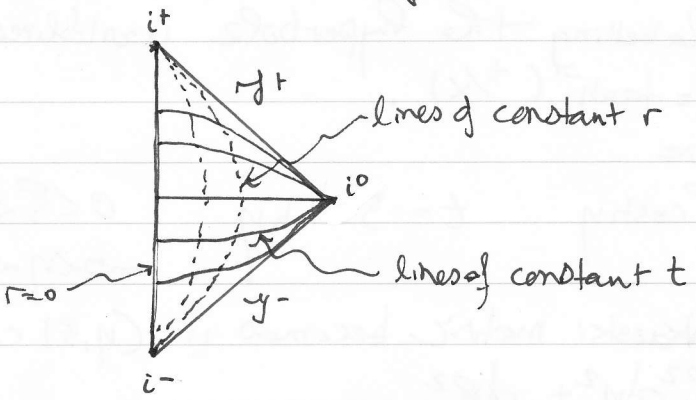
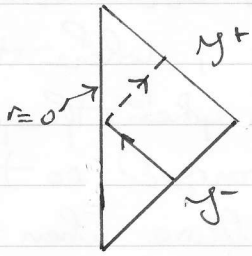
Carter-Penrose diagram for 3+1 dim Minkowski space



Every point represents an S^2

Radial null geodesics in 3+1 dim Minkowski space



Radial null geodesics are straight 45° lines coming in from $\mathcal{I}^-$, passing through $r=0$ and going out to $\mathcal{I}^+$.

Note that each point of the diagram represents a 2-sphere in the 3+1 spacetime. On the outgoing leg to $\mathcal{I}^+$, the (θ,ϕ) point on S^2 must be the antipode of the (θ,ϕ) point on S^2 for the incoming leg from $\mathcal{I}^-$.

Rindler Spacetime : Minkowski viewed by a uniformly accelerated observer

Rindler space is a subregion of Minkowski space in 1+1 dims.
Consider motion along a hyperbolic worldline

$$X_\mu X^\mu = -t^2 + x^2 = \xi^2, \text{ constant} \qquad \mu = 0, 1$$

Then $-t\,dt + x\,dx = 0$ so $\frac{dx}{dt} = \frac{t}{x}$

and $d\tau^2 = dt^2 - dx^2$

$$= dt^2\left(1 - \frac{t^2}{x^2}\right) = \frac{\xi^2 dt^2}{x^2} \quad \text{so} \quad \frac{dt}{d\tau} = \frac{x}{\xi}$$

Thus $$\frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt}\frac{dt}{d\tau} = (1, \tfrac{t}{x})\frac{x}{\xi} = (x, t)\cdot\frac{1}{\xi}$$

So $$a^\mu = \frac{d^2x^\mu}{d\tau^2} = \frac{d}{dt}\left(\frac{dx^\mu}{d\tau}\right)\frac{dt}{d\tau} = \left(\frac{t}{x}, 1\right)\frac{1}{\xi}\cdot\frac{x}{\xi} = \frac{x^\mu}{\xi^2}$$

Hence $a_\mu a^\mu = \frac{X_\mu X^\mu}{\xi^4} = \xi^2/\xi^4 = 1/\xi^2$ Constant norm acceleration.

Now choose new coordinates. Let one coordinate be $\xi = (x^2 - t^2)^{1/2}$ labelling the hyperbolic worldlines; let the other be $\eta = \tanh^{-1}(t/x)$.

(η, ξ) Rindler coords

Then we have

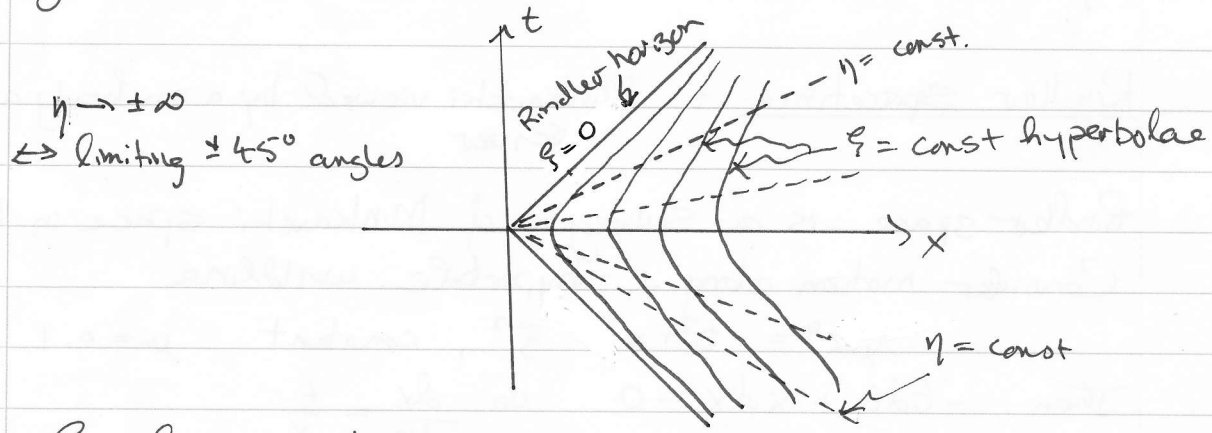
$$x = \xi \cosh\eta \qquad t = \xi \sinh\eta \qquad \begin{matrix} 0 < \xi < \infty \\ -\infty < \eta < \infty \end{matrix}$$

The 1+1 dim Minkowski metric becomes in (η, ξ) coordinates

$$ds^2 = -\xi^2 d\eta^2 + d\xi^2$$

An accelerated observer travelling on a $\xi = (x^2 - t^2)^{1/2} = \text{const}$ worldline is stationary in Rindler coordinates, so then $d\tau^2 = \xi^2 d\eta^2$, i.e. $\tau = \xi\eta$, so for the accelerated observer on the $\xi = \text{const}$ worldline, the proper time evolved along that worldline is $\tau = \xi\eta$, synchronised so that $\tau = 0$ corresponds to the point on the worldline when the observer passes the x axis (ie $t = 0$).

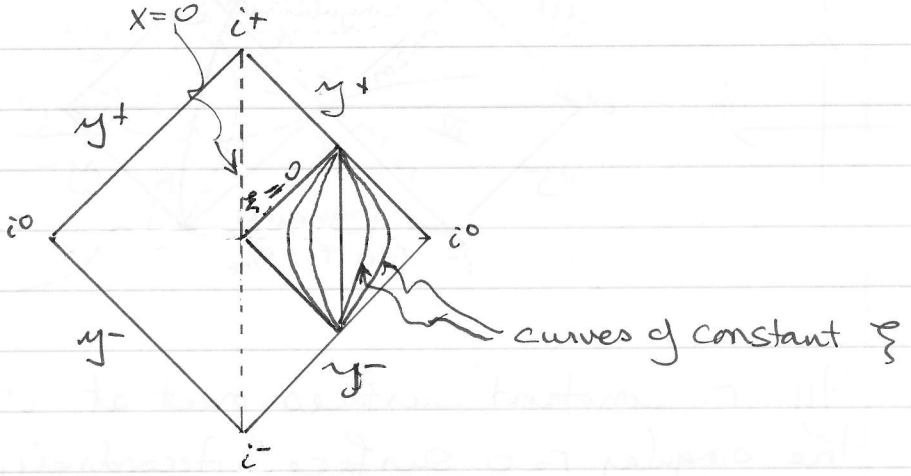
In terms of the original Minkowski coordinates, the portion of 1+1 dim Minkowski space that can be accessed uniformly by such an accelerated observer has $x \geq t$.



