), e(\lambda)] = \frac{1}{2}\int_{\lambda_A}^{\lambda_B} d\lambda \left[e^{-1}(\lambda) g_{\mu\nu}(x(\lambda))\dot{x}^\mu\dot{x}^\nu - m^2 e(\lambda)\right]$$

The einbein $e(\lambda)$ is to be varied independently of $x^\mu(\lambda)$ in this formulation. We then need to impose two extremisation conditions: $\frac{\delta\hat{I}}{\delta x^\mu(\lambda)} = 0$ and $\frac{\delta\hat{I}}{\delta e(\lambda)} = 0$.

Varying $e(\lambda)$, one finds $\delta\hat{I} = \int_{\lambda_A}^{\lambda_B} d\lambda \left[-e^{-2} g_{\mu\nu}(x(\lambda))\dot{x}^\mu\dot{x}^\nu - m^2\right]\delta e(\lambda)$

So one obtains the algebraic equation

$$-e^{-2} g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - m^2 = 0 \implies e = \frac{1}{m}\left[-g_{\mu\nu}(x(\lambda))\dot{x}^\mu\dot{x}^\nu\right]^{1/2} = \frac{1}{m}\frac{d\tau}{d\lambda}$$

Since this is an algebraic equation, it is permissable to substitute the solution for $e(\lambda)$ back into $\hat{I}$ and one finds $\hat{I}[x^\mu(\lambda), e[x(\lambda)]] = -I[x(\lambda)]$ so the $\hat{I}[x,e]$ and $I[x]$ formulations give equivalent equations of motion. Check this:

$$-\frac{\delta I}{\delta x^\mu(\lambda)} = \frac{\delta\hat{I}}{\delta x^\mu(\lambda)}\Big|_{e=e[x]} + \int_{\lambda_A}^{\lambda_B} d\lambda' \frac{\delta\hat{I}}{\delta e(\lambda')}\Big|_{e[x]}\frac{\delta e[x(\lambda')]}{\delta x^\mu(\lambda)}$$

$$= \frac{\delta\hat{I}}{\delta x^\mu(\lambda)}\Big|_{e=e[x]}$$

(the integral term vanishes by $\frac{\delta\hat{I}}{\delta e(\lambda')} = 0$ for $e = e[x(\lambda')]$)

So the condition $\frac{\delta I}{\delta x^\mu(\lambda)} = 0$ is equivalent to $\frac{\delta\hat{I}}{\delta x^\mu(\lambda)}\Big|_{e=e(x)} = 0$.

(Note that the vanishing of the 2nd RHS term above happens for a similar reason to the vanishing of the $\frac{\delta I_{EHGF}}{\delta\Gamma^\rho_{\mu\nu}}\frac{\delta\Gamma^\rho_{\mu\nu}}{\delta g_{\sigma\lambda}}\delta g_{\sigma\lambda}$ term in the 1.5 order formalism derivation of Einstein's equation.)

Performing the $\delta x^\mu(\lambda)$ variation in $\hat{I}[x^\mu(\lambda), e(\lambda)]$ one finds that $\frac{\delta \hat{I}}{\delta x^\mu(\lambda)} = 0$ implies, after integrating $\frac{d}{d\lambda}$ by parts

$$e^{-2}\dot{e}\, g_{\mu\nu}\dot{x}^\nu - e^{-1} g_{\mu\nu}\ddot{x}^\nu - e^{-1} g_{\mu\nu,\rho}\dot{x}^\rho\dot{x}^\nu + \tfrac{1}{2} e^{-1} g_{\rho\nu,\mu}\dot{x}^\rho\dot{x}^\nu = 0$$

so

$$\ddot{x}^\nu + g^{\nu\sigma} g_{\sigma\mu,\rho}\dot{x}^\rho\dot{x}^\mu - \tfrac{1}{2} g^{\nu\sigma} g_{\rho\mu,\sigma}\dot{x}^\rho\dot{x}^\mu = e^{-1}\dot{e}\,\dot{x}^\nu$$

i.e.

$$\ddot{x}^\nu + \Gamma^\nu_{\mu\rho}\dot{x}^\mu\dot{x}^\rho = e^{-1}\dot{e}\,\dot{x}^\nu \quad \text{geodesic equation}$$

which can be rewritten $\frac{D}{d\lambda}\dot{x}^\mu = e^{-1}\dot{e}\,\dot{x}^\mu$

where

$$\frac{D}{d\lambda}\dot{x}^\mu = \ddot{x}^\mu + \Gamma^\mu_{\lambda\rho}\dot{x}^\lambda\dot{x}^\rho \quad \text{path covariant derivative of } \dot{x}^\mu$$

$e(\lambda)$ transforms like a worldline density

Using the worldline reparametrisation invariance, which in terms of the $\hat{I}$ action requires $e'(\lambda') = \frac{d\lambda}{d\lambda'} e(\lambda)$ (see the solution $e(x(\lambda)) = \frac{1}{m}\left[-g_{\mu\nu}(x(\lambda))\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right]^{1/2}$), one may pick a worldline parametrisation in which $\frac{de}{d\lambda} = \dot{e} = 0$. This gives an affinely parametrised geodesic for which $\frac{D}{d\lambda}\dot{x}^\mu = 0$. On such an affinely parametrised geodesic, one has $\lambda = a\tau + b$ where $a, b \in \mathbb{R}$ and τ is the proper time evolved along the geodesic, since $e = \frac{1}{m}\frac{d\tau}{d\lambda}$ is a constant function of λ.

- Massless particles

Note that setting $m = 0$ in the original $I[x(\lambda)]$ action sets that action to zero, so it cannot be used. However, the $\hat{I}[x(\lambda), e(\lambda)]$ action can still be used:

$$\hat{I}_{|m=0} = \frac{1}{2}\int_{\lambda_A}^{\lambda_B} d\lambda\, e^{-1}(\lambda)\, g_{\mu\nu}(x(\lambda))\,\dot{x}^\mu\dot{x}^\nu .$$

One still obtains $\frac{D}{d\lambda}\dot{x}^\mu = 0$ but in that case $ds^2 = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0$ and one has a null affinely parametrised geodesic.

For $m \neq 0$, one has $ds^2 = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -d\tau^2$ and $\lambda = a\tau + b$, a time-like affinely parametrised geodesic.

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General coordinate covariance for particle motion

We saw above that the particle action — either $I[x(\lambda)]$ or $\hat{I}[x(\lambda), e(\lambda)]$ — has a one-dimensional worldline reparametrisation invariance $\lambda \to \lambda'(\lambda)$. Under this, $x^\mu(\lambda)$ transforms like a worldline scalar, $x^\mu(\lambda) \to x^{\mu\prime}(\lambda') = x^\mu(\lambda)$. What about spacetime general coordinate transformations, $x^\mu \to x^{\mu\prime}(x)$? Clearly, x^μ is not tensorial under such transformations. However, $g_{\mu\nu}(x)$ is a $(0,2)$ tensor and both $I[x]$ and $\hat{I}[x,e]$ depend explicitly only on $\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$. This transforms according to the chain rule: $\dot{x}^{\mu\prime} = \frac{dx^{\mu\prime}}{d\lambda} = \frac{\partial x^{\mu\prime}(\lambda)}{\partial x^\rho(\lambda)} \frac{dx^\rho(\lambda)}{d\lambda} = \frac{\partial x^{\mu\prime}}{\partial x^\rho} \dot{x}^\rho$, so $\dot{x}^\mu$, unlike x^μ, actually is tensorial, of type $(1,0)$. Moreover, in $\hat{I}[x,e]$ the einbein $e(\lambda)$ does not have any spacetime transformation — it is a general coordinate scalar. Consequently, both $I[x(\lambda)]$ and $\hat{I}[x(\lambda), e(\lambda)]$ are also invariant actions under spacetime general coordinate transformations.

• Killing vectors and conservation laws

Suppose one has found a Killing vector V. Consider then the infinitesimal change (constant in λ) $x^\mu \to x^{\mu\prime} = x^\mu + \epsilon V^\mu(x)$, where ϵ is an infinitesimal parameter of the transformation. The effect on $\hat{I}[x^\mu(\lambda), e(\lambda)]$ comes from the changes in $x^\mu(\lambda)$ and $\dot{x}^\mu(\lambda)$ (i.e. $e(\lambda)$ doesn't vary): $\delta\hat{I} = \hat{I}[x^{\mu\prime}(\lambda), e(\lambda)] - \hat{I}[x^\mu(\lambda), e(\lambda)]$

$\dot{x}^{\mu\prime} = \dot{x}^\mu + \partial_\sigma V^\mu \dot{x}^\sigma \epsilon$

so $\delta_\epsilon \hat{I} = \int d\lambda\, e^{-1}(\lambda) \left[\partial_\mu V^\rho \dot{x}^\mu \dot{x}^\nu g_{\rho\nu} + \frac{1}{2} \partial_\rho g_{\mu\nu} V^\rho \dot{x}^\mu \dot{x}^\nu \right] \epsilon$

$= \int d\lambda\, e^{-1}(\lambda)\, \nabla_\mu V_\rho \dot{x}^\mu \dot{x}^\nu g_{\rho\nu}\, \epsilon = \int d\lambda\, e^{-1} \nabla_{(\mu} V_{\nu)} \dot{x}^\mu \dot{x}^\nu \epsilon = 0$

Thus, one has a symmetry of the action $\hat{I}$.

Then use Noether's theorem. Let $\epsilon \to \epsilon(\lambda)$ now λ-dependent and calculate again: $\delta_\epsilon \frac{dx^{\mu\prime}}{d\lambda} = \partial_\sigma V^\mu \dot{x}^\sigma \epsilon + V^\mu \dot{\epsilon}$, so $\delta_{\epsilon(\lambda)} \hat{I} = \int d\lambda\, \dot{\epsilon}\, e^{-1} g_{\mu\nu}(x) V^\mu \dot{x}^\nu$. The cofactor of $\dot{\epsilon}$ is the Noether charge $Q = e^{-1} V_\nu \dot{x}^\nu$. Verify by calculation using the $\hat{I}$ field equations that $\frac{dQ}{d\lambda} = 0$, so Q is a conserved quantity.

Source:
Nakahara: Geometry, Topology and Physics

• Differential forms

We now return to general structure independent of a metric.

As above, let M be an n-dimensional manifold. An r-form is a totally antisymmetric $(0,r)$ co-tensor; in components one has $A_{\mu_1\cdots\mu_r} = A_{[\mu_1\cdots\mu_r]}$

In order to be non-vanishing, one must have $r \leq n$.

Basis forms and wedge products

For general co-tensors ω of type $(0,r)$, one expands in components $\omega = \omega_{i_1\cdots i_r}\, dx^{i_1}\otimes\cdots\otimes dx^{i_r}$, where $\otimes$ denotes a naive tensor product. Now suppose that $\omega_{i_1\cdots i_r}$ happens to be totally antisymmetric, i.e. $\omega_{i_1\cdots i_r} = \omega_{[i_1\cdots i_r]}$ (antisymmetrization "strength one" as usual). Then the appropriate basis for such a totally antisymmetric r-form may itself be totally antisymmetrised. The corresponding r-form basis is written $dx^{i_1}\wedge dx^{i_2}\wedge\cdots\wedge dx^{i_r}$, where $\wedge$ is the wedge product. Adopting the convention of writing $\omega = \frac{1}{r!}\omega_{i_1 i_2\cdots i_r}\, dx^{i_1}\wedge dx^{i_2}\wedge\cdots\wedge dx^{i_r}$ and setting this equal to the $(0,r)$ co-vector expansion in terms of a naive tensor product, one has

$$dx^{i_1}\wedge\cdots\wedge dx^{i_r} := dx^{i_1}\otimes\cdots\otimes dx^{i_r} + \text{even permutations of } i_1\cdots i_r - \text{odd permutations of } i_1\cdots i_r$$

Eg. $dx^i\wedge dx^j \equiv dx^i\otimes dx^j - dx^j\otimes dx^i$ 2-form basis

$$dx^i\wedge dx^j\wedge dx^k \equiv dx^i\otimes dx^j\otimes dx^k + dx^j\otimes dx^k\otimes dx^i + dx^k\otimes dx^i\otimes dx^j - dx^i\otimes dx^k\otimes dx^j - dx^j\otimes dx^i\otimes dx^k - dx^k\otimes dx^j\otimes dx^i$$

The wedge product has the properties

i) $dx^{i_1}\wedge\cdots\wedge dx^{i_r} = 0$ if some index i appears at least twice

ii) For a permutation $i_1\cdots i_r \to P(i_1)\cdots P(i_r)$, one has
$dx^{i_1}\wedge\cdots\wedge dx^{i_r} = \mathrm{sgn}(P)\, dx^{P(i_1)}\wedge\cdots\wedge dx^{P(i_r)}$
$\mathrm{sgn}(P) = +1$ even perms
$\mathrm{sgn}(P) = -1$ odd perms

iii) $dx^{i_1}\wedge\cdots\wedge dx^{i_r}$ is linear in each dx^i

If one denotes the vector space of r-forms at a point $p\in M$ by Ω^r_p, and if M is a manifold of dimension n, then there are $\binom{n}{r}$ choices of the set $(i_1\cdots i_r)$ out of $(1,2,\ldots,n)$ so the dimension of the vector space Ω^r_p is $\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$

- Exterior product

The exterior product $\wedge$ of a q-form $\xi \in \Omega^q_p$ and an r-form $\omega \in \Omega^r_p$ is a $(q+r)$-form $\xi \wedge \omega \in \Omega^{q+r}_p$. For $\xi = \frac{1}{q!}\xi_{i_1 \cdots i_q} dx^{i_1} \wedge \cdots \wedge dx^{i_q}$ and writing $\omega = \frac{1}{r!}\omega_{i_{q+1} \cdots i_{q+r}} dx^{i_{q+1}} \wedge \cdots \wedge dx^{i_{q+r}}$ then for

$$f = \xi \wedge \omega = \frac{1}{(q+r)!} f_{i_1 \cdots i_{q+r}} dx^{i_1} \wedge \cdots \wedge dx^{i_q} \wedge dx^{i_{q+1}} \wedge \cdots \wedge dx^{i_{q+r}}$$

one has $f_{i_1 \cdots i_{q+r}} = \frac{(q+r)!}{q!\,r!} \xi_{[i_1 \cdots i_q} \omega_{i_{q+1} \cdots i_{q+r}]}$

Note that if $q+r > n$, then $\xi \wedge \omega$ vanishes identically.