The line $x = t$, i.e. $\xi = 0$, is the boundary of the portion of the 1+1 dim spacetime that is accessible to any uniformly accelerated observer. The region $x \leq t$ is hidden to such observers. Accordingly, the $x = t \leftrightarrow \xi = 0$ line is the <u>Rindler horizon</u>. <u>Rindler space</u> is the $x > |t|$ wedge, foliated by the $\xi = \text{const}$ worldlines of accelerated observers.

- Carter-Penrose diagram for Rindler space

The Rindler wedge is the region $x>|t|>0$, so $\tan\tilde{v} = v = t+x > 0$ and $\tan\tilde{u} = u = t-x < 0$, so the compactified Rindler space (including $\mathcal{I}^\pm$) is the region $-\pi/2 \le \tilde{u} < 0$, $0 < \tilde{v} \le \pi/2$, a subregion of $1+1$ dim Minkowski space



Carter-Penrose diagram for Kruskal spacetime $(3+1)$ dims

The KS metric is

$$ds^2 = -\frac{32M^3}{r} e^{-r/2M} dU dV + r^2 d\Omega^2 \qquad UV = -\exp(r_*/2M)$$

$$r_* = r + 2M \ln\left|\frac{r}{2M} - 1\right|$$

Define new null coordinates $\tilde{U}, \tilde{V}$: $-\infty < U, V < \infty$ original KS coords

$$U = \tan\tilde{U} \qquad V = \tan\tilde{V} \qquad -\pi/2 < \tilde{U}, \tilde{V} < \pi/2$$

then the metric becomes

$$ds^2 = (\cos\tilde{U}\cos\tilde{V})^{-2}\left[-\frac{32M^3}{r} e^{-r/2M} d\tilde{U} d\tilde{V} + r^2 \cos^2\tilde{U}\cos^2\tilde{V}\, d\Omega^2\right]$$

after which we perform a Weyl transformation with $\Lambda = \cos\tilde{U}\cos\tilde{V}$:

$$d\tilde{s}^2 = \Lambda^2 ds^2 = -\frac{32M^3}{r} e^{-r/2M} d\tilde{U} d\tilde{V} + r^2 \cos^2\tilde{U}\cos^2\tilde{V}\, d\Omega^2$$

after which one adds points at infinity, so $-\pi/2 \le \tilde{U}, \tilde{V} \le \pi/2$

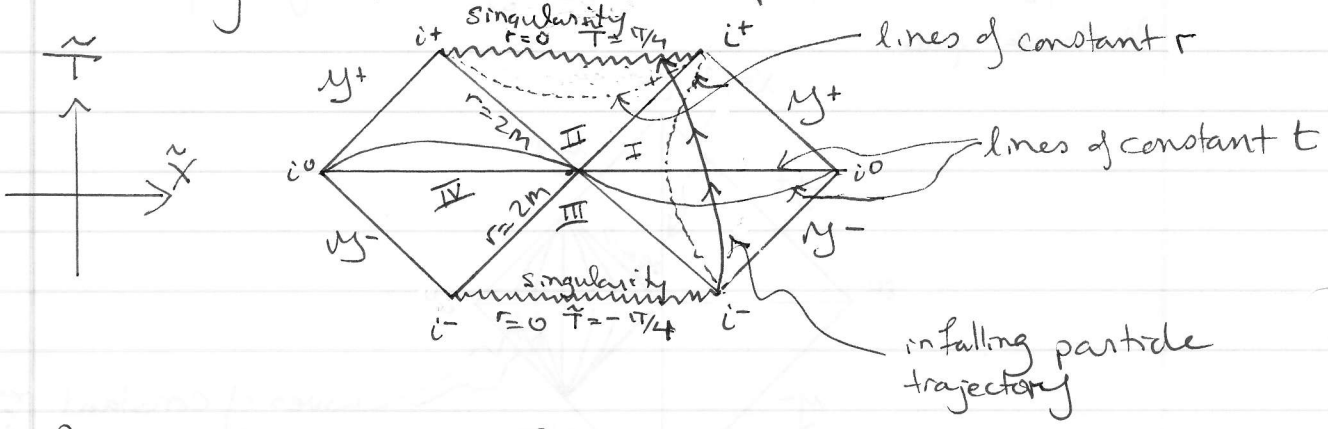
The curvature singularity locus $UV = 1$ corresponds to $\tan\tilde{U}\tan\tilde{V} = 1 \Leftrightarrow \cos\tilde{U}\cos\tilde{V} - \sin\tilde{U}\sin\tilde{V} = 0 \Leftrightarrow \cos(\tilde{U}+\tilde{V}) = 0$

So the singularity locus is $\tilde{U} + \tilde{V} = \pm\pi/2$.

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It can be useful at this stage to define $\tilde{T}$ and $\tilde{X}$ by $\tilde{U} = \tilde{T} - \tilde{X}$ and $\tilde{V} = \tilde{T} + \tilde{X}$, so the singularity locus is at $\tilde{T} = \pm \pi/4$

- CP diagram for Kruskal spacetime



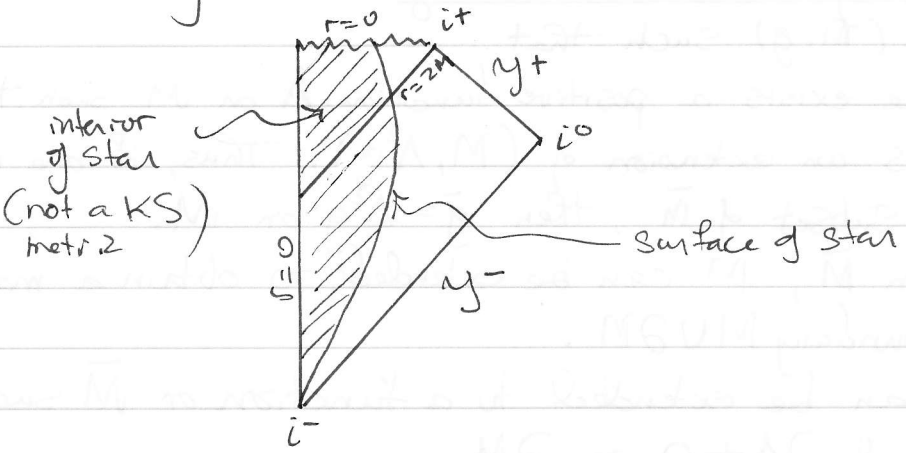
All r = constant surfaces meet at i^+ and i^-, including the singular $r=0$ surface. Accordingly, $i^\pm$ are singular points at future/past timelike infinity, and they therefore are not to be added to the spacetime manifold M.

As one approaches $\mathcal{I}^\pm$, i.e. $r \to \infty$ for any t, the KS spacetime becomes asymptotically flat. Consequently, the far RH side of the Kruskal CP diagram has the same structure as the CP diagram of (3+1) dim Minkowski spacetime (and similarly for the far LH region IV side). Accordingly, one can add edge points i^0 and $\mathcal{I}^\pm$ as for (3+1) Minkowski space.

- CP diagram for a collapsing star

The example trajectory for an infalling particle above can also be viewed as the trajectory of the surface of a radially collapsing star. The only difference in that case is that the region to the left of that surface trajectory is then inside the star, where $T_{\mu\nu} \neq 0$. Consequently, in that region the spacetime is not described by the KS metric. The exterior metric must be KS, however, by Birkhoff's theorem.

Accordingly, for a collapsing star one may draw the CP diagram



Asymptotic Flatness

We have used the asymptotic flatness of the Kruskal spacetime in drawing the corresponding Carter-Penrose diagram. Let us now be more formal about what is meant by asymptotic flatness.

Firstly, a bit more differential geometry

- Manifold with a boundary: M with boundary ∂M ($\dim n$, $\dim n-1$) is defined in the same way as a manifold without boundary, except that the coordinate charts $\{\vartheta_\alpha, \psi_\alpha\}$ now fall into two types. The interior of M is the set of points which have neighbourhoods homeomorphic to an open subset of $\mathbb{R}^n$ (e.g. an open n-ball), i.e. there are corresponding open subsets $\vartheta_\alpha \subset M$ with maps ψ_α to open subsets of $\mathbb{R}^n$. The interior of M is denoted $\mathrm{Int}\,M$. The boundary of M, denoted ∂M, is the complement of $\mathrm{Int}\,M$ in M. Every boundary point has a neighbourhood in M homeomorphic to the "half" n-ball $\{(x_1, x_2 \cdots, x_n) \mid \sum_{i=1}^{n} x_i^2 < 1 \text{ and } x_1 \geq 0\}$. The homeomorphism must send each point of ∂M to a point with $x_1 = 0$.

Definition: A time-orientable spacetime (M, g) is <u>asymptotically flat at null infinity</u> if there exists a spacetime $(\bar{M}, \bar{g})$ such that

mainly important: $\Lambda \neq 0$ on M

1. There exists a positive function Ω on M such that $(\bar{M}, \bar{g})$ is an extension of $(M, \Omega^2 g)$. Thus, if one regards M as a subset of $\bar{M}$, then $\bar{g} = \Omega^2 g$ on M.
2. Within $\bar{M}$, M can be extended to obtain a manifold with boundary $M \cup \partial M$.
3. Ω can be extended to a function on $\bar{M}$ such that $\Omega = 0$ and $d\Omega \neq 0$ on ∂M.
4. ∂M is the disjoint union of two components $\mathcal{I}^+$ and $\mathcal{I}^-$, each diffeomorphic to $\mathbb{R} \times S^2$.
5. No past (future) directed causal curve starting in M intersects $\mathcal{I}^+$ ($\mathcal{I}^-$).
6. $\mathcal{I}^\pm$ are complete. This means that the range of the u affine parameter on $\mathcal{I}^+$ is $(-\infty, \infty)$ and that of the v affine parameter on $\mathcal{I}^-$ is $(-\infty, \infty)$. This eliminates the possibility of $\mathcal{I}^+$ or $\mathcal{I}^-$ "ending" at some finite value of the corresponding affine parameter.

—

Conditions 1, 2 & 3 are just the requirement that there exist an appropriate conformal compactification. The requirement that $d\Omega \neq 0$ on ∂M ensures that Ω have a first order zero on ∂M. This is needed to

key point → ensure that the metric g_{ab} approaches the Minkowski metric sufficiently fast near $\mathcal{I}^\pm$.

Conditions 4, 5 & 6 ensure that infinity has the same structure as null infinity in Minkowski spacetime. Condition 5 ensures that $\mathcal{I}^+$ lies to the future of M and $\mathcal{I}^-$ lies to the past of M.

Example: Schwarzschild in outgoing EF coordinates (u, r, θ, ϕ), for which $\mathcal{I}^+$ corresponds to letting $r \to \infty$ with finite coordinate u.

Let $r = 1/x$ with $x > 0$. Then

$$ds^2 = -(1-2Mx)du^2 + 2\frac{dudx}{x^2} + \frac{1}{x^2}(d\theta^2 + \sin^2\theta d\phi^2)$$

Choose then a conformal factor $\Lambda = x$ to obtain the unphysical metric

$$d\bar{s}^2 = -x^2(1-2Mx)du^2 + 2dudx + d\theta^2 + \sin^2\theta d\phi^2 .$$

This metric can now smoothly be extended to $x = 0$, which is the surface $\mathcal{I}^+$. This surface is parametrized by (u, θ, ϕ) and is diffeomorphic to $\mathbb{R} \times S^2$ with a full range $-\infty < u < \infty$.

This verifies the conditions of the asymptotic flatness definition at $\mathcal{I}^+$. One can similarly verify these conditions at $\mathcal{I}^-$ using ingoing EF coordinates and the same conformal factor $\Lambda = 1/r$, noting that the coordinate r is the same in both ingoing and outgoing EF coordinates.

Thus, the Schwarzschild solution is asymptotically flat at $\mathcal{I}^\pm$ null infinities. The Kruskal spacetime is similarly asymptotically flat, in both regions I and IV.

A more detailed analysis shows that the metric ds^2 approaches the Minkowski metric near $\mathcal{I}^\pm$. This holds if (M, g) satisfies the vacuum Einstein equations or in spacetimes in which $T_{\mu\nu}$ decays sufficiently rapidly near $\mathcal{I}^\pm$.

The Killing symmetries of flat space carry over as asymptotic symmetries of asymptotically flat space: there are vectors of such a spacetime which asymptotically approach flat-space Killing vectors near i^0.

Bondi-Metzner-Sachs (BMS) symmetry group

Asymptotic symmetries as one approaches $\mathscr{I}^+$ are represented by a vector field V^μ in the physical spacetime (M, g) such that Killing's equation $\mathcal{L}_V g_{\mu\nu} = 0$ is satisfied to "as good an approximation as possible" as one approaches $\mathscr{I}^+$. For this purpose, one requires that V^μ, viewed as a vector in the unphysical (Weyl rescaled) spacetime have a smooth extension to $\mathscr{I}^+$. In addition, one requires that the tensor field $\Lambda^2 \mathcal{L}_V g_{\mu\nu}$ also have a smooth extension to $\mathscr{I}^+$ that vanishes on $\mathscr{I}^+$. Two vector fields V^μ and V'^μ on the physical spacetime satisfying these conditions are considered to generate the same infinitesimal asymptotic symmetry if their extensions to $\mathscr{I}^+$ are equal there.

The resulting asymptotic symmetry group is universal in the sense that one gets the same abstract group for all asymptotically flat spacetimes. What is surprising, however, is that this is not just the 10 parameter Poincaré group, but an infinite dimensional group known as the BMS group.

$u = t - r$
$v = t + r$

In Minkowski (3+1) dim spacetime with coordinates (u, v, θ, ϕ) we had $ds^2 = -du\,dv + \frac{(v-u)^2}{4}(d\theta^2 + \sin^2\theta\, d\phi^2)$. For an arbitrary function $f(\theta, \phi)$ of the angular coordinates,

$$V^\mu = f\left(\frac{\partial}{\partial u}\right)^\mu + \frac{v^2}{2r^2}\frac{\partial f}{\partial \theta}\left(\frac{\partial}{\partial \theta}\right)^\mu + \frac{v}{2r^2 \sin^2\theta}\frac{\partial f}{\partial \phi}\left(\frac{\partial}{\partial \phi}\right)^\mu$$

is itself nonvanishing at $\mathscr{I}^+$, but $\Lambda^2 \mathcal{L}_V g_{\mu\nu}$ vanishes at $\mathscr{I}^+$ for $\Lambda = 2\cos\tilde{u}\cos\tilde{v}$ as before, with $u = \tan\tilde{u}$, $v = \tan\tilde{v}$.

In a nonflat but asymptotically flat spacetime, one cannot in general find any vector field which satisfies Killing's equation near infinity to a better approximation than this.

This infinite dimensional family of "angle dependent translations" generates the BMS group.

Null hypersurfaces

Let $S(x)$ be a smooth function of the spacetime coordinates x^μ. Consider the family of hypersurfaces $S = \text{constant}$. Vector fields normal to such hypersurfaces are

$$l = \tilde{f}(x)(g^{\mu\nu}\partial_\nu S)\frac{\partial}{\partial x^\mu}$$ where $\tilde{f}$ is some arbitrary non-zero function.