Using the exterior product, one defines an algebra $\Omega^*_p(M) = \Omega^0_p(M) \oplus \Omega^1_p(M) \oplus \cdots \oplus \Omega^n_p(M)$, where for convenience one defines $\Omega^0_p(M) = \mathbb{R}$. $\Omega^*_p(M)$ is the space of all differential forms at the point $p \in M$. It is closed under the exterior product.

Assigning an r-form smoothly to each point $p \in M$, one denotes the space of smooth r-forms on M by $\Omega^r(M)$. $\Omega^0(M)$ is the algebra $\mathcal{F}(M)$ of smooth functions on M; $\Omega^1(M)$ is the co-tangent space T^*M (space of co-vectors), etc

r-forms	basis	dimension
$\Omega^0(M) = \mathcal{F}(M)$	$\{1\}$	1
$\Omega^1(M) = T^*M$	$\{dx^i\}$	n
$\Omega^2(M)$	$\{dx^i \wedge dx^j\}$	$n(n-1)/2$
$\Omega^r(M)$	$\{dx^{i_1} \wedge \cdots \wedge dx^{i_r}\}$	$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$
$\Omega^n(M)$	$\{dx^1 \wedge \cdots \wedge dx^n\}$	$\binom{n}{n} = 1$

$\Omega^*(M) = \Omega^0 \oplus \Omega^1 \oplus \cdots \oplus \Omega^n$ is the exterior algebra on M.

The exterior product has the properties

i) associativity: $f \wedge (g \wedge h) = (f \wedge g) \wedge h$

ii) homogeneity: $(cf) \wedge g = c(f \wedge g) = f \wedge (cg)$, $c \in \mathbb{R}$

iii) distributivity: if f and $g \in \Omega^k_p(M)$ and $h \in \Omega^\ell_p(M)$, then

$$(f+g) \wedge h = (f \wedge h) + (g \wedge h) \in \Omega^{k+\ell}_p(M)$$

iv) anticommutativity: if $f \in \Omega^k_p(M)$ and $g \in \Omega^\ell_p(M)$, then

$$g \wedge f = (-1)^{k\ell} f \wedge g \qquad \text{(Note: hence } f \wedge f = 0 \text{ if } k \text{ is odd)}$$

v) $(dx^{i_1} \wedge \cdots \wedge dx^{i_k}) \wedge (dx^{i_{k+1}} \wedge \cdots \wedge dx^{i_{k+\ell}}) = dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{i_{k+1}} \wedge \cdots \wedge dx^{i_{k+\ell}}$

• Exterior derivative

d acts like $dx^i \partial_i$

The exterior derivative d is a differential operator which takes an r-form A to an $(r+1)$-form dA. It works similarly to the exterior product of a 1-form with an r-form. The components of dA are

$$(dA)_{i_1 \cdots i_{r+1}} = (r+1)\, \partial_{[i_1} A_{i_2 \cdots i_{r+1}]}$$

eg if $A = A_\mu dx^\mu$ on a Lorentzian manifold, one has

Maxwell field strength

$$F = dA = \tfrac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \quad \text{where } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu ;$$

or if $B = \frac{1}{2} B_{\mu\nu} dx^\mu \wedge dx^\nu$ on a Lorentzian manifold, then $G = dB = \frac{1}{3!} G_{\mu\nu\rho} dx^\mu \wedge dx^\nu \wedge dx^\rho$ where $G_{\mu\nu\rho} = (\partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu})$ ($B_{\mu\nu}^{(x)}$ is an important field in $D = 10$ string theory).

If A is an r-form and B is an l-form, one has the Leibniz rule $d(A \wedge B) = (dA) \wedge B + (-1)^r A \wedge (dB)$, an $(r+l+1)$ form.

Nilpotence: the exterior derivative has the key property that $d^2 \equiv 0$.

One says that a form A is closed if $dA = 0$. A form is exact if $A = dB$ for some smooth form B. Every exact form is thus closed, but the converse is not in general true. The study of closed but not exact forms is cohomology.

Now re-introduce the metric $g_{\mu\nu}$ on the manifold M. Owing to the symmetry $\Gamma^\mu_{\nu\rho} = \Gamma^\mu_{\rho\nu}$ of the Christoffel connection, one also has $(dA)_{\nu_1 \cdots \nu_{r+1}} = (r+1)\, \nabla_{[\nu_1} A_{\nu_2 \cdots \nu_{r+1}]}$.

• Volume n-form

Using the metric, one may define a volume n-form ϵ by

$$\epsilon = \sqrt{|g|}\, dx^1 \wedge \cdots \wedge dx^n = \frac{1}{n!} \sqrt{|g|}\, \varepsilon_{\mu_1 \cdots \mu_n} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n}$$

($|g| = g$ for Riemannian manifolds; $|g| = -g$ for Lorentzian)

where $\varepsilon_{\mu_1 \cdots \mu_n}$ is the totally antisymmetric object that equals $+1$ if $(\mu_1 \cdots \mu_n)$ is an even permutation of $(1, 2, \cdots n)$ and equals -1 if $(\mu_1 \cdots \mu_n)$ is an odd permutation of $(1, 2, \cdots, n)$ and equals zero if any index is repeated.

As usual, ϵ has components given by $\epsilon = \frac{1}{n!}\epsilon_{\mu_1\cdots\mu_n} dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_n}$.
Since $\epsilon_{\mu_1\cdots\mu_n} = \frac{1}{\sqrt{|g|}}\epsilon_{\mu_1\cdots\mu_n}$, where $\epsilon_{\mu_1\cdots\mu_n}$ are the components of an honest (co)tensor, $\epsilon_{\mu_1\cdots\mu_n}$ is not itself a tensor. One says that it is a tensor density of weight (-1).

Raising indices on $\epsilon_{\mu_1\cdots\mu_n}$ using the (inverse) metric $g^{\mu\nu}$, one has the tensor $\epsilon^{\mu_1\cdots\mu_n} = g^{\mu_1\nu_1}\cdots g^{\mu_n\nu_n}\epsilon_{\nu_1\cdots\nu_n} = (\det g)^{-1}\hat{\epsilon}^{\mu_1\cdots\mu_n}\sqrt{|g|}$ where $\hat{\epsilon}^{\mu_1\cdots\mu_n}$ is numerically the same as $\epsilon_{\mu_1\cdots\mu_n}$. Since $\hat{\epsilon}^{\mu_1\cdots\mu_n} = (-1)^{[t]}\sqrt{|g|}\,\epsilon^{\mu_1\cdots\mu_n}$ where $[t]$ is the number of time coordinates on the manifold M, one says that $\hat{\epsilon}^{\mu_1\cdots\mu_n}$ is a tensor density of weight 1.

Note: $\hat{\epsilon}^{\mu_1\cdots\mu_n}$ is not $\epsilon_{\mu_1\cdots\mu_n}$ with indices raised by the metric. Both are numerically invariant

The volume n-form $\epsilon = \sqrt{|g|}\,dx^1\wedge\cdots\wedge dx^n$ is coordinate independent because the $\left|\frac{\partial x'}{\partial x}\right|$ factors in the transformations of $\sqrt{|g|}$ and $dx^1\wedge\cdots\wedge dx^n$ cancel out. Accordingly, the weight (-1) density $\epsilon_{\mu_1\cdots\mu_n}$ and the weight 1 density $\hat{\epsilon}^{\mu_1\cdots\mu_n}$ are both numerically invariant when transformed under the corresponding rules for a covariant tensor density of weight (-1) or a contravariant tensor density of weight 1.

Some useful facts: $\nabla_\rho \epsilon_{\mu_1\cdots\mu_n} = 0$

partial contraction rule: $\epsilon^{\mu_1\cdots\mu_r\nu_{r+1}\cdots\nu_n}\epsilon_{\rho_1\cdots\rho_r\nu_{r+1}\cdots\nu_n} = (-1)^{[t]}\,r!\,(n-r)!\,\delta^{\mu_1\cdots\mu_r}_{\rho_1\cdots\rho_r}$
where $\delta^{\mu_1\cdots\mu_r}_{\rho_1\cdots\rho_r} = \delta^{\mu_1}_{[\rho_1}\cdots\delta^{\mu_r}_{\rho_r]}$.

- Hodge dual

Using the volume form, constructed using the metric, one may start from an r-form A and obtain an $(n-r)$ form $*A$, called the Hodge dual of A, defined by

$$*A_{\mu_1\cdots\mu_{n-r}} = \frac{1}{r!}\epsilon_{\mu_1\cdots\mu_{n-r}\nu_1\cdots\nu_r}A^{\nu_1\cdots\nu_r}$$

where $A^{\nu_1\cdots\nu_r} = g^{\nu_1\sigma_1}\cdots g^{\nu_r\sigma_r}A_{\sigma_1\cdots\sigma_r}$.

Then one finds $*(*A) = (-1)^{[t]}(-1)^{r(n-r)}A$ (as above, $[t]$ is the number of time coords)

and $(*d*A)_{\mu_1\cdots\mu_{r-1}} = (-1)^{[t]}(-1)^{r(n-r)}\nabla^\nu A_{\mu_1\cdots\mu_{r-1}\nu}$

Note that, starting from the trivial 0-form $\mathbb{1}$ (just 1 everywhere on M) one has $*\mathbb{1} = \epsilon$.

Integration of forms

Note that all r-forms $A = \frac{1}{r!} A_{\mu_1 \ldots \mu_r} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r}$ are coordinate invariant because all $dx^{\mu'}/\partial x^{\nu}$ transformation matrices cancel between the covariant indices on $A_{\mu_1 \ldots \mu_r}$ and the contravariant indices on $dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r}$. Accordingly, A is directly ready to be interpreted as an invariant integrand for integration over an r-dimensional manifold, or over an r-dimensional submanifold of an n-dimensional manifold. One can write

$$A = \frac{1}{r!} A_{\mu_1 \ldots \mu_r} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r} = A_{1 \ldots r} \, dx^1 \wedge \cdots \wedge dx^r$$

so one defines the form integral over an r-dimensional $M_{(r)}$ manifold $\int_{M_{(r)}} A := \int dx^1 \cdots dx^r \, A_{1 \ldots r} = \int d^r x \, A_{1 \ldots r}$, the usual integral. Note that this definition of the integral of an r-form over $M_{(r)}$ doesn't require a metric — it is already invariant under general coordinate transformations.

Given a metric, and hence a volume form for an n-dimensional manifold M, one can define the volume of M to be $\mathrm{Vol}(M) = \int_M \epsilon$ (which could be infinite). One can also use the volume form to define the integral of a function $f(x)$ on M: $\int_M f \epsilon = \int d^n x \sqrt{|g|} \, f(x) = \int_M f * \mathbb{1}$.

Stokes Theorem

If M is an oriented n-dimensional manifold with boundary ∂M, then for an $(n-1)$ form A on M one has

$$\int_M dA = \int_{\partial M} A .$$

Gauss Law

Closely related to Stokes theorem is Gauss's law or the Divergence Theorem. Again, let M be an n-dimensional manifold with boundary ∂M, endowed now with a metric $g_{\mu\nu}$ and the corresponding volume form ϵ.

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If V is a one-form on M, then $A = *V$ is an $(n-1)$ form and one may use Stokes theorem:

$$\int_M dA = \int_{\partial M} A = \int_{\partial M} *V$$

Now write this in standard notation. For an $r=1$ form one has $*d*V = (-1)^{[t]}(-1)^{1\cdot(n-1)} \nabla^\nu V_\nu \mathbb{1}$ then apply $*$ again to this zero-form: $**\mathbb{1} = (-1)^{[t]}\mathbb{1}$ so

$$d*V = (-1)^{(n-1)}(\nabla^\mu V_\mu)\epsilon$$

consequently $\int_M dA = \int_M d*V = (-1)^{(n-1)} \int_M d^n x \sqrt{|g|}\, \nabla_\mu V^\mu$.