If $l^2 = l^\mu l^\nu g_{\mu\nu} = 0$ for a hypersurface N in this family, then N is said to be a <u>null hypersurface</u>

EF: ingoing $ds^2 = -(1-\frac{2M}{r})dv^2 + 2dvdr + r^2 d\Omega^2$

<u>Eg.</u> Schwarzschild in ingoing EF coordinates (v, r, θ, ϕ) and the surface $S = r - 2M = 0$.

Then $$l = \tilde{f}(r)\left[(1-\frac{2M}{r})\frac{\partial S}{\partial r}\frac{\partial}{\partial r} + \frac{\partial S}{\partial r}\frac{\partial}{\partial v} + \frac{\partial S}{\partial v}\frac{\partial}{\partial r}\right]$$

$$= \tilde{f}(r)\left[(1-\frac{2M}{r})\frac{\partial}{\partial r} + \frac{\partial}{\partial v}\right]$$

Note: $\begin{pmatrix} -(1-\frac{2M}{r}) & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & (1-\frac{2M}{r}) \end{pmatrix}$ (v, r)

for which $l^2 = g^{\mu\nu}\partial_\mu S \partial_\nu S \tilde{f}^2 = g^{rr}\tilde{f}^2 = (1-\frac{2M}{r})\tilde{f}^2 = 0$ for $r = 2M$

so $r = 2M$ is a null hypersurface and the normal vectors to it are $l = \tilde{f}\frac{\partial}{\partial v}$.

Let N be a null hypersurface with normal l. A vector t which is tangent to N is one for which $t \cdot l = 0$. However, since N is a null hypersurface, $l \cdot l = 0$ as well. So the normal vector l is <u>itself a tangent vector</u>, i.e.

$$l^\mu = \frac{dx^\mu}{d\lambda} \text{ for some null curve } x^\mu(\lambda) \text{ in } N.$$

Moreover, this null curve is a <u>geodesic</u>. Recall that a geodesic is a curve $x^\mu(\lambda)$ for which $t^\mu = \dot{x}^\mu(\lambda)$ satisfies $D(\lambda)t^\mu = f(x)t^\mu$ for some $f(x)$, where for any vector field $V^\mu(x(\lambda))$ one has $D(\lambda)V^\mu(\lambda) = \frac{d}{d\lambda}V^\mu + t^\nu \Gamma^\mu_{\nu\rho}V^\rho = t^\rho \nabla_\rho V^\mu$.

To see that $x^\mu(\lambda)$ is a geodesic, let N be the $S = 0$ member of the $S = \text{const.}$ hypersurfaces. Then $l^\mu = \tilde{f}(x) g^{\mu\nu}\partial_\nu S$.

$l^\rho = \frac{dx^\rho}{d\lambda}$

$l_\rho = \tilde{f}\partial_\rho S$

Then $l^\rho \nabla_\rho l^\mu = (l^\rho \partial_\rho \tilde{f})\, g^{\mu\nu}\partial_\nu S + \tilde{f}\, g^{\mu\nu} l^\rho \nabla_\rho \partial_\nu S$ (Leibniz)

$= (l^\rho \partial_\rho \ln \tilde{f})\, l^\mu + \tilde{f}\, g^{\mu\nu} l^\rho \nabla_\nu \partial_\rho S$ $\quad \Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$ sym.

$= \left(\frac{d}{d\lambda}\ln\tilde{f}\right) l^\mu + l^\rho \tilde{f}\, \nabla^\mu (\tilde{f}^{-1} l_\rho)$

$= \left(\frac{d}{d\lambda}\ln\tilde{f}\right) l^\mu + l^\rho \nabla^\mu l_\rho - (\partial^\mu \ln\tilde{f})\, l^\rho l_\rho$

$= \left(\frac{d}{d\lambda}\ln\tilde{f}\right) l^\mu + \frac{1}{2}\partial^\mu (l^2) - (\partial^\mu \ln\tilde{f})\, l^2$

Now, N is by assumption a null hypersurface with normal l, so $l^2_{|N} = 0$. The quantity $\partial_\mu(l^2)$ is not in general vanishing, unless the whole family S = const. of hypersurfaces is null. However, one does have that $l^2 = 0$ is constant on N, so $t^\mu \partial_\mu (l^2) = 0$ for any vector t tangent to N. A vector V_μ satisfying $t^\mu V_\mu$ for any vector t^μ tangent to N must be proportional to the normal l_μ, i.e. for $V_\mu = \partial_\mu(l^2)$ one must have $\partial_\mu(l^2) = \alpha(x)\, l_\mu$ for some $\alpha(x)$.

Thus $l^\rho \nabla_\rho l^\mu_{|N} \propto l^\mu$ with $l^\mu = dx^\mu/d\lambda$ for the curve $x^\mu(\lambda)$. So $x^\mu(\lambda)$ is a geodesic with tangent vector l.

The function $\tilde{f}(x)$ can finally be chosen so that

$$D_{(\lambda)} l^\mu_{|N} = l^\rho \nabla_\rho l^\mu_{|N} = 0$$

in which case λ is an affine parameter.

Generators of a null hypersurface N

The null geodesics $x^\mu(\lambda)$ with affine parameter λ, for which the tangent vectors $l^\mu = dx^\mu/d\lambda$ are normal to a null hypersurface N, are called the generators of N.

Kruskal: $ds^2 = \frac{-32M^3}{r} e^{-r/2M}\, dU dV + r^2 d\Omega^2$

Ex. Let N be the $U = 0$ hypersurface in Kruskal spacetime. A normal vector to an $S = UV$ = constant hypersurface is

$$l = -\frac{\tilde{f} r}{16M^3} e^{r/2M} \partial_V \quad (\text{since } g^{VU} = -\tfrac{r}{16M^3} e^{r/2M} \text{ for Kruskal})$$

so $l_{|N} = \frac{-\tilde{f} e}{8M^2}\partial_V$ since $r = 2M$ on N.

Then since $g_{\mu\nu}$ is purely off-diagonal for $\mu,\nu = U, V$ in Kruskal coords and l has only an l^V component, one has $l^2 = l_\mu l^\rho g_{\mu\rho} = 0$.

This is also the case for any UV = constant hypersurface, not just $UV = 0$. Hence $\partial_\mu(l^2) = 0$ as well in this case.

Consequently $l^\rho \nabla_\rho l^\mu = 0$ provided $\tilde{f}$ is constant so $\partial_\rho \tilde{f} = 0$. Choose $\tilde{f} = -8M^2 e^{-1}$ so then $l = \partial_V$ is normal to $U=0$ and V is an affine parameter for the generator of the $U=0$ null hypersurface. The $U=0$ null hypersurface is denoted $\mathcal{H}^+$ and the $V=0$ null hypersurface is denoted $\mathcal{H}^-$, both at $r=2M$.

$l^\mu = l^V$ only Kruskal coords

- Killing horizons

A null hypersurface N is a Killing horizon of a Killing vector field ξ if $\xi|_N$ is normal to N

Let l be a normal vector to N such that for $l^\mu = dx^\mu/d\lambda$ $l^\rho \nabla_\rho l^\mu = 0$, i.e. λ is an affine parametrization of $x^\mu(\lambda)$. Then, as we have seen above $\xi = f l$ for some function $f(x)$, one must have $\xi^\rho \nabla_\rho \xi^\mu = \kappa \xi^\mu$ on N where $\kappa = \xi^\rho \partial_\rho \ln|f|$. The quantity κ is called surface gravity. Surface gravity takes its name from the fact that κ is constant over the horizon and it equals the force that an observer at infinity would have to exert to keep a unit mass at the horizon, eg. using a massless inelastic string.

Example: Schwarzschild in ingoing EF coordinates

We saw earlier using $S = r - 2M$ that the surface $S=0$ is a null hypersurface, $\mathcal{H}^+$, in ingoing EF coordinates.

The normal is $l^\mu = \tilde{f}(g^{vr}\partial_r S, g^{rr}\partial_r S, 0, 0) = \tilde{f}(1, (1-\frac{2M}{r}), 0, 0)$
$= \tilde{f}(1, 0, 0, 0) = \tilde{f}\partial_v$ at $r = 2M$;

$l^2 = l_\mu l_\nu g^{\mu\nu} = \tilde{f}^2 g^{rr} = \tilde{f}^2(1 - \frac{2M}{r}) = 0$ at $r = 2M$, so $\mathcal{H}^+$ is null.

Since $l = \tilde{f}\partial_v$ and $\xi = k = \partial_v$ is a Killing vector, the $\mathcal{H}^+$ null hypersurface is a Killing horizon.

Now evaluate the surface gravity κ on $\mathcal{H}^+$. Note the difference between the forms of the geodesic equations for l and ξ. With a proper choice of $\tilde{f}$, the integral curve $x^\mu(\lambda)$ with $\dot{x}^\mu = l^\mu$ is affinely parametrized, $l^\rho \nabla_\rho l^\mu = 0$.

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For the integral curves with tangent vector ξ^μ on the other hand, one has $\xi^\rho \nabla_\rho \xi^\mu = \kappa \xi^\mu$, so κ measures the failure of the integral curves of ξ^μ to be affinely parametrized.

$\xi^\mu = (1,0,0,0)$ IEF coords: $g_{vv} = -(1 - \frac{2M}{r})$ and $g^{vr} = 1$

For $\xi = \partial_v$, the $\mu = v$ component of $\xi^\rho \nabla_\rho \xi^\mu$ is

$$\xi^\rho \nabla_\rho \xi^v = \xi^\rho \partial_\rho \xi^v + \xi^\rho \Gamma^v_{\rho\sigma} \xi^\sigma = \Gamma^v_{vv}$$

and $\Gamma^v_{vv} = \frac{1}{2} g^{v\rho}(g_{v\rho,v} + g_{\rho v,v} - g_{vv,\rho}) = \frac{1}{2} g^{vr}(-g_{vv,r})$

$$= -\left(-\frac{M}{r^2}\right) = \frac{1}{4M} \quad \text{at } r = 2M.$$

So $\kappa = \frac{1}{4M}$.

Note that the normalization of ξ is important. If one had used a vector with a different normalization $\tilde{\xi} = c\xi$, one would then have $\tilde{\xi}^\rho \nabla_\rho \tilde{\xi}^\mu = (c\kappa)\tilde{\xi}^\mu$, giving $\tilde{\kappa} = c\kappa$. The normalization of ξ leading to $\kappa = \frac{1}{4M}$ in Schwarzschild spacetime is fixed by requiring that $\xi^2 = -1$ asymptotically as $r \to \infty$.

Event horizons and the definition of a Black Hole

eg for spacetimes without Killing symmetries

– A more general concept than that of Killing horizons –

We need to make some definitions

- Chronological future (past) of a point x^μ: $I^+(x)$ ($I^-(x)$): The set of points which can be reached from x^μ by a future (past) directed timelike curve.

- Causal future (past) of a point x^μ: $J^+(x)$ ($J^-(x)$): the set of points which can be reached from x^μ by a future (past) directed timelike or null (i.e. causal) curve.

So $I^\pm(x) \subset J^\pm(x)$

For a set $U \subset M$, define $I^\pm(U) = \{\bigcup_i I^\pm(x_i) \; \forall x_i \in U\}$

$J^\pm(U) = \{\bigcup_i J^\pm(x_i) \; \forall x_i \in U\}$

We then define $\overline{J}^+(U)$ to be the topological closure of $J^+(U)$, i.e. including limit points and similarly define $\overline{J}^-(U)$ to be the topological closure of $J^-(U)$.

Note
Set difference $B \setminus A := \{x \in B \text{ but } x \notin A\}$

- The boundary of $\overline{J}^+(U)$, denoted $\dot{J}^+(U)$ is defined to be $\dot{J}^+(U) := \overline{J}^+(U) \setminus J^+(U)$, and similarly the boundary $\dot{J}^-(U)$ is defined to be $\dot{J}^-(U) := \overline{J}^-(U) \setminus J^-(U)$.

We are now in a position to define the future event horizon of an asymptotically flat (at null infinity) spacetime M $\mathcal{H}^+ = \dot{J}^-(\mathcal{I}^+)$ (we will say this more precisely below), i.e. the boundary of the closure of the causal past of $\mathcal{I}^+$.

Note that, strictly speaking, $\mathcal{I}^+$ is a set of points which is not part of the original spacetime manifold (M, g). However, $\mathcal{I}^+$ is contained in the conformal compactification $(\tilde{M}, \tilde{g})$. Since causal structure is preserved under Weyl (conformal) transformations, the set of points $\mathcal{H}^+$ obtained by taking the boundary of the causal past of $\mathcal{I}^+$ will be contained in the original spacetime manifold (M, g).

The set of points of M that can send a signal to $\mathcal{I}^+$ is $M \cap J^-(\mathcal{I}^+)$.

- The Black Hole region $B = M \setminus [M \cap J^-(\mathcal{I}^+)]$.

The future event horizon is $\mathcal{H}^+ = \dot{B}$ (the boundary of B in M). Equivalently, $\mathcal{H}^+ = M \cap \dot{J}^-(\mathcal{I}^+)$ (expressed more precisely).

Similarly

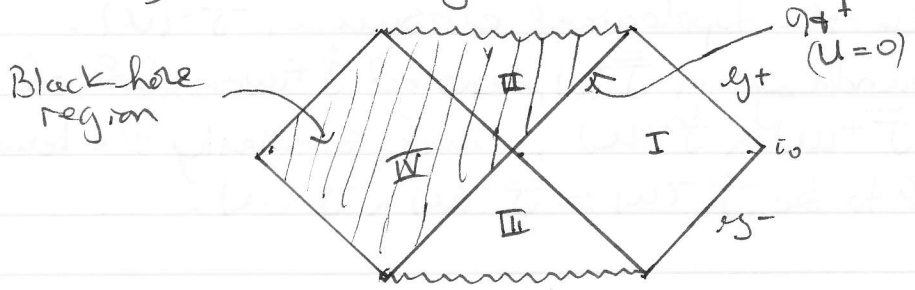
- The White Hole region $W = M \setminus [M \cap J^+(\mathcal{I}^-)]$.

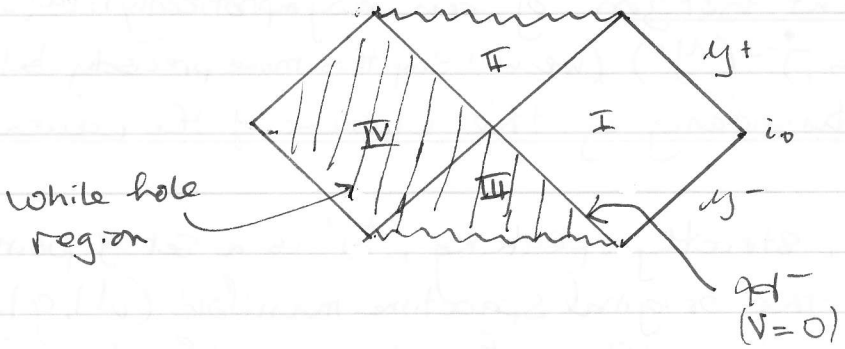
The past event horizon is $\mathcal{H}^- = \dot{W} = M \cap \dot{J}^+(\mathcal{I}^-)$.