Now rewrite $\int_{\partial M} *V$. Assume ∂M has an outward pointing normal vector n^μ, with $n^2 = n^\mu n^\nu g_{\mu\nu} = \mp 1$ according to whether n^μ is timelike or spacelike. On ∂M there is an induced metric $h_{\mu\nu}$ which is related to $g_{\mu\nu}$ by $g_{\mu\nu} = h_{\mu\nu} \mp n_\mu n_\nu$, noting that $h_{\mu\nu} n^\nu = 0$. Use the induced metric to define an $(n-1)$ form volume form $\bar{\epsilon}$ on ∂M. Then

$n_\mu = g_{\mu\nu} n^\nu$

$$*V = \frac{1}{(n-1)!} \epsilon_{\mu_1 \cdots \mu_{n-1}\nu} V^\nu dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{n-1}}$$
$$= (-1)^{(n-1)} \frac{1}{(n-1)!} \epsilon_{\nu\mu_1\cdots\mu_{n-1}} V^\nu dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{n-1}}$$

Now the volume form ϵ on M and the volume form $\bar{\epsilon}$ on ∂M are related by $\epsilon = n_\mu dy^\mu \wedge \bar{\epsilon}$, where the boundary ∂M is given (at least locally) by $y^\mu = \text{constant}$, so the normal one-form is $n = n_\mu dy^\mu$. Then $V^\nu \epsilon_{\nu\mu_1\cdots\mu_{n-1}} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{n-1}}$ is obtained by taking the contraction $\left\langle V^\nu \frac{\partial}{\partial y^\nu}, \epsilon \right\rangle$ noting that $\left\langle \frac{\partial}{\partial y^\mu}, dy^\nu \right\rangle = \delta_\mu^\nu$ and $\left\langle \frac{\partial}{\partial y^\mu}, dx^{\nu_i} \right\rangle = 0 \quad i = 1, \cdots, n-1$.

Thus, $*V = (-1)^{(n-1)} \frac{1}{(n-1)!} n_\mu V^\mu \bar{\epsilon}_{\nu_1 \cdots \nu_{n-1}} dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{n-1}}$

$$= (-1)^{n-1} n_\mu V^\mu \bar{\epsilon}$$

So, putting things together one has

$$\int_M d^n x \sqrt{|g|}\, \nabla_\mu V^\mu = \int_{\partial M} d^{n-1} x \sqrt{|h|}\, n_\mu V^\mu$$

which is the ordinary form of the divergence theorem.

H. Weyl, Ann Phys (Leipzig) 54 (117) 1917 and in "Space-Time-Matter" (Methuen, London, 1922)
S. Deser and J. Franklin, "Schwarzschild and Birkhoff à la Weyl", Am. J. Phys. 73 (2005) 261 and gr-qc/0408067

Deriving the Schwarzschild Solution

Ansätze

Unlike linear theories such as electrodynamics ↔ Maxwell's theory, where at least in principle there exists a general solution to the field equations (via Fourier transforms, etc), the highly nonlinear nature of general relativity makes this impossible. Often, the best one can do is to propose some general form of solution, or ansatz (starting point) and try one's luck within that restricted context.

We will be interested, in particular, in spherically symmetric solutions, so this gives us a basis for constructing an ansatz.

Consistent truncations and reduced actions

One perfectly acceptable way to implement an ansatz is first to derive the general field equations for a theory, to substitute the ansatz form of sought-for solutions and then to search for solutions. However, this procedure can be a long road.

Instead, we are going to follow a more delicate approach: inserting a carefully chosen ansatz into the action, then deriving restricted field equations appropriate to this ansatz, and then finding more quickly the corresponding solutions

Method in GR originates with H. Weyl

When is such a procedure admissable? The subject is rather delicate. But one situation where it can be guaranteed to work is when the ansatz restricts the field variables to the most general form invariant with respect to a symmetry, for example — spherical symmetry. This guarantees a consistent truncation to the ansatz form.

Electrodynamics — the Coulomb solution

In electrodynamics, we impose spherical symmetry by allowing functional dependence of $A_\mu(x^\nu)$ on $r = \sqrt{x^i x^i}$ and t. Thus $\vec{A} = A_r(r,t)\hat{r}$ $A_0 = A_0(r,t)$. Note that other components of $\vec{A}$ in polar coordinates like A_ϕ, A_θ would transform non-trivially under spatial rotations, so spherical symmetry requires these to be set to zero.

The $F_{\mu\nu}$ field strength components in this ansatz have the structure $F_{ij} = \epsilon_{ijk} B_k = 0$; $F^{0i} = E^i$; $E_r = (A_0' - \dot{A}_r) = (\partial_r A_0 - \partial_0 A_r)$; $\vec{E} = E_r \hat{r}$. Note that in setting $B_i = \frac{1}{2}\epsilon_{ijk} F_{jk} = 0$ we are restricting to a situation without a spherically symmetric magnetic field B_r.

$A_0' = \partial_r A_0$
$\dot{A}_r = \partial_t A_r$

Note that the gauge field component A_r can be arbitrarily changed by a Maxwell gauge transformation: $\delta A_r = \partial_r \Lambda$. This can be used to set A_r to zero, but we will not do so yet.

Substituting the spherically symmetric ansatz into the Maxwell action $I_{max} = -\frac{1}{4}\int d^4x\, F_{\mu\nu} F^{\mu\nu} = \frac{1}{2}\int d^4x (E \cdot E - B \cdot B)$ one obtains the reduced action

$$I_{max}^{red} = 2\pi \int (\dot{A}_r - A_0')^2 r^2 dr dt$$

where the $2\pi = 4\pi/2$ factor comes from $\int_{S^2} d^2\Omega$.

Treat source-free - only I_{max} -

Now vary I_{max}^{red}. Note that the δA_r variation produces a contentful field equation $\frac{\delta I_{max}^{red}}{\delta A_r} = 4\pi r^2 \frac{\partial}{\partial t}(A_0' - \dot{A}_r) = 0$

Having obtained this field equation, we may now choose to pick Coulomb gauge $A_r = 0$, simplifying the $\delta/\delta A_r$ equation to $\frac{\delta I_{max}^{red}}{\delta A_r}\Big|_{A_r=0} = 4\pi r^2 \dot{A}_0' = 0$.

The $\delta/\delta A_0$ equation gives, similarly setting A_r to zero after variation, $\frac{\delta I_{max}^{red}}{\delta A_0}\Big|_{A_r=0} = 4\pi \frac{\partial}{\partial r}(r^2 A_0') = 0$.

The solution to this equation is $A_0 = -\frac{q(t)}{r}$.
However the $\delta/\delta A_r$ equation then becomes

$$4\pi r^2 \dot{A}_0' = 4\pi \dot{q} = 0$$

So one obtains both the Coulomb solution $A_0 = -\frac{q}{r}$ and the constancy in time of the charge, $\dot{q} = 0$.
Then, of course $E^i = F^{0i}$, so $E_r = q/r^2$ as usual.

A few comments need to be made about this derivation. Firstly, the equation $\partial I_{max}^{fixed}/\delta A_0 = 0$ is the remaining content under the spherically symmetric ansatz of the covariant source-free equation $\nabla_i E^i = 0$. Why the covariant derivative? Because we are using polar coordinates, $ds^2 = dr^2 + r^2 d\Omega^2$. So, implementing spherical symmetry by dropping ∂_θ and ∂_φ derivatives, one may write in 3 dimensions $\nabla_i E^i = \frac{1}{\sqrt{g}} \nabla_i (\sqrt{g} E^i)$ but $\sqrt{g} E^i$ is a contravariant density, for which $\nabla_i (\sqrt{g} E^i) = \partial_i (\sqrt{g} E^i)$. Moreover, $\sqrt{g} = r^2 \sin\theta$ and $g^{rr} = 1$ so $\nabla_i E^i = \frac{1}{r^2 \sin\theta} \partial_r (r^2 \sin\theta E_r) = \frac{1}{r^2} \partial_r (r^2 E_r)$. Since $E_r = A_0'$ in the gauge $A_r = 0$, one sees the equivalence between $\nabla_i E^i = 0$ and $\partial_r (r^2 A_0') = 0$.

Of course, in deriving the Coulomb solution $A_0 = -q/r$, one has told a traditional fib. This is not really a solution of $\nabla_i E^i = 0$. If it were truly such a solution for the whole of space, one would have $\int d^3x \, \nabla_i E^i = 0$, but using Gauss's Law, we have, for a surface at $r = \infty$ with normal one-form $n = dr$ $\int d^3x \, \nabla_i E^i = \int r^2 d^2\Omega \, n^r E_r |_{r \to \infty} = 4\pi q$. In order to genuinely obtain the Coulomb solution, we should have included a delta-function source for the electric field. This situation is identical to that of the Laplace equation in $\mathbb{R}^3$, where $V = \frac{-k}{r}$ is not really a solution of $\nabla^2 V = 0$ unless $c = 0$. Obtaining a potential $V = -c/r$ requires $\nabla^2 V = 4\pi k \delta^3(x)$ with a source at the origin.

Note that an even simpler approach would be to restrict the ansatz even further, and to require t-independence in addition to spherical symmetry. In that case, one could implement the gauge choice $A_r = 0$ already in the ansatz, before variation of the action. The A_0 equation would then give $\partial_r (r^2 A_0') = 0$, solved as before by $A_0 = -q/r$. However, by doing so one loses the possibility to derive $\dot{q} = 0$. Retaining A_r, varying and then setting $A_r \to 0$ allows this derivation

Spherically symmetric metric

Now continue on to the analogous problem in general relativity. Requiring spherical symmetry in the spatial submanifolds is equivalent to demanding the existence of a set of three Killing vectors $V_\mu^{(a)}$, $\nabla_{(\mu} V_{\nu)}^{(a)} = 0$, $a = 1, 2, 3$, generating the $SO(3)$ rotation group. It is simpler, however to choose <u>adapted coordinates</u> to the extent possible. It is not possible to choose coordinates making all three $SO(3)$ symmetries manifest (since $SO(3)$ is non-abelian), but ordinary polar coordinates are the easiest to work with.

Focus on the 3-dimensional spatial submanifolds. In Cartesian coordinates, a spherically symmetric metric takes the form $ds^2_{(3)} = A(r,t)\, dx^i dx^i + B(r,t)\, x^i dx^i x^j dx^j$, where $r^2 = x^k x^k$. Then go over to polar coordinates using $dx^i dx^i = dr^2 + r^2 d\Omega^2$, $x^i dx^i = r dr$, where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ is the usual $SO(3)$ invariant metric on the unit 2-sphere. Including the time coordinate, the corresponding spherically symmetric spacetime metric is $ds^2 = -D(r,t)\, dt^2 + (A(r,t) + B(r,t) r^2)\, dr^2 + E(r,t) r^2 d\Omega^2 + 2C(r,t)\, r\, dr\, dt$

Note: A and B occur only in combination $A + Br^2 = \tilde{A}$

Of these five (A, B, C, D, E) metric functions, E and C can be modified by coordinate transformations which preserve the spherically symmetric form: $A(r,t)$ by an $r \to r'(r,t)$ coordinate change followed by a $t \to t'(r,t)$ coordinate change.

The $r \to r'(r,t)$ coordinate change can be used to set $E = 1$. This is a type of gauge change

$Er^2 = (r')^2$

called a "proper" gauge change (Benguria, Cordero + Teitelboim Nucl. Phys. B122 (1977) 61) which can be made without changing any physical content, and in particular without losing any physically important equations if made in the action even before variation. One could keep the E function throughout the analysis, but this just makes things more complicated.

The gauge-modifiable function $C(r,t) = g_{0r}$ is more akin to the A_r component in the Maxwell analysis: its variation yields a physically important equation. Transforming $x^0(x^{0'}, r')$, $r = x^{r'}$

$g_{0'r'} = \frac{\partial x^0}{\partial x^{0'}}\frac{\partial x^0}{\partial x^{r'}} g_{00} + \frac{\partial x^0}{\partial x^{0'}}\frac{\partial x^r}{\partial x^{r'}} g_{0r}$ can be used to set $C = g_{0r} \to 0$, infinitesimally: $x^0 = x^{0'} + \xi^{0'}$, $x^r = x^{r'}$ then $\delta g_{0r} = \pounds_\xi g_{0r} = \xi^0 \partial_0 g_{0r} + \partial_r \xi^0 g_{00} + \partial_0 \xi^0 g_{0r}$

but, as in the Maxwell analysis, we will do this only after variation of the ansatz-reduced action.

Accordingly, setting $E = 1$ already as part of the initial ansatz, and further redefining metric functions $D = ab^2$, $\tilde{A} = A + Br^2 = a^{-1}$, $C = bf/r$ where a, b and f are so-far undetermined functions of r and t. The metric then becomes $ds^2 = -ab^2 dt^2 + a^{-1} dr^2 + r^2 d\Omega^2 + 2bf\, dr dt$.

After variation, we will avail ourselves of the gauge freedom to set $f(r,t) = 0$, but we need to obtain the $\frac{\delta I}{\delta f}$ field equation first. When written in terms of a, b and f, the reduced gravitational action is still a nonlinear mess. However, since we intend to set $f \to 0$ eventually after variation, all we really need to retain are the linear terms in f in the action. Higher powers of f will vary to yield terms in the $\delta I^{red}/\delta f = 0$ equation that still contain powers of f, which are then set to zero by a gauge choice. Keeping the full dependence on a and b, but just linear dependence on f leads to the reduced action

$$I_{EH}^{red}(a,b,f) = 8\pi \int dr dt \left(-b(ar - r)' + a^{-1} r f \dot{a}\right).$$

Varying and then setting $f = 0$ yields

$$\frac{\delta I}{\delta a}\Big|_{f=0} = rb' = 0 \Rightarrow b = b(t)$$

$$\frac{\delta I}{\delta b}\Big|_{f=0} = -(ar - r)' = 0 \Rightarrow a = 1 - \frac{2m(t)}{r}$$

$$\frac{\delta I}{\delta f}\Big|_{f=0} = r a^{-1} \dot{a} = 0 \Rightarrow \dot{m} = 0$$

Finally, the time dependence of $b(t)$ can be removed by a pure $t \to t'(t)$ time transformation. Use this to set $b = 1$.

Accordingly, we have arrived at the Schwarzschild metric

$$ds^2 = -\left(1-\frac{2m}{r}\right)dt^2 + \frac{1}{1-\frac{2m}{r}}dr^2 + r^2 d\Omega^2$$

and in addition, we have learned

$$\frac{dm}{dt} = 0.$$

The time independence of m, analogous to the $\dot{q}=0$ result in Maxwell's theory, is Birkhoff's theorem.

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ref. S. Deser and B. Tekin, "Shortcuts to spherically symmetric solutions", Class. Quantum Grav. 21 (2004) 5295 and gr-qc/0404120

Schwarzschild - de Sitter

Now extend the above discussion to include a cosmological constant. For a simpler discussion, we now forgo derivation of the corresponding Birkhoff theorem. (Although the discussion can be generalized to include that, along the above lines for standard Schwarzschild.)

The action including the cosmological term is

$$I = \int d^4x \sqrt{-g}\,(R+2\Lambda).$$

We now take the spherically symmetric ansatz

$$ds^2 = -a(r)\,b(r)^2 dt^2 + a(r)^{-1}dr^2 + r^2 d\Omega^2$$

where we have implemented an additional assumption of t-invariance and we have correspondingly gauged $g_{0r} \to 0$ from the start. For this ansatz, $\sqrt{-g} = br^2 \sin\theta$, $\int \sin\theta d\theta d\phi = 4\pi$ the reduced action becomes $I^{red}_{sds} = 8\pi \int dr\,[-b(ar-r)' + \Lambda b r^2]$ dropping $\int dt$

From which $\frac{\delta I^{red}}{\delta b} = 0 \Rightarrow 1 - a'r - a + \Lambda r^2 = 0$

and $\frac{\delta I^{red}}{\delta a} = 0 \Rightarrow rb' = 0 \Rightarrow b = \text{const.}$

The $\frac{\delta I}{\delta b} = 0$ equation is solved by $a(r) = 1 - \frac{2m}{r} + \frac{\Lambda r^2}{3}$.

The constant b can then be gauged to 1 by $t \to kt$ transformation.

The Schwarzschild – de Sitter solution is thus

$$ds^2 = -\left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right)dt^2 + \left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2 d\Omega^2$$

Schwarzschild – Reissner – Nordstrom

One can then extend this further by including the gravitationally coupled Maxwell action:

$$I = \int d^4x \sqrt{-g}(R + 2\Lambda) - \frac{1}{4}\int d^4x \sqrt{-g} F_{\mu\nu} F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma}$$

Implementing the same ansatz for A_μ as we had before, but now requiring time independence and setting $A_r \to 0$ one obtains the action

$$I = 8\pi \int dr \left[-b(ar - r)' + \Lambda b r^2 + \frac{1}{4b} r^2 (A_0')^2\right]$$

Varying with respect to A_0 yields $\partial_r\left(\frac{r^2 A_0'}{b}\right) = 0$ while varying with respect to a again yields $b' = 0$, so one may solve already $A_0 = -\frac{q}{r}$ as before.

The $\delta/\delta b$ equation is then

$$1 - a'r - a + \Lambda r^2 - \frac{1}{4b^2} r^2 (A_0')^2 = 0$$

Substituting what we have found from A_0 and a sets $-\frac{1}{4b^2} r^2 (A_0')^2 = -\frac{q^2}{4b^2 r^2}$, where $\partial_r b = 0$

Finally, the solution to the $\frac{\delta}{\delta b}$ equation is then

$$a(r) = 1 - \frac{2m}{r} + \frac{\Lambda r^2}{3} + \frac{q^2}{4b^2 r}$$

And, as for Schwarzschild, one may finally use a $t \to t'$ coordinate change to set $b = 1$, giving

$$a(r) = 1 - \frac{2m}{r} + \frac{\Lambda r^2}{3} + \frac{q^2}{4r^2}$$

This is the cosmological Schwarzschild – Reisner – Nordstrom solution

$$ds^2 = -a(r) dt^2 + \frac{dr^2}{a(r)} + r^2 d\Omega^2$$

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Falling into a Schwarzschild black hole

Now explore radial motion in the original Schwarzschild geometry

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ round S^2 metric

$\frac{dM}{dt} = 0$

Killing vectors: we derived this solution assuming spherical symmetry, i.e. symmetry under the spatial rotation group $SO(3)$. The (θ, ϕ) coordinates are adapted to one of the three $SO(3)$ generators, since the metric has no ϕ dependence. The other two one just has to work out. On S^2, one has

$$V^{(1)} = \partial_\phi$$
$$V^{(2)} = \sin\phi\,\partial_\theta + \cot\theta\cos\phi\,\partial_\phi$$
$$V^{(3)} = \cos\phi\,\partial_\theta - \cot\theta\sin\phi\,\partial_\phi$$

These three generate the Lie algebra of $SO(3)$:

$$\mathcal{L}_{V^{(1)}} V^{(2)} = V^{(3)}$$
$$\mathcal{L}_{V^{(1)}} V^{(3)} = -V^{(2)}$$
$$\mathcal{L}_{V^{(2)}} V^{(3)} = V^{(1)}$$

Another adapted coordinate is t — the metric is <u>stationary</u> (i.e. t-independent). So ∂_t is another Killing vector. It is timelike, i.e. $\|\partial_t\|^2 < 0$ for $r > 2M$ but spacelike for $r < 2M$. There is, in addition, a <u>discrete symmetry</u> $t \to -t$ arising from the absence of a g_{tr} component. Absence of a g_{tr} component means the metric is <u>static</u>. As we have seen from Birkhoff's theorem, the static nature of the Schwarzschild metric is a direct consequence of spherical symmetry.

Now consider a point particle falling radially towards the center of the black hole (not recommended for health). As we have seen, the worldline path of such a particle will be a radial timelike geodesic $\frac{d}{d\tau}\theta = \frac{d}{d\tau}\phi = 0$, so

$$X^\mu(\tau) = [t(\tau), r(\tau), \theta_0, \phi_0]$$

where we choose to parametrize the particle's path by proper time τ — an affine parametrization. ($\lambda = \tau$)

In terms of coordinate (t) time, one will have a radial position $R(t) = r(\tau(t))$. One has

$$d\tau^2 = -ds^2 = \left(1-\frac{2M}{R}\right)dt^2 - \left(1-\frac{2M}{R}\right)^{-1}dR^2 \qquad \text{since } d\theta = d\phi = 0$$

and for this motion $dR = \frac{dR}{dt}dt = \dot{R}\,dt$ ($\dot{R} = \frac{dR}{dt}$), so, dividing by $d\tau^2$ one has

$$1 = \left[\left(1-\frac{2M}{R}\right) - \left(1-\frac{2M}{R}\right)^{-1}\dot{R}^2\right]\left(\frac{dt}{d\tau}\right)^2 \qquad (*)$$

Now, $k = \partial_t$ is a Killing vector and we saw earlier that $Q = e^{-1}V_\nu \frac{dx^\nu}{d\lambda}$ is a Noether charge, i.e. a conserved quantity for motion $X^\nu(\lambda)$ satisfying the particle field equations. Here, the Killing vector is $V = V^\mu\partial_\mu = \partial_t$ so $V^0 = 1$ and $V_\mu = g_{\mu\nu}V^\nu$ so $V_0 = g_{tt}$. Moreover, we are using affine $\lambda = \tau$, proper time parametrization, for which the einbein $e = \frac{1}{m}\frac{d\tau}{d\lambda}$ is constant. Hence

$$Q = e^{-1}g_{tt}\frac{dt}{d\tau} \text{ is a constant of the motion.}$$

Moreover, since e is constant in the affine $\lambda = \tau$ worldline parametrization, we conclude that

$$\mathcal{E} = -g_{tt}\frac{dt}{d\tau} \text{ is a constant of the motion along the geodesic.}$$

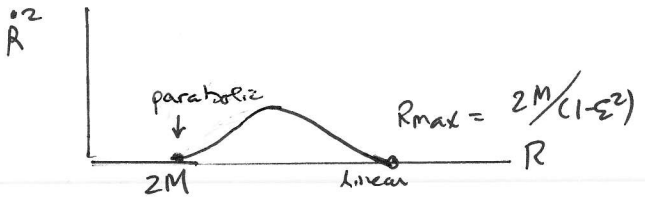
$\mathcal{E}$ is the energy/unit mass of the particle.

Then, substituting into the above equation ($*$) one has

$$\dot{R}^2 = \frac{1}{\mathcal{E}^2}\left(1-\frac{2m}{R}\right)^2\left(\frac{2m}{R} - 1 + \mathcal{E}^2\right), \qquad \dot{R} = \frac{dR}{dt}$$

Then $\dot{R} = 0$ at $R_{max} = \frac{2m}{1-\mathcal{E}^2}$. This is where the particle starts falling from rest. Alternatively, one may view this equation as giving the value $\mathcal{E} = \sqrt{1-\frac{2m}{R_{max}}}$ for a given path starting from rest at R_{max}.

here $\dot{x}^\nu = \frac{dx^\nu}{d\tau}$



For gravitationaly bound motion, ie. $R_{max} < \infty$ one has $0 < \varepsilon < 1$. Note that $\dot{R} = 0$ again at $R = 2M$. Calculate the time t_{2M} taken to fall from $R = R_{max}$ to $R = 2M$:

$$t_{2M} = \int_0^{t_{2M}} dt = -\int_{R_{max}}^{2M} \frac{dR}{\dot{R}} \qquad \text{Note: } \dot{R} < 0 \text{ for infall}$$

$$= -\varepsilon \int_{R_{max}}^{2M} dR \left(1 - \frac{2M}{R}\right)^{-1} \left(\frac{2M}{R} - 1 + \varepsilon^2\right)^{-1/2}$$

let $R = 2M + y$ and change top integration limit to $2M + \delta$. Then

$$t_{2M} = -2M \int_{R_{max} - 2M}^{\delta} \frac{dy}{y} \left(1 + \frac{y}{2M}\right)^{3/2} \left(1 + \frac{y}{2M}\left(1 - \frac{1}{\varepsilon^2}\right)\right)^{-1/2}$$

Near the upper integration limit, for δ infinitesimal, the leading order part of this, as $\delta \to 0$ is $-2M \ln \delta \to \infty$ as $\delta \to 0$ So the time taken to fall into $R = 2M \leftrightarrow y = 0$ is infinite

So in terms of the t coordinate (which out at $r \to \infty$ becomes regular Minkowski metric time), the particle is seen to approach $R \to 2M$ but taking an infinite amount of t time to get there.

For an observer travelling along with the particle, however, and using $\lambda = \tau$ parametrization, the dependence $R(\tau)$ is instead relevant. Then $\frac{d}{dt} = \left(\frac{dt}{d\tau}\right)^{-1} \frac{d}{d\tau} = \varepsilon^{-1}\left(1 - \frac{2M}{R(\tau)}\right) \frac{d}{d\tau}$

$$\text{so} \quad \left(\frac{dR}{d\tau}\right)^2 = \left(\frac{2M}{R} - 1 + \varepsilon^2\right) = (1 - \varepsilon^2)\left(\frac{R_{max}}{R} - 1\right)$$

Thus the proper time evolved for a comoving observer falling from R_{max} to R is

$$\tau(R) = \int_0^{\tau} d\tau = -\int_{R_{max}}^{R} dR (1-\varepsilon^2)^{-1/2} \left(\frac{R_{max}}{R} - 1\right)^{-1/2} \qquad \text{let } z = \frac{R}{R_{max}}$$

$$= -(1-\varepsilon^2)^{-1/2} R_{max} \int_1^{R/R_{max}} \frac{dz\, \sqrt{z}}{(1-z)^{1/2}}$$

$$= (1-\varepsilon^2)^{-1/2} R_{max} \left[(z(1-z))^{1/2} + \arcsin\left[(1-z)^{1/2}\right] \right]_1^{R/R_{max}}$$

Note: $[\ ]$ vanishes at $z = 1$

Note that this remains finite for an observer falling to $R = 2M$ or even all the way to $R = 0$. Proper time taken to fall to $R = 0$: $\tau(0) = (1-\varepsilon^2)^{-1/2} R_{max} \arcsin 1 = \pi M (1-\varepsilon^2)^{-3/2}$

$R_{max} = \frac{2M}{1-\varepsilon^2}$

Particle motion in Schwarzschild geometry

(Reference: Misner, Thorne & Wheeler, Chapter 25)

Recall the action for a massive particle in "einbein" form: Let the particle mass now be μ

$$\hat{I}[x^\mu(\lambda), e(\lambda)] = \frac{1}{2}\int d\lambda \left[e^{-1}(\lambda)\, g_{\mu\nu}(x(\lambda))\dot{x}^\mu \dot{x}^\nu - \mu^2 e(\lambda)\right]$$

λ: worldline parameter

Varying $\delta e(\lambda)$, we found the algebraic equation

$$-e^{-2} g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - \mu^2 = 0 \Rightarrow e = \frac{1}{\mu}\left[-g_{\mu\nu}(x(\lambda))\dot{x}^\mu\dot{x}^\nu\right]^{1/2}$$

where $\dot{x}^\mu = dx^\mu/d\lambda$ $\qquad = \frac{1}{\mu}\frac{d\tau}{d\lambda}$ $\quad \tau$ = proper time

The δx^μ equation is $\frac{D}{d\lambda}\dot{x}^\mu = e^{-1}\dot{e}\dot{x}^\mu$ geodesic equation

where $\frac{D}{d\lambda}\dot{x}^\mu = \ddot{x}^\mu + \Gamma^\mu_{\lambda\rho}\dot{x}^\lambda\dot{x}^\rho$ path covariant derivative of $\dot{x}^\mu$

Choosing a worldline parametrisation where $\dot{e} = 0$ gives an affinely parametrised geodesic $\frac{D}{d\lambda}\dot{x}^\mu = 0$.
On such a geodesic, $\lambda = a\tau + b$

Moreover, if there is a Killing symmetry of the spacetime metric, with Killing vector V, we saw that there is a conserved Noether charge $Q = e^{-1} V_\nu \dot{x}^\nu$, $dQ/d\lambda = 0$.