In Kruskal spacetime, no causal curve from region II or IV can reach $\mathcal{I}^+$, so B is the union of regions II and IV, including the boundary $U = 0$ where $r = 2M$. $\mathcal{H}^+$ is the surface $U = 0$.

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Similarly, W is the union of regions III and IV, including the boundary $V=0$. $\mathcal{H}^+$ is the surface $V=0$.





Properties of the Future Event Horizon $\mathcal{H}^+$

i) i_0 and $\mathcal{I}^-$ are contained in $J^-(\mathcal{I}^+)$, so they are not part of $\mathcal{H}^+$.

ii) $\mathcal{H}^+$ is a null hypersurface.

iii) No two points of $\mathcal{H}^+$ are timelike separated. For nearby points, this follows from property ii), but this is also true globally.

iv) The null geodesic generators of $\mathcal{H}^+$ may have past endpoints, ie where the continuation of the geodesic further into the past is no longer in $\mathcal{H}^+$. This can be, for example, the point in a collapsing star spacetime at which a black hole forms.

v) If a generator of $\mathcal{H}^+$ had a future endpoint, on the other hand, the future continuation of that null geodesic would leave $\mathcal{H}^+$ beyond that endpoint. However, there is a Theorem of Roger Penrose:

The generators of $\mathcal{H}^+$ have no future endpoints.

Singularities

A metric is singular in some basis if its components are not smooth or if its determinant vanishes. Some singularities are just artifacts of a particular choice of coordinates; these are called coordinate singularities and can be eliminated by a change of coordinates. The $r = 2M$ coordinate singularity in Schwarzschild coordinates is a classic example.

Curvature singularities are genuine defects in a spacetime. One needs, however, to develop a way to separate them from problems with a particular coordinate choice. One rather thorough approach is to introduce vierbeins, i.e. orthonormal frames e_μ^a, relating coordinate frame indices to tangent-space indices, with the metric given by $g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}$ (where η_{ab} is the ordinary Minkowski metric). Vierbeins transform as vectors under general coordinate transformations on their world (greek) indices: $e_{\mu'}^{\ a}(x') = \frac{\partial x^\rho}{\partial x^{\mu'}} e_\rho^{\ a}(x)$. On their tangent space (roman) indices, they transform under local Lorentz transformations: $e_\mu^{\ a'}(x) = \Lambda^{a'}_{\ b}(x)\, e_\mu^{\ b}(x)$, where $\Lambda^{a'}_{\ b} \in SO(3,1)$.

In the vierbein formalism, covariant derivatives are constructed using the Levi-Civita spin connection $\omega_\mu^{\ ab}$, which is the analogue of the Christoffel connection in the pure metric formalism. The covariant derivative of a local Lorentz vector field $V_a(x)$ is $(\nabla_\mu V)_a = \partial_\mu V + \omega_\mu{}_a^{\ b} V_b$, where the raised b index is raised simply with η^{ab}.

Vierbeins and spin connections naturally can be expressed as 1-forms: $e^a = e_\mu^{\ a} dx^\mu$, $\omega^{ab} = \omega_\mu^{\ ab} dx^\mu$. The analogue of the metricity condition $\nabla_\rho g_{\mu\nu} = 0$ which determines $\Gamma^\rho_{\mu\nu}$ algebraically is the condition of vanishing torsion: $de^a + \omega^a_{\ b} \wedge e^b = 0$, which determines $\omega_\mu^{\ ab}$ algebraically. The curvature 2-form is then $R^{ab} = d\omega^{ab} + \omega^{ac} \wedge \omega_c^{\ b}$.

Written out in mixed world + tangent space coordinates, the curvature is $R_{\mu\nu}{}^{ab}$. One may also change all its indices to tangent-space form:

$R_{abcd} = e^{\mu}{}_a e^{\nu}{}_b R_{\mu\nu}{}^{ef} \eta_{ec}\eta_{fd}$, where $e^{\mu}{}_a$ is the matrix inverse of $e_{\mu}{}^a$: $e^{\mu}{}_a e_{\mu}{}^b = \delta_a{}^b$; $e^{\mu}{}_a e_{\nu}{}^a = \delta^{\mu}{}_{\nu}$.

This tangent-space form of the curvature is a general coordinate scalar: $R_{abcd}(x') = R_{abcd}(x)$. It is also a local Lorentz tensor. Singularities arising from unfortunate world coordinate choices are eliminated, however. Most types of spacetime singularity are revealed by studying R_{abcd}.

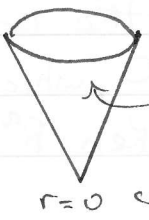
One type of singularity which can be tricky to reveal is a <u>conical singularity</u>. Consider the manifold $M = \mathbb{R}^2$ with a metric written in polar coordinates (r, ϕ) where $\phi \sim \phi + 2\pi$: $ds^2 = dr^2 + \lambda r^2 d\phi^2$ $0 < r < \infty$, $0 \le \phi < 2\pi$.

If $\lambda = 1$, one can just convert back to Cartesian coordinates $ds^2 = dx^2 + dy^2$ to show that $r = 0$ is just a coordinate singularity, ie. the metric can smoothly be extended to $r = 0$.

The case where $\lambda \neq 1$ is different, however. Let $\phi' = \lambda\phi$ yielding $ds^2 = dr^2 + r^2 d(\phi')^2$, which is clearly <u>locally</u> isometric to flat Euclidean space. However, there is still a singularity and the space is <u>not globally</u> isometric to Euclidean $\mathbb{R}^2$, because the period of ϕ' is now $2\pi\lambda$. Consider a small circle of radius $r = \epsilon$ surrounding the point $r = 0$. For this circle, $\frac{\text{circumference}}{\text{radius}} = \frac{2\pi\lambda\epsilon}{\epsilon} = 2\pi\lambda$. This does not tend to zero as $\epsilon \to 0$.

Any smooth Riemannian manifold must be locally flat, i.e. one must recover the structure of Euclidean geometry on sufficiently small scales. This is not true for the above example, which is an example of a <u>conical singularity</u>.

Eg.
$\lambda < 1$



$2\pi(1-\lambda)$: deficit angle

We need an appropriate definition of "singularity" more generally.

• Geodesic completeness

We first need a clear notion of endpoints of a curve. A curve is a smooth map from an open interval $(a,b) \subset \mathbb{R}$ to M: $\gamma:(a,b)\to M$. Then a point $p \in M$ is a future endpoint of a future-directed causal curve γ if, for any neighborhood $\mathcal{O}$ of p, there exists t_0 such that $\gamma(t) \in \mathcal{O}$ for all $t > t_0$ (i.e. the curve "stops" at p).

A curve γ is future-inextendible if it has no future endpoint. It goes similarly for past endpoints and past-inextendible. A curve γ is inextendible if it is both future and past inextendible.

Eg. Minkowski spacetime (M,g). Let $\gamma:(-\infty,0)\to M$ with $\gamma(t) = (t,0,0,0)$. Then $(0,0,0,0)$ is a future endpoint of γ. On the other hand if one takes (M,g) to be Minkowski spacetime with the origin deleted, then γ is future-inextendible.

Then: a geodesic is complete if an affine parameter for it extends to the full range $(-\infty,\infty)$.

A spacetime is geodesically complete if all inextendible causal geodesics are complete.