Schwarzschild geometry $ds^2 = -(1-2M/r)dt^2 + \frac{dr^2}{1-2M/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$ has time translation and spherical symmetries. Of these, two are associated to the cyclic coordinates t and ϕ, with Killing vectors $V_{(t)} = \partial_t$ and $V_{(\phi)} = \partial_\phi$. Accordingly, one has conserved quantities $E = -g_{00}\, dt/d\lambda \equiv (1 - \frac{2M}{r})\, dt/d\lambda$ and $L = (d\phi/d\lambda)\, r^2 \sin^2\theta$.

Now, define rescaled quantities $\tilde{E} = E/\mu$ and $\tilde{L} = L/\mu$, ie. "energy and "angular momentum" per unit mass of particle. at infinity"

Also, choose a worldline affine parameter $\lambda = \tau/\mu$: proper time per unit mass of particle, for which $e = 1$.
The 5e equation then becomes, solving for
$dt/d\tau = \frac{\tilde{E}}{(1-2M/r)}$ and $d\phi/d\tau = \frac{\tilde{L}}{r^2 \sin^2\theta}$ and squaring

$\tau = \lambda\mu$

$$-\frac{\tilde{E}^2}{(1-2M/r)^2}\left(1-2M/r\right) + \left(\frac{dr}{d\tau}\right)^2 \frac{1}{1-2M/r} + \left(\frac{d\theta}{d\tau}\right)^2 r^2 + \frac{r^2 \sin^2\theta\, \tilde{L}^2}{(r^2\sin^2\theta)^2} = -1$$

re.

$$-\frac{\tilde{E}^2}{1-2M/r} + \frac{\left(\frac{dr}{d\tau}\right)^2}{1-2M/r} + \left(\frac{d\theta}{d\tau}\right)^2 r^2 + \frac{\tilde{L}^2}{r^2\sin^2\theta} = -1 \quad .$$

{ N.B. this is the GR generalisation of $\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + \mu^2 = 0$ in flat space }

Now specialise to particle motion in the equatorial plane of the S^2 sphere, for which $\theta = \pi/2$ and $d\theta/d\tau = 0$.
so $\sin\theta = 1$

For such motion, one consequently has

$$\left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - (1-2M/r)\left(1 + \tilde{L}^2/r^2\right)$$

$$= \tilde{E}^2 - \tilde{V}^2(r)$$

where $\tilde{V}(r) = \left[(1-2M/r)\left(1+\tilde{L}^2/r^2\right)\right]^{1/2}$ is an "effective potential" for the particle motion.

This equation, together with the supplementary equations

$$d\phi/d\tau = \frac{\tilde{L}}{r^2} \quad \text{and} \quad dt/d\tau = \frac{\tilde{E}}{(1-2M/r)}$$

determine the particle's orbit.

Note that the particle's rest mass μ makes no appearance in these equations

Newtonian limit : $|\tilde{E}-1| \ll 1$, $M/r \ll 1$, $\tilde{L}/r \ll 1$

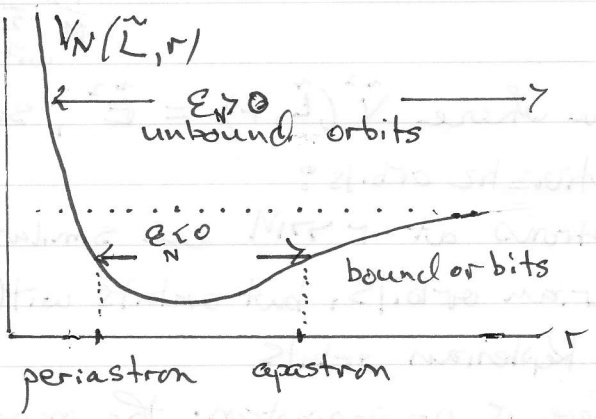
$\tilde{E}$ has the interpretation as $r \to \infty$ as energy per unit rest mass μ.
$\tilde{E} = E/\mu = (1-v_\infty^2)^{-1/2}$ (with $c=1$).

In the nonrelativistic limit, it is more appropriate to consider $\mathcal{E}_N = \frac{1}{2}(\tilde{E}^2-1) = \frac{1}{2}\left(\frac{1}{1-v_\infty^2}-1\right) \approx \frac{1}{2}v_\infty^2 \approx \tilde{E}-1$

Also, instead of $\tilde{V}^2(\tilde{L},r)$ it is more appropriate to consider $V_N(\tilde{L},r) = \frac{1}{2}(\tilde{V}^2-1) \approx -\frac{M}{r} + \frac{\tilde{L}^2}{2r^2}$.

Then, the effective potential equation becomes, in the nonrelativistic limit,

$$\frac{1}{2}\left(\frac{dr}{d\tilde{\tau}}\right)^2 + V_N(\tilde{L},r) = \mathcal{E}_N$$

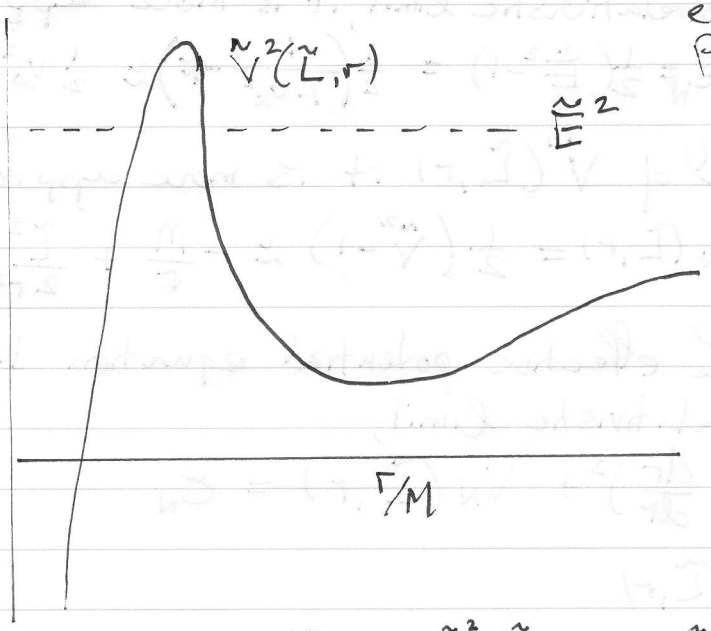


Unbound orbits : particles with $\mathcal{E}_N \geq 0$ (i.e. $\tilde{E} \geq 1$) come in from $r=\infty$ along hyperbolic or parabolic (for $\mathcal{E}=0$) orbits, then they are reflected off the effective potential at turning point $\mathcal{E}_N = V_N$ (i.e. $\tilde{E}^2 = \tilde{V}^2$), and finally return to $r=\infty$.

Bound orbits : particles with $\mathcal{E}_N < 0$ (i.e. $\tilde{E} < 1$) move back and forth between periastron (inner) and apastron (outer) turning points. The Newtonian orbit is an ellipse.

Relativistic orbits in Schwarzschild geometry

In Schwarzschild geometry, the effective potential $\tilde{V}^2(\tilde{L}, r) = (1 - 2M/r)(1 + \tilde{L}^2/r^2)$ has a "pit" as $r \to 0$:



effective potential for $L^+ = \frac{\tilde{L}}{M} > 2\sqrt{3}$

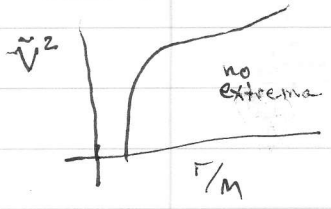
extrema:

$r \frac{d}{dr}(\tilde{V}^2) = r^2 - \frac{\tilde{L}^2}{M} r + 3\tilde{L}^2 = 0$

$\Rightarrow$

$\frac{r}{M} = \frac{L^+}{2}\left(L^+ \pm (L^{+2} - 12)^{1/2}\right)$ with $L^+ = \tilde{L}/M$

For $\frac{\tilde{L}}{M} < 2\sqrt{3}$ have instead



Turning points occur where $\tilde{V}^2(\tilde{L}, r) = \tilde{E}^2$, so $\frac{dr}{d\tau} = 0$ there

General features of relativistic orbits:

- Orbits with periastrons at $r \gg M$ are similar to the nonrelativistic Keplerian orbits, but orbits with $r \lesssim 10M$ differ markedly from Keplerian orbits
- For $\tilde{L}/M = L^+ \leq 2\sqrt{3}$ ($2\sqrt{3} \approx 3.46$), there is no periastron: the incoming particle is unavoidably pulled into $r = 2M$.
- For $2\sqrt{3} < \tilde{L}/M < 4$, there are bound orbits, but any particle coming in from infinity ($\tilde{E}^2 \geq 1$) gets pulled in to $r = 2M$.
- For $\tilde{L}/M = L^+ > 4$, particles coming in from ∞ with $\tilde{E}^2 < \tilde{V}^2_{max} = (1 - 2u_m)(1 + L^{+2} u_m^2)$ with $u_m = \frac{1}{6}\left(1 + \left(1 - 12/L^{+2}\right)^{1/2}\right)$ reach a periastron and return to ∞, but particles with $\tilde{E}^2 > \tilde{V}^2_{max}$ get pulled into $r = 2M$. (+ extremum)
- There are stable circular orbits at the minimum of $\tilde{V}^2$. The minimum moves inward from $r = \infty$ for $\tilde{L} = \infty$ to $r = 6M$ for $L^+ = 2\sqrt{3}$. (– extremum)
- There are unstable circular orbits at the maximum of $\tilde{V}^2$. This moves outward from $r = 3M$ for $L^+ = \infty$ to $r = 6M$ for $L^+ = 2\sqrt{3}$. Outward perturbation with $\tilde{E}^2 > 1$: escape to ∞. If $\tilde{E}^2 < 1$, either spiral into $r = 2M$ or have a bound orbit. Inward perturbation: spiral into $r = 2M$.

Eddington-Finkelstein coordinates

The Schwarzschild solution in Schwarzschild coordinates looks singular at $r=2M$. There are a variety of ways to see that this is not an essential physical singularity (although $r=0$ is). For example, in $\sqrt{-g}$ the $(1-\frac{2M}{r})^{\pm 1}$ factors cancel out, so nothing untoward is happening with the volume element at $r=2M$.

It is convenient, however, to look for coordinates where the apparent singularity ($r=2M$) disappears.

The idea of Eddington-Finkelstein coordinates is to replace the time coordinate t by a coordinate v adapted to ingoing radial null geodesics. On a radial null geodesic ($d\theta=d\phi=0$)

$$ds^2=0 \Rightarrow dt^2=(1-\tfrac{2M}{r})^{-2}dr^2=dr_*^2$$

Solve $\frac{dr_*}{dr}=\left(1-\frac{2M}{r}\right)^{-1}$: $r_*=\int\frac{dr}{1-\frac{2M}{r}}=\int\frac{r\,dr}{r-2M}$

Let $y=r-2M$ so $r_*=\int\frac{(y+2M)dy}{y}=2M\ln|y|+y$

ie $r_*=r-2M+2M\ln|r-2M|+\text{const}$. Choose the constant to make $r_*=r+2M\ln\left|\frac{r-2M}{2M}\right|$ — Regge-Wheeler radial coordinate

(r_* also called "tortoise coordinate")

Then for $2M<r<\infty$ one has $-\infty<r_*<\infty$.

On a radial null geodesic, one has $d(t+r_*)=0$ or $d(t-r_*)=0$

Define $u:=t-r_*$ so lines of constant u are outgoing radial null geodesics

$v:=t+r_*$ so lines of constant v are ingoing radial null geodesics

Make a coordinate change

$(t,r,\theta,\phi)\longrightarrow(v,r,\theta,\phi)$ Ingoing Eddington-Finkelstein coords

So $ds^2=-(1-\frac{2M}{r})dt^2+\frac{dr^2}{1-2M/r}+r^2d\Omega^2$

and $dr_*=(1-\frac{2M}{r})^{-1}dr$

So $dt=d(v-r_*)=dv-\frac{dr}{1-2M/r}$

$ds^2=-(1-\frac{2M}{r})dv^2+2dv\,dr\,\frac{(1-\frac{2M}{r})}{1-\frac{2M}{r}}-\frac{dr^2}{1-2M/r}+\frac{dr^2}{1-2M/r}+r^2d\Omega^2$

i.e.

$$ds^2=-\left(1-\frac{2M}{r}\right)dv^2+2dv\,dr+r^2d\Omega^2$$

Schwarzschild in Eddington-Finklestein ingoing coordinates

Although the EF form of the Schwarzschild metric was initially defined only for $r > 2M$, it is now plain to see that it can be analytically continued to cover all $r > 0$.

Write the metric as a matrix:

$$g_{\mu\nu} = \begin{pmatrix} -(1-\frac{2M}{r}) & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \quad \text{at } r=2M \text{ becomes} \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 4M^2 & 0 \\ 0 & 0 & 0 & 4M^2\sin^2\theta \end{pmatrix}$$

This is finite and nonsingular even at $r=2M$ since $\det g_{|r=2M} = -16M^4\sin^2\theta \neq 0$

Earlier, we considered radial infall of a particle in the Schwarzschild geometry. Extend this picture to a spherical shell of particles, viewed as the surface of a radially collapsing star. Exterior to this surface, by Birkhoff's theorem, the metric is unavoidably just Schwarzschild (since spherical symmetry + Einstein's equations $\Rightarrow$ Schwarzschild with constant M). In this picture, the surface of a collapsing spherically symmetric star collapses through $r=2M$ right down to $r=0$.

Recall the Killing vector $k = \partial_t$, timelike for $r > 2M$

$$k = \frac{\partial x^{\mu'}}{\partial t}\frac{\partial}{\partial x^{\mu'}}$$ where for EF coords, $x^{\mu} = \begin{pmatrix} v = t + r_* \\ r = r \\ \theta = \theta \\ \phi = \phi \end{pmatrix}$

$\frac{\partial v}{\partial t} = 1$

and $dv = dt + (1-\frac{2M}{r})dr$

so $k = \frac{\partial}{\partial t} = \frac{\partial v}{\partial t}\frac{\partial}{\partial v} + \frac{\partial r'}{\partial t}\frac{\partial}{\partial r'}$

$$k = \frac{\partial}{\partial v}$$

So the $k = \frac{\partial}{\partial v}$ Killing vector can now be defined over the whole spacetime $0 < r < \infty$.

Note the norm of k: $\|k\|^2 = g_{\mu\nu}k^\mu k^\nu$

$= g_{vv} = -(1-\frac{2M}{r})$

So k is timelike for $r > 2M$ $\quad \|k\|^2 < 0$

null for $r = 2M$ $\quad \|k\|^2 = 0$

spacelike for $r < 2M$ $\quad \|k\|^2 > 0$

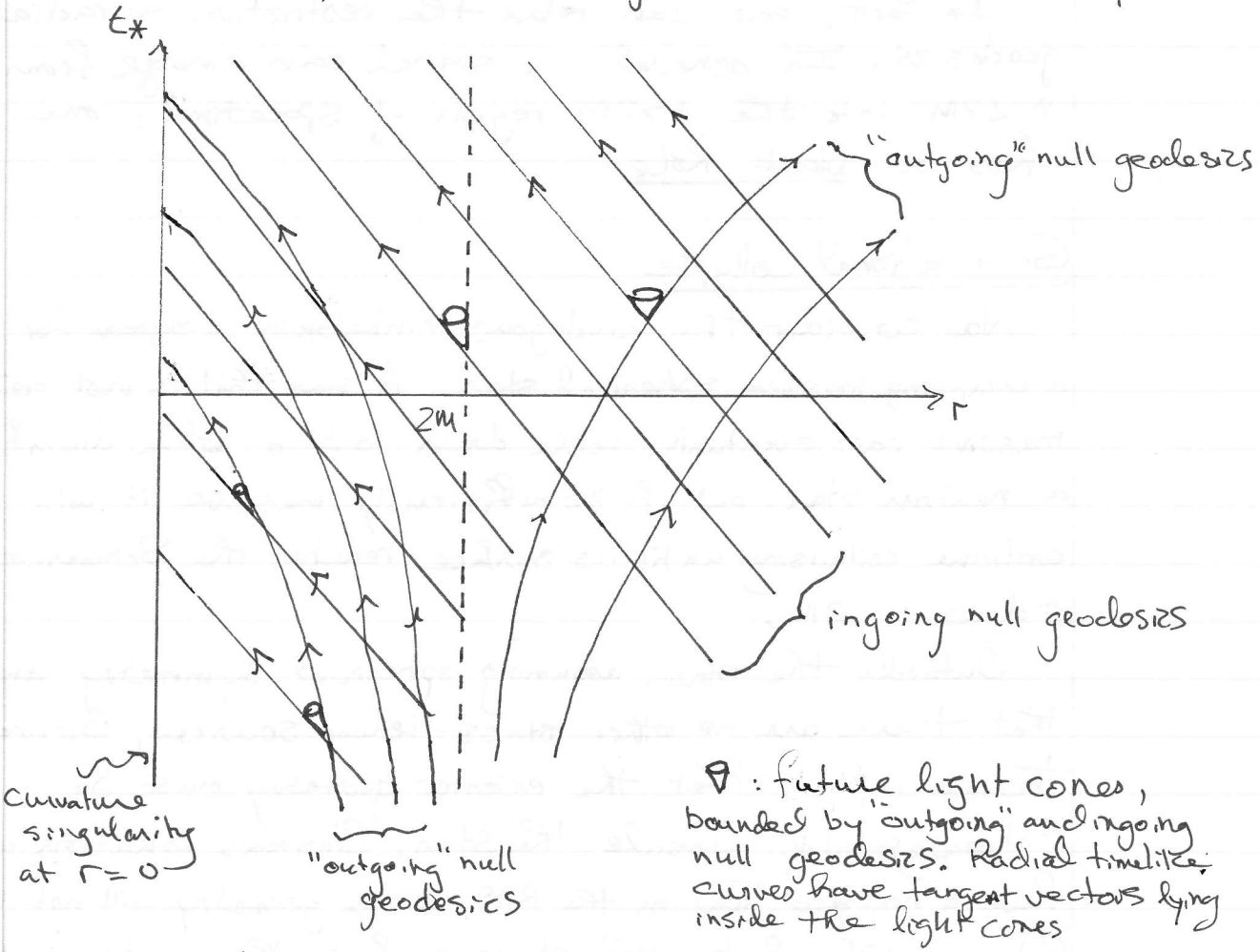
- Neither t nor v is a "time" coordinate for $r < 2M$ – but r is.

Geodesics

The Eddington-Finkelstein coordinates are designed to make one family of radial null geodesics transparent – the ingoing radial null geodesics $v = t + r^* = \text{constant}$.

The other family of radial null geodesics comprises the outgoing ones. For them, $u = t - r_* = \text{constant}$. Write this in EF coordinates: $t = v - r_*$ so $v - 2r_* = \text{constant}$, i.e. $v = 2r + 4M \ln\left|\frac{r}{2M} - 1\right| + \text{constant}$.

A plot of these two families of geodesics is given in the Finkelstein diagram. For this purpose, one may define $t_* = v - r$, so the ingoing geodesics with $v = \text{const}$ become $t_* = -r + \text{const}$, i.e. 45° straight lines in the (t_*, r) plane



▽ : future light cones, bounded by "outgoing" and ingoing null geodesics. Radial timelike curves have tangent vectors lying inside the light cones

Note that the "outgoing" radial null geodesics have r increasing with t_* for $r > 2M$. However, for $r < 2M$, both families of null radial geodesics have r decreasing with t_*. Hence "outgoing". Both families reach the $r=0$ curvature singularity in finite affine parameter.

Consider the tangential derivative $\frac{\partial r}{\partial t_*}$ to the "outgoing" family of null geodesics. We have $t_* = v - r$, so $t_* = r + 4M \ln\left|\frac{r}{2M} - 1\right|$. Then $\frac{dt_*}{dr} = \frac{r+2M}{r-2M}$, or $\frac{dr}{dt_*} = \frac{r-2M}{r+2M}$ for the "outgoing" family. At $r = 2M$, one has $dr/dt_* = 0$, i.e. a vertical line at $r = 2M$.

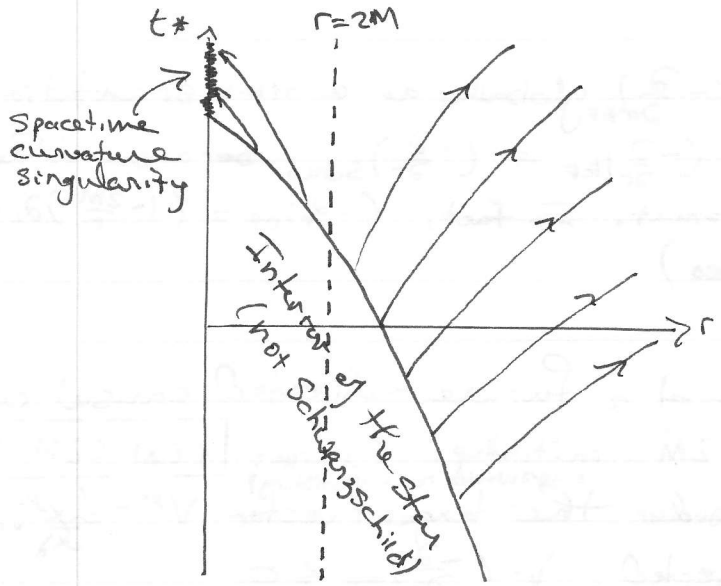
The surface at $r = 2M$ is a separator between the $r > 2M$ outgoing null geodesics and the infalling "outgoing" null geodesics for $r < 2M$. No radial timelike curves come from $r < 2M$ to points with $r > 2M$.

In fact, one can relax the restriction to radial geodesics. In general, no signal can emerge from $r < 2M$ into the $r > 2M$ region of spacetime: one has a <u>black hole</u>.

Gravitational Collapse

Now consider the analogous Finkelstein diagram for a collapsing massive spherical star. A star that is not too massive can eventually settle down to be a white dwarf or a neutron star, but if it is sufficiently massive it will continue collapsing until its surface reaches the Schwarzschild radius $r = 2M$.

Outside the star, assuming spherical symmetry and that there are no other stress-tensor sources, Birkhoff's theorem implies that the exterior geometry must be Schwarzschild. Inside the star, Einstein's equation will have a non-zero $T_{\mu\nu}$ on the RHS, so the geometry will not be Schwarzschild. But on the star's surface, the geometry will be Schwarzschild, and, by continuity, points on the surface will follow radial timelike curves.



Finkelstein diagram for gravitational collapse

Fate of Future-directed causal curves

Some definitions:

- Causal vector: A vector V is causal if it is timelike or null, $V^\mu V^\sigma g_{\mu\sigma} \leq 0$.
- Causal curve: A curve $x^\mu(\lambda)$ is causal if its tangent vector is everywhere causal, $\frac{dx^\mu}{d\lambda}\frac{dx^\sigma}{d\lambda} g_{\mu\sigma}(x(\lambda)) \leq 0 \ \forall \lambda$.
- Time orientation: a causal vector field $T^\mu(x)$.
- Time orientable spacetime: a spacetime that admits a time orientation.
- Future directed causal vector: a causal vector V that lies in the same light cone as the time orientation T^μ, $V^\mu T^\sigma g_{\mu\sigma} \leq 0$. (If $V^\mu T^\sigma g_{\mu\sigma} > 0$, it is past directed)

Now, for Schwarzschild we need to choose a time orientation. For $r > 2M$, an obvious time orientation vector field is $k = \partial_t$ (Schwarzschild coords) $= \partial_r$ (EF coordinates). However, for $r < 2M$ k becomes spacelike since then $-(1-\frac{2m}{r}) > 0$. Consequently, k cannot serve as a time orientation globally. Another choice could be $\pm\partial_r$ in EF coordinates, since these vectors are globally null (since $g_{rr} = 0$ in EF coords). The choice should be made to agree with the k time orientation for $r > 2M$. $T = -\partial_r$ works since $k \cdot (-\partial_r) = k^\mu(-\partial_r)^\sigma g_{\mu\sigma} = -g_{rr} = -1$: agrees.

So one can choose $(-\frac{\partial}{\partial r})_{EF}$ globally as a time orientation for all $r > 0$ (note: $(-\frac{\partial}{\partial r})_{EF} \neq (-\frac{\partial}{\partial r})_{Schw}$ because different coordinates are held constant. In fact, $(-\partial_r)_{EF} = (1-\frac{2M}{r})\partial_t - \partial_r$ in Schwarzschild coordinates)

—

Now consider the fate of a future-directed causal curve $x^\mu(\lambda)$ where $r(\lambda_0) \leq 2M$ initially. Claim: $\boxed{r(\lambda) \leq 2M \;\; \forall \lambda \geq \lambda_0}$ (assumed non-vanishing)

To prove this, consider the tangent vector $V^\mu = \frac{dx^\mu}{d\lambda}$

Since V is future directed, $V \cdot (-\frac{\partial}{\partial r})_{EF} \leq 0$

i.e. $-V^\mu g_{\mu r} \leq 0 \Rightarrow -V^v \leq 0$ since only $g_{vr} = 1 \neq 0$

Thus $\frac{dv(\lambda)}{d\lambda} \geq 0$. Now, one also has

$$V^2 = V^\mu V^\sigma g_{\mu\sigma} = -(1-\tfrac{2M}{r})(\tfrac{dv}{d\lambda})^2 + 2\tfrac{dv}{d\lambda}\tfrac{dr}{d\lambda} + r^2\left((\tfrac{d\theta}{d\lambda})^2 + \sin^2\theta(\tfrac{d\phi}{d\lambda})^2\right)$$

rearrange:

$$-2\tfrac{dv}{d\lambda}\tfrac{dr}{d\lambda} = -V^2 + (\tfrac{2M}{r} - 1)(\tfrac{dv}{d\lambda})^2 + r^2\left((\tfrac{d\theta}{d\lambda})^2 + \sin^2\theta(\tfrac{d\phi}{d\lambda})^2\right) \quad (*)$$

Moreover $-V^2 \geq 0$ since V is by assumption causal. So the above RHS ≥ 0 $\forall r \leq 2M$. Then, since $dv(\lambda)/d\lambda \geq 0$ one concludes that $dr/d\lambda \leq 0$ for any starting $r(\lambda_0) \leq 2M$.

More strictly, if $r(\lambda_0) < 2M$ then $r(\lambda) < 2M$ $\forall \lambda \geq \lambda_0$, because if $dr/d\lambda = 0$ on the LHS of $(*)$, then on the RHS of $(*)$ one would have to have $\dot{v} = \dot{\theta} = \dot{\phi} = 0$ so $V^\mu = 0$ but this contradicts the assumption of a curve with a non-vanishing $dx^\mu/d\lambda = V^\mu$.