Examples: $\mathbb{M}^4$ - Minkowski spacetime is geodesically complete

$\mathbb{M}^4 \setminus$ (point) is geodesically incomplete

a cone is geodesically incomplete

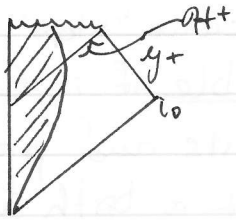
Kruskal spacetime is geodesically incomplete because there exist geodesics that tend towards $r=0$ at finite affine parameter and so cannot be extended to infinite affine parameter.

A spacetime that is itself extendible will be geodesically incomplete, but that is just because one is not considering the whole spacetime. One can regard a spacetime as singular if it is geodesically incomplete and inextendible. Kruskal spacetime is thus singular according to this definition.

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Clothed versus naked singularities

From the above definition, Schwarzschild spacetime is singular. In the portion of the CP diagram relevant to gravitational collapse, however — regions I and II — the $r=0$ singularity is "clothed" by the $\mathcal{H}^+$ horizon: no signal from $r=0$ can reach $\mathcal{I}^+$. This is equally true when one covers part of the CP diagram corresponding to the region where $T_{\mu\nu} \neq 0$, inside a collapsing star.



$r=0$ singularity clothed by the $\mathcal{H}^+$ horizon

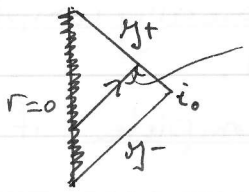
If one searches for solutions with naked singularities not clothed by a horizon, it is easy to find one:

$M<0$ Schwarzschild

$$ds^2 = -\left(1 + \frac{2|M|}{r}\right)dt^2 + \frac{1}{\left(1+\frac{2|M|}{r}\right)}dr^2 + r^2 d\Omega^2$$

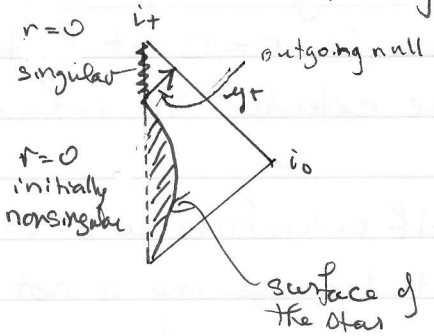
For all $r>0$, this solves Einstein's equations, just as well as the $M>0$ solution. Here is its CP diagram:

$M<0$



outgoing null geodesic from $r=0$ to $\mathcal{I}^+$: a signal from $r=0$ can reach $\mathcal{I}^+$

Since such "vacuum" (i.e. with $T_{\mu\nu}=0$) solutions to Einstein's equations exist, one must confront the question of their relevance in physically achievable situations. Particularly bad would be spacetimes with a CP diagram like



At late times in such a spacetime, the solution would be $M<0$ Schwarzschild with a naked singularity, but at earlier times the spacetime would be non-singular, with $r=0$ just being the center of the $T_{\mu\nu}\neq 0$ matter distribution.

However, this $M<0$ possibility is ruled out by the Positive Mass Theorem.

Positive mass Theorem

R. Schoen & S-T Yau, CMP 65 (1):45-76 (1979);
CMP 79(2): 231-260 (1981)

- Dominant energy condition:
  Let a^μ be any causal 4-vector, ie
  $g_{\mu\nu} a^\mu a^\nu \leq 0$ and $a^0 > 0$ (So a^μ is either timelike or lightlike).
  Then the dominant energy condition requires the stress tensor $T^{\mu\nu}$ to satisfy the condition that
  $b^\mu = -T^{\mu\nu} a_\nu$ is also a causal 4-vector.

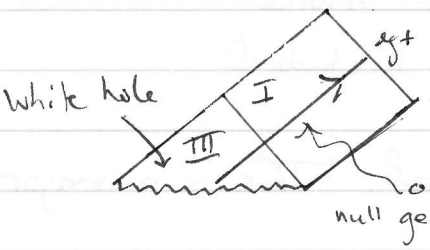
$a_\nu = g_{\nu\rho} a^\rho$

The dominant energy condition says that taking the scalar product of $T^{\mu\nu}$ with a causal vector cannot produce a spacelike (non-causal) vector.

The positive mass theorem states that for an isolated asymptotically flat physical system with $T^{\mu\nu}$ satisfying the dominant energy condition, the total energy, including contributions from both matter and gravitation, is non-negative. Moreover, the energy is zero only for Minkowski spacetime.

A subsequent simpler proof was given using spinors, motivated by the supersymmetry algebra in supergravity:
E. Witten, CMP 80 (3): 381-402 (1981)

Another context in which a naked singularity occurs is in the white hole part of Kruskal spacetime



Although this might be relevant for singular cosmology, it is not relevant for gravitational collapse.

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Cosmic Censorship conjecture (Penrose 1969)

There are two versions of this conjecture: weak and strong, both concerned with the global geometry of spacetime.

- Weak cosmic censorship:
  The assertion that there can be no singularity visible from $\mathcal{I}^+$. In other words, singularities need to be hidden from an observer at infinity by an event horizon.

- Strong cosmic censorship:
  The occurrence of a naked singularity formed in some kind of gravitational collapse would lead to an inability to predict fully the future from nonsingular data given on an initial spacelike hypersurface (ie. a Cauchy initial surface). The strong cosmic censorship hypothesis asserts that this cannot happen, i.e. that the (classical) fate of all observers should be predictable from such initial data.

impossible to predict the behaviour of spacetime in the causal future of a singularity

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The two versions of the cosmic censorship conjecture are mathematically independent, and both are problematic. There exist spacetimes for which the weak conjecture is valid but the strong one is violated, and, conversely, there exist spacetimes for which the weak conjecture is violated, but the strong one is valid.

So a key problem is how to formulate the conjecture properly.

1) It is not difficult to construct counterexamples, but these may be considered "non-generic" or "trivial"
2) Cosmological singularities are excluded

There is, in any case, no proof. This is a major unsolved problem in classical GR.

a) The equivalence of inertial mass M and total energy (Special relativity)

b) The equivalence of inertial mass and the total energy, including gravitational self-interaction (general relativity).

Consider then the total energy required to assemble a sphere of rest mass M_0 carrying a charge Q in geometrised units, bringing this matter shell in from infinite radius to radius r:

$$M_{total} = M_0 + GQ^2/r - GM_{total}^2/r$$

total energy = rest mass energy + Coulomb energy + gravitational self-interaction energy

This is a quadratic equation for M_{total}, with solutions

$$M_{total}(r) = \frac{1}{2G}\left[-r \pm \left(r^2 + 4M_0Gr + 4G^2Q^2\right)^{1/2}\right]$$

choose + soln.

The solution that tends to M_0 as $r \to \infty$ is the + solution.

Now consider whether such a shell will undergo gravitational collapse. This can only happen if $M_{total}(r)$ decreases with decreasing r, so that the kinetic energy can increase. From the above expression for M_{total}, find

require $\frac{dM_{tot}}{dr} > 0$

$$\frac{dM_{total}}{dr} = \frac{M_0 - M}{2GM + r} = \frac{G(M^2 - Q^2)}{(2GMr + r^2)}$$

Consequently, gravitational collapse only occurs if $M > |Q|$ in this simplified analysis.

Note another aspect of this model of spherical charged shell assembly: as $r \to 0$, $M_{total} \to |Q|$, independent of M_0

The above argument is simplified by its "enhanced Newtonian" framework, but, in fact, the same results are obtained from a full treatment in general relativity.

See: R. Arnowitt, S. Deser & C.W. Misner, "The Dynamics of General Relativity" in L. Witten, ed. "Gravitation, an introduction to current research" (Wiley, New York, 1962); also now gr-qc 0405109.