Now consider the case where $r(\lambda_0) = 2M$. If $dr/d\lambda < 0$ at $\lambda = \lambda_0$ then $r(\lambda_0 + \delta\lambda) < 2M$ and the above analysis takes over. Also, if $dr/d\lambda = 0$ for all $\lambda > \lambda_0$ then one has a curve stuck at $r = 2M$, consistent with the above claim. If $dr/d\lambda > 0$ at $\lambda = \lambda_0$, then for $\lambda' = \lambda_0 - \delta\lambda$ one has $r < 2M$ at λ' and the above analysis takes over again. The remaining case to consider is then the case where $dr/d\lambda = 0$ at $\lambda = \lambda_0$ but assume $dr/d\lambda > 0$ for $\lambda = \tilde{\lambda} = \lambda_0 + \tilde{\delta}\lambda$. Now, at $\lambda = \lambda_0$, the LHS hence the RHS of $(*)$ vanishes.

$\tilde{\lambda}$ close to λ_0

With the RHS of (*) vanishing at $\lambda = \lambda_0$ and (with) $r = 2M$, one must have $V^2 = \dot\theta = \dot\phi = 0$. Then either $V^\mu = 0$, contrary to our assumption of a non-vanishing tangent vector, or $\dot v = dv/d\lambda \neq 0$. However, we have already shown that $dv/d\lambda \geq 0$. So one must have $dv/d\lambda > 0$ at $\lambda = \lambda_0$. Near $\lambda = \lambda_0$ one can then use v as a parameter for the x^μ causal curve instead of λ. Divide (*) by $(dv/d\lambda)^2$:

$$-2\, dr/dv \geq \frac{2M}{r} - 1 \Rightarrow 2\, dr/dv \leq 1 - \frac{2M}{r}$$

Now $r = 2M$ at $v = v_0 = v(\lambda_0)$. Consider $v_2 > v_1 > v_0$, v_2 and $v_1 = v_1(\tilde\lambda)$ slightly greater than v_0. Then $2\int_{r(v_1)}^{r(v_2)} \frac{dr}{1 - 2M/r} \leq v_2 - v_1$

Reductio ad absurdum: $p \to q$ $\cong$ $\neg q \to \neg(\neg p)$

Then take $\tilde\delta\lambda \to 0$ so $\tilde\lambda \to \lambda_0$, hence $v_1 \to v_0$ and $r(v_1) \to 2M$. Now we have a contradiction with the assumption that $dr/d\lambda > 0$ at $\tilde\lambda$ because $v_2 - v_1$ tends to $v_2 - v_0$, a finite limit, while $\int_{2M}^{r(v_2)} dr/(1 - 2M/r)$ diverges. So one must be stuck at $r = 2M$ $\forall \lambda$, the previous case.

Consequently, no future-directed causal curve connects a point at $r \leq 2M$ to a point with $r > 2M$, ie. it is impossible to send a signal from a point with $r \leq 2M$ to a point $r > 2M$. In particular, one cannot send such a signal out to $r = \infty$.

A <u>black hole</u> is defined to be a region of spacetime from which it is impossible to send a signal to infinity. We have thus established that the region $r \leq 2M$ of the Schwarzschild spacetime, as extended using I-F coordinates, lies inside a black hole.

On the other hand, there do exist future-directed causal curves from points $r > 2M$ out to $r = \infty$ (eg. a radial null curve). So points with $r > 2M$ do not lie inside the black hole. The surface at $r = 2M$ is called the <u>event horizon</u>.

<u>Falling to $r = 0$ from $r = 2M$</u>: we had before $\tau(r_{max} \to r, -0) = \pi M (1 - \mathcal{E}^2)^{-3/2}$ in Schwarzschild coords, and $r_{max} = \frac{2M}{1 - \mathcal{E}^2}$ so for $r_{max} = 2M$, $\mathcal{E} = 0$. Then $\pi M \approx 10^5 \left(\frac{M}{M_\odot}\right)$ seconds.

Outgoing EF coordinates & white holes

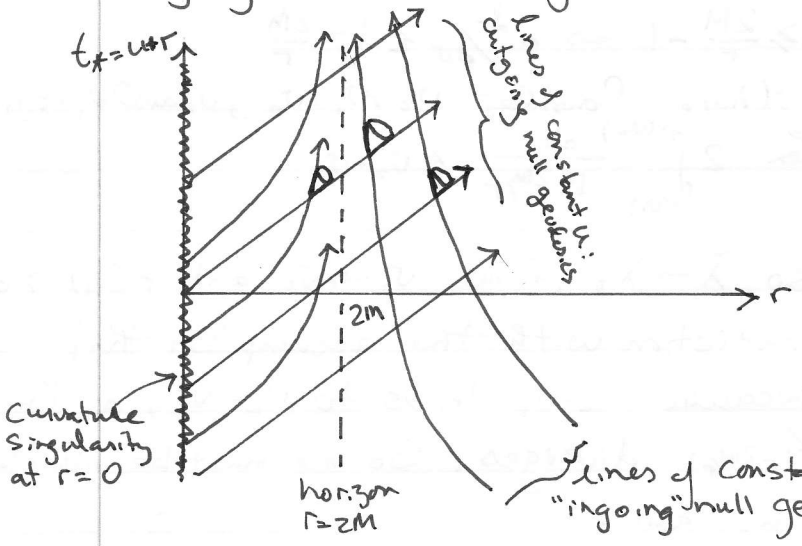
We had $u = t - r_*$. Change coordinates now to (u, r, θ, ϕ):

$$ds^2 = -\left(1 - \frac{2M}{r}\right)du^2 - 2dr\,du + r^2 d\Omega^2 \qquad -\infty < u < \infty$$

As for the ingoing EF coordinates, one can analytically continue down to all $r > 0$. Note, however, that the regions

$$(r < 2M)_{\text{outgoing EF}} \neq (r < 2M)_{\text{ingoing EF}}.$$

Outgoing Finkelstein diagram



Note: outgoing Finkelstein diagram is just the ingoing diagram turned upside down

"Ingoing" radial null geodesics with $v = \text{const}$ approach $r = 2M$ while outgoing radial null geodesics with $u = \text{const}$ escape to $r = \infty$.

This is the time reverse of a black hole, guaranteed to exist because Einstein's equations are time reversal invariant. This is called a white hole. Radial null geodesics inside $r = 2M$ start at the $r = 0$ singularity. Study of the stability properties of black holes shows them to be stable, ie. small perturbations away from the Schwarzschild solution decay. Conversely, in the time reversed white hole solutions, small perturbations will grow and consequently such solutions are unstable.

Kruskal-Szekeres coordinates (KS)

We have now seen that there are two different ways to extend the $r>2M$ region of Schwarzschild spacetime to regions inside the horizon. Both the black hole and white hole regions can be covered together by yet another set of coordinates.

Firstly, note that in the outer region $r>2M$ one can use coordinates (u,v,θ,ϕ), in terms of which (recall $u=t-r_*$, $v=t+r_*$)

$$ds^2 = -\left(1-\frac{2M}{r}\right)du\,dv + r^2 d\Omega^2$$

However, this is still singular at $r=2M$. To remove the singularity, define

$$U = -\exp\left(\frac{-u}{4M}\right) \quad ; \quad V = \exp\left(\frac{v}{4M}\right)$$

note that $U<0$ and $V>0$ so far. The coordinate $r=r(U,V)$ is defined implicitly by

$$UV = -\exp(r_*/2M) = -e^{r/2M}\left(\frac{r}{2M}-1\right) \quad (*)$$

Note also that $V/U = -e^{t/2M}$.

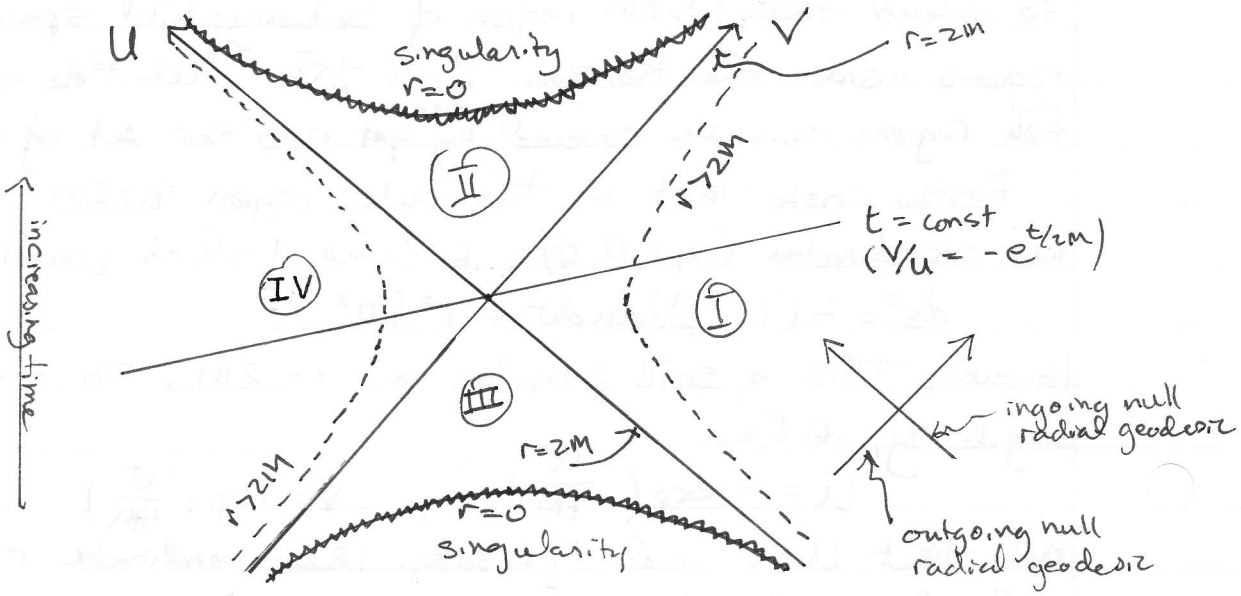
Then $ds^2 = -\frac{32M^3}{r}e^{-r/2M}dU\,dV + r^2 d\Omega^2$ with $r=r(U,V)$

The coordinates (U,V,θ,ϕ) are the Kruskal-Szekeres (KS) coordinates. Having defined them in the region $r>2M$ where we had $U<0$ and $V>0$, one can now extend the Schwarzschild coordinate $r>2M$ region to full coordinate ranges $-\infty<U,V<\infty$ with the above metric and $r(U,V)$ defined implicitly by $(*)$ above.

Note that when $r=2M$ in $(*)$ above, one has $UV=0$ so either $U=0$ or $V=0$. When $r=0$, one has $UV=1$. Radial null geodesics have either U or V = constant.

This extended structure reveals that the horizon surface $r=2M$ actually corresponds to two surfaces in KS coordinates, intersecting at $U=V=0$ — a 2d subsurface known (recalling the remaining θ and ϕ dependence) as the bifurcation 2-sphere. Similarly, the curvature singularity at $r=0$ corresponds to $UV=1$, a hyperbola with two branches.

Kruskal diagram



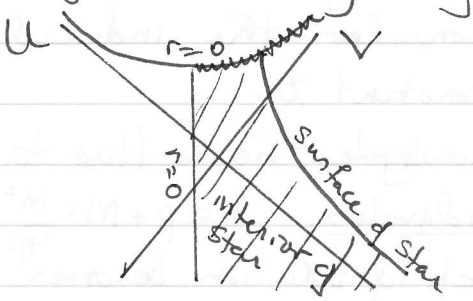
Region I is the original $r > 2M$ region in Schwarzschild coordinates.
Regions I & II are covered by ingoing EF coordinates
Regions I & III are covered by outgoing EF coordinates
Region IV is a new region, isometric to region I under $(U, V) \to (-U, -V)$.

Region II is the $r \leq 2M$ interior of the black hole that we discussed using ingoing EF coordinates.
Region III is the interior of the white hole discussed using outgoing EF coordinates.

Each point of the Kruskal diagram can be viewed as representing a 2-sphere of radius $r(U, V)$. Alternatively, the diagram can be interpreted as the causal structure of radial motion at a fixed set $\{\theta_0, \phi_0\}$ of polar angles.

Note that the singularity in region II is to the future of any point in region II. It is therefore more appropriate to think of the singularity as a "time" at which tidal forces become infinite. Inside region II it is r, not t, that plays the role of time.

Most of the Kruskal diagram is unphysical, as far as black holes or gravitational collapse are concerned. Replacing a region inside the surface of a collapsing star by a non-vacuum spacetime with $T_{\mu\nu} \neq 0$, what is physically relevant is a diagram involving only regions I and II.



inside the collapsing star, until the matter hits the singularity $r=0$ is just the origin of polar coordinates

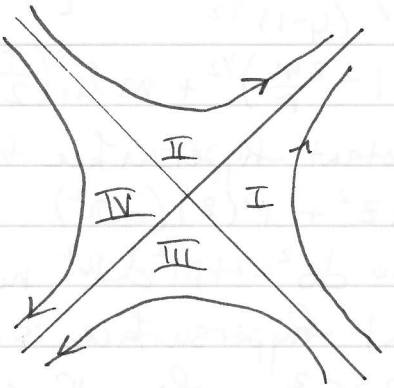
Translations generated by the Killing vector k:

We had $k = \partial_t$ in Schwarzschild coordinates. In KS coordinates, this becomes

$$k = \frac{1}{4M}\left(V\frac{\partial}{\partial V} - U\frac{\partial}{\partial U}\right)$$

with (norm)^2 $k^2 = -\left(1 - \frac{2M}{r}\right)$

So k is timelike in regions I and IV, spacelike in regions II and III and null on the lines $U=0$ or $V=0$. At the subregion $U=V=0$, one has $k=0$. This is the bifurcation 2-sphere.



Orbits of k

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The Einstein-Rosen bridge

In region I, one has $V/U = -e^{t/2M}$ so a surface of constant coordinate t is a straight line through the origin of the Kruskal diagram. Such lines extend across into region IV. Consider the induced geometry on such a subsurface at constant t.

One way to investigate this is first to change coordinates to isotropic coordinates $r = \rho + M + \frac{M^2}{4\rho} = \rho\left(1+\frac{M}{2\rho}\right)^2$ in which the Schwarzschild solution becomes

$$ds^2 = -\frac{\left(1-\frac{M}{2\rho}\right)^2}{\left(1+\frac{M}{2\rho}\right)^2}dt^2 + \left(1+\frac{M}{2\rho}\right)^4\left(d\rho^2+\rho^2 d\Omega^2\right).$$

This has a discrete isometry $\rho \to \frac{M^2}{4\rho}$ which interchanges regions I and IV. A surface at constant t then has the metric

$$ds^2 = \left(1+\frac{M}{2\rho}\right)^4\left(d\rho^2+\rho^2 d\Omega^2\right).$$

The surface $r=2M$ becomes $\rho = \frac{M}{2}$ in these coordinates, where the 4-geometry looks singular. However, this is just a coordinate singularity.

Another way to look at the $t=$ const slice is to use coordinates defined by $dz = \left(1-\frac{2M}{r}\right)^{-1/2}dr$. Integrate this, setting first $r = 2My$, so

$$z = 2M\int \frac{dy\sqrt{y}}{(y-1)^{1/2}} = 2M\left[\left(y(y-1)\right)^{1/2} + \frac{1}{2}\ln y + \ln\left(1+(1-\tfrac{1}{y})^{1/2}\right)\right]$$

so

$$z = r\left(1-\frac{2M}{r}\right)^{1/2} + M\ln\left(\frac{r}{2M}\right) + 2M\ln\left(1+\left(1-\frac{2M}{r}\right)^{1/2}\right)$$

$r=2M \leftrightarrow z=0$

Then the $t=$ constant hypersurface has

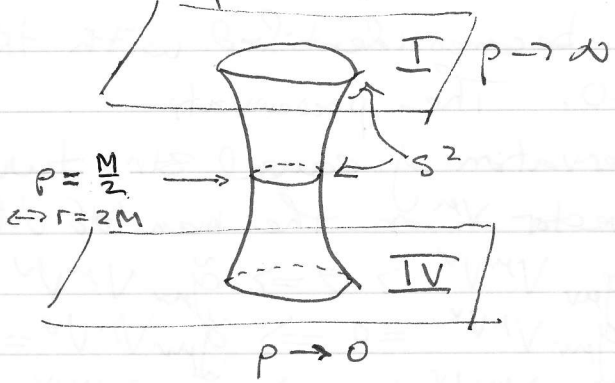
$$ds^2 = dz^2 + r^2(z)\left(d\Omega^2\right)$$

and at $z=0 \leftrightarrow r=2M$ $ds^2 = 4M^2 d\Omega^2$

The $t=$ constant hypersurface is a Riemannian 3-manifold with topology $\mathbb{R}\times S^2$, where $\mathbb{R}$ is parametrized by ρ or z.

Regions I and IV have identical $r=2M \leftrightarrow \rho = \frac{M}{2}$ hypersurfaces which can be smoothly joined at the bifurcation 2-sphere $U=V=0$.

The t = constant geometry has two asymptotically flat regions ($\rho \to \infty$ and $\rho \to 0$) connected by a "throat" with minimum radius $r = 2M$ at $\rho = \frac{M}{2}$. Note that the inside of region II corresponds to complex values of ρ, so it is not seen in the isotropic coordinates or z coordinates



Einstein-Rosen Bridge

Although this looks like an attractive "wormhole" between regions I and IV, remember that it is a subsurface at constant t, and one cannot make a causal trip through it. Crossing the $r = 2M$ horizon, as we have seen, leads inevitably to getting crushed in the Schwarzschild singularity at $r = 0$.

Carter-Penrose Diagrams (aka "Penrose Diagrams")

We have seen that solutions to general relativity can have a rather complicated causal and singularity structure. It is very useful to have a graphical method of summarising the essential structure of a solution.

Causal structure is left unchanged by a <u>Weyl transformation</u>:

often called "conformal transformation" but that is inaccurate

$$ds^2 \longrightarrow d\tilde{s}^2 = \Lambda^2(x^\mu)\, ds^2 \qquad \Lambda \neq 0 \,.$$

i.e. $\tilde{g}_{\mu\nu}(x) = \Lambda^2(x)\, g_{\mu\nu}(x)$, where $\Lambda(x)$ is smooth & non-vanishing $\forall x \in \mathcal{M}$.

$\Lambda(x)$ some arbitrary function

Such Weyl transformations map from one spacetime $(\mathcal{M}, g)$ to a different spacetime $(\mathcal{M}, \tilde{g})$. They preserve the <u>causal structure</u> of the spacetime, but curvature is <u>not</u> preserved.

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new coordinates and

One can choose the Weyl transformation function $\Lambda(x)$ such that it brings all points at ∞ in the original metric into points at finite affine parameter in the Weyl transformed metric $\tilde{g}_{\mu\nu}$. In order for this to happen, one needs to choose $\Lambda(\vec{r}, t) \to 0$ as $|\vec{r}| \to \infty$ and/or $|t| \to \infty$. Then "infinity" becomes identified with the surface where $\Lambda(x) = 0$.

The preservation of causal structure is seen as follows. Consider a vector V^μ on the manifold M. Since $\Lambda(x)^2 > 0$, one has

$$g_{\mu\nu} V^\mu V^\nu > 0 \Rightarrow \tilde{g}_{\mu\nu} V^\mu V^\nu > 0$$
$$g_{\mu\nu} V^\mu V^\nu = 0 \Rightarrow \tilde{g}_{\mu\nu} V^\mu V^\nu = 0$$
$$g_{\mu\nu} V^\mu V^\nu < 0 \Rightarrow \tilde{g}_{\mu\nu} V^\mu V^\nu < 0.$$

So curves that are timelike/null/spacelike in the metric $g_{\mu\nu}$ are timelike/null/spacelike with respect to $\tilde{g}_{\mu\nu}$. Curves that happen to be geodesics for the metric $g_{\mu\nu}$ will not in general be geodesics for the transformed metric $\tilde{g}_{\mu\nu}$. However, there is a special case to be made for null geodesics, since a vanishing norm for a vector V^μ is preserved under such a Weyl transformation, $\tilde{g}_{\mu\nu} V^\mu V^\nu = 0$.

$g_{\mu\nu} V^\mu V^\nu = 0$

The idea of a Carter-Penrose diagram involves these steps: 1) make an ordinary coordinate transformation on the spacetime (M, g) such that points at infinity are mapped to points at a finite coordinate distance location, so that the entire spacetime can be drawn on a piece of paper. Typically, when one does this the metric g will have divergences in the new coordinates as one approaches the "points at infinity", ie. the edges of the new finite diagram. Then 2) make a Weyl transformation on the metric g with a scale factor $\Lambda(x)$ in order to obtain a new metric $\tilde{g}$ which is regular at the edges of the diagram.

The "points at infinity" are not actually part of the original manifold (M, g). After the coordinate change

and Weyl transformation, however, they are represented by the edges of the Carter-Penrose diagram and it is natural 3) to include them. This is called a "conformal compactification" of the resulting spacetime $(\mathcal{M}, \tilde{g})$. The points at infinity are the loci when $\Lambda = 0$, as needed to make $\tilde{g}$ regular.

Example: 1+1 dimensional Minkowski space

$$ds^2 = -dt^2 + dx^2 \qquad -\infty < t, x < \infty$$

Introduce light-cone coordinates $u = t - x$, $v = t + x$.

Then $ds^2 = -du\,dv$

Next, shrink "infinity" down to a finite coordinate distance by defining $u = \tan\tilde{u}$, $v = \tan\tilde{v}$

where the coordinate ranges are now $-\pi/2 < \tilde{u}, \tilde{v} < \pi/2$.

Note the strict inequalities so far, since points with u or v equal to $\tan(\pm\pi/2) = \pm\infty$ are not points in the initial 1+1 dimensional spacetime.

In terms of $\tilde{u}$ & $\tilde{v}$, the line element is

$$ds^2 = -\frac{1}{(\cos\tilde{u}\cos\tilde{v})^2}\, d\tilde{u}\, d\tilde{v}$$

and this diverges as $\tilde{u}, \tilde{v} \to \pm\pi/2$.

Next, make a Weyl transformation with $\Lambda = \cos\tilde{u}\cos\tilde{v}$:

$$d\tilde{s}^2 = (\cos\tilde{u}\cos\tilde{v})^2 ds^2 = -d\tilde{u}\,d\tilde{v}.$$

The metric $\tilde{g}_{\mu\nu}$ is now regular at points where $\tilde{u}$ or $\tilde{v}$ equal $\pm\pi/2$. These points can now be included in the conformally compactified spacetime $(\mathcal{M}, \tilde{g})$:

$$d\tilde{s}^2 = -d\tilde{u}\,d\tilde{v}, \qquad -\pi/2 \le \tilde{u}, \tilde{v} \le \pi/2.$$

Points at infinity: $(\tilde{u}, \tilde{v}) = (\pi/2, \pi/2)$ and $(-\pi/2, -\pi/2)$ are denoted $i^{\pm}$: future and past timelike infinity. All future or past directed timelike curves end up at i^+ or i^-. In the original coordinates, these points correspond to $t \to \pm\infty$ with x finite.

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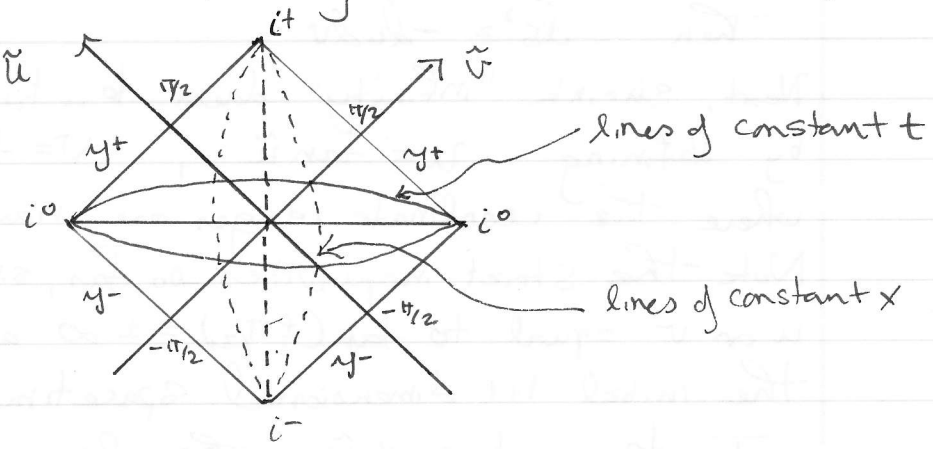
Points with $(\tilde{u},\tilde{v}) = (\pi/2, -\pi/2)$ or $(\tilde{u},\tilde{v}) = (-\pi/2, \pi/2)$ are denoted i^0. These are the endpoints of spacelike geodesics $x \to \pm\infty$, t finite, ie. spatial infinity.

Points with $\tilde{v} = \pi/2$ and constant $|\tilde{u}| < \pi/2$ or with $\tilde{u} = \pi/2$ and constant $|\tilde{v}| < \pi/2$ are denoted $\mathcal{I}^+$ (scri-plus), future null infinity. They are the locus of endpoints of future directed null geodesics.

Analogously, points with $\tilde{v} = -\pi/2$ and constant $|\tilde{u}| < \pi/2$ or $\tilde{u} = -\pi/2$ and constant $|\tilde{v}| < \pi/2$ are denoted $\mathcal{I}^-$: past null infinity.

Here is the corresponding Carter-Penrose diagram:

for 1+1 Minkowski spacetime



3+1 dimensional Minkowski spacetime

$$ds^2 = -dt^2 + dx^i dx^i \qquad i = 1,2,3$$
$$= -dt^2 + dr^2 + r^2 d\Omega^2$$

(choose some arbitrary pt as origin of polar coords)

Then, as in 1+1 Minkowski spacetime

let $u = t - r \qquad v = t + r \qquad -\infty < u \leq v < \infty$

↑ since $r \geq 0$

Then $ds^2 = -du\,dv + \frac{(v-u)^2}{4} d\Omega^2$

Then let $u = \tan\tilde{u}$, $v = \tan\tilde{v} \qquad -\pi/2 < \tilde{u} \leq \tilde{v} < \pi/2$

↑ still strict inequalities

$$ds^2 = \frac{1}{(2\cos\tilde{u}\cos\tilde{v})^2}\left[-4\,d\tilde{u}\,d\tilde{v} + \sin^2(\tilde{v}-\tilde{u})\,d\Omega^2\right]$$

Then take $\Lambda = 2\cos\tilde{u}\cos\tilde{v}$ to obtain

$$d\tilde{s}^2 = -4\,d\tilde{u}\,d\tilde{v} + \sin^2(\tilde{v}-\tilde{u})\,d\Omega^2 \qquad -\pi/2 \leq \tilde{u} \leq \tilde{v} \leq \pi/2$$

conformally compactified 3+1 dim Minkowski spacetime.