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One might have worried that, owing to the negative gravitational self-interaction ($-GM^2/r$) the total mass could become less than $|Q|$, or, in the $Q=0$ case, less than zero. However, one sees that, at least in this simple spherically symmetrical enhanced-Newtonian analysis, this cannot happen. Moreover, GR in this sense resolves the infinite self-energy problem of point particles in classical electromagnetism — this $r \to 0$ model of a point particle becomes an extreme $M = |Q|$ RN black hole.

What about electrons? An electron has charge $e = 1.6 \times 10^{-19}$ Coulombs and mass $m_e = 9.1 \times 10^{-31}$ kg. In geometrised units, this gives $Q = e/(4\pi\epsilon_0 G)^{1/2} = 1.4 \times 10^{-29}$ kg. So for an electron, $Q \gg M$, clearly in the $M < |Q|$ range. Does this mean that an electron must display a naked singularity? No! in quantum mechanics, the electron wavefunction spreads out over a volume with Compton wavelength scale $\lambda = \hbar/mc = 2.4 \times 10^{-12}$ m, which is much larger than its Schwarzschild radius $r_s = 2Gm_e/c^2 = 1.4 \times 10^{-57}$ m.

- Case ii) $M > |Q|$ Sub-extremal

In this case, Δ has two real roots $r_+ > r_-$. At both of these, there are coordinate singularities. We can repeat the strategy used for removing such coordinate singularities from the Schwarzschild metric, first defining a tortoise coordinate r_* by setting

$$\frac{\Delta}{r^2} dr_*^2 = \frac{r^2}{\Delta} dr^2 \quad , \text{ after which the metric}$$

takes the form $ds^2 = -\frac{\Delta}{r^2}(dt^2 - dr_*^2) + r^2 d\Omega^2$

where one solves $dr_* = \frac{r^2 dr}{\Delta}$ in a fashion similar to the Schwarzschild case, making an appropriate choice of integration constant to give $r_* = \frac{1}{2\kappa_+} \ln\left(\frac{|r - r_+|}{r_+}\right) + \frac{1}{2\kappa_-} \ln\left(\frac{|r - r_-|}{r_-}\right)$

where $\kappa_+ = \frac{(r_+ - r_-)}{2r_+^2} > 0$ and $\kappa_- = \frac{r_- - r_+}{2r_-^2} < 0$.

Einstein-Maxwell Theory: Reissner-Nordstrom Black Holes

From the Einstein-Maxwell action

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$I_{EM} = \int d^4x \sqrt{-g} R - \frac{1}{4} \int d^4x \sqrt{-g} F_{\mu\nu} F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma}$$

one derives the field equations

$$-R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R + \frac{1}{2} \left(F_{\mu\rho} F_{\nu\sigma} g^{\rho\sigma} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right) = 0$$

i.e. $G_{\mu\nu} = T_{\mu\nu}$ where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$

and $T_{\mu\nu} = \frac{1}{2} \left(F_{\mu\rho} F_{\nu}{}^{\rho} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right)$

and $\nabla^\mu F_{\mu\nu} = 0$.

Spherically symmetric solutions of this system may be obtained by trying the metric ansatz

$\tilde{A} = A + Br^2$ from before

$$ds^2 = -D(r,t) dt^2 + A(r,t) dr^2 + r^2 d\Omega^2$$ together with

$A_0 = C(r,t)$ $A_r = 0$ and then proceeding to a general analysis of these equations. Alternatively, one can exploit the expedited treatment of inserting an ansatz (including a g_{rt} component in order not to loose a time dependence equation) as we discussed early in the course.

The resulting solution family is (Note: t dependence has gone from this solution, a generalization of Birkhoff.)

$$ds^2 = -a(r) dt^2 + \frac{dr^2}{a(r)} + r^2 d\Omega^2$$

and

$$A_0 = -\frac{q}{r}$$

with $a(r) = 1 - \frac{2M}{r} + \frac{q^2}{4r^2}$, where M and q are parameters of this family of solutions.

The RN metric can then be written

$$ds^2 = -\frac{\Delta}{r^2} dt^2 + \frac{r^2}{\Delta} dr^2 + r^2 d\Omega^2$$

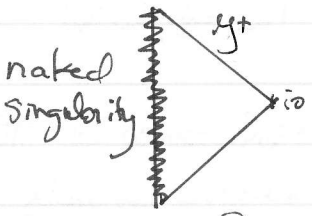
where $\Delta = r^2 - 2Mr + \frac{q^2}{4} = (r - r_+)(r - r_-)$

where $r_\pm = M \pm (M^2 - q^2/4)^{1/2}$.

As for pure Schwarzschild, there is a genuine curvature singularity at $r = 0$, which one can verify by calculating Riemann in tangent space components. Similarly, $F_{ab} = e_a^\mu e_b^\nu F_{\mu\nu}$ is singular at $r = 0$. That is of course familiar from flat-space Maxwell theory.

For the RN solution, there are 3 parametric ranges to consider.

- Case i) $M < |Q/2|$. This case is similar to $M<0$ pure Schwarzschild: Δ has no real $r_\pm$ roots and the singularity at $r=0$ is naked. According to the cosmic censorship hypothesis, this case should not occur as the endpoint solution of a gravitational collapse. The CP diagram for the RN solution in this case is the same as for $M<0$ Schwarzschild:



Note on conventions: So far, we have suppressed the Newton's constant prefactor of the gravitational action. Now that we are including matter in Maxwell's theory, we should have restored the prefactor $\frac{1}{16\pi G}$. In discussing the RN solution, relativists conventionally write the action

$$\frac{I}{conv} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,(R - F_{\mu\nu}F^{\mu\nu}) \qquad \text{geometrised units of charge}$$

with the rescaling $A_\mu^{(\text{Particle Physics})} = 2A_\mu^{(\text{grav})}$ converting the $-\frac{1}{4}$ coefficient of the Maxwell action to -1, the charge becomes $Q = 2q$ and the RN solution becomes

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}dr^2 + r^2 d\Omega^2$$

$$A_0 = -\frac{Q}{r}. \qquad \Delta \text{ is now } \Delta = r^2 - 2Mr + Q^2.$$

- Can such a $M < |Q|$ solution arise in gravitational collapse?

Using the geometrised units of charge, the Coulomb force between two charges Q_1 and Q_2 separated by a distance r in flat spacetime is $F = G|Q_1Q_2|/r^2$. Similarly, the Coulomb energy, again in flat spacetime, required to assemble a spherical shell of charge Q in from infinite radius to radius r is GQ^2/r.

One can accordingly make a simple argument in Newtonian gravity enhanced by two features of relativistic theories:

One then defines radial null coordinates $u = t - r_*$; $v = t + r_*$.
Ingoing Eddington-Finkelstein coordinates (IEF) are (v, r, θ, ϕ); the metric then becomes in IEF coordinates

$$ds^2 = -\frac{\Delta}{r^2} dv^2 + 2 dr dv + r^2 d\Omega^2 .$$

$\begin{pmatrix} -\frac{\Delta}{r^2} & 1 \\ 1 & 0 \end{pmatrix}^{-1} =$ This is now regular for all $r > 0$, including $r = r_+$ and $r = r_-$.

$\begin{pmatrix} 0 & 1 \\ 1 & \frac{\Delta}{r^2} \end{pmatrix}$ Hypersurfaces of constant r are null when $g^{rr} = \frac{\Delta}{r^2} = 0$, i.e. when $\Delta = 0$. Thus $r = r_+$ and $r = r_-$ are null hypersurfaces $N_\pm$.

Killing horizons $N_\pm$

In the original (t, r, θ, ϕ) "static" coordinates, the metric $g_{\mu\nu}$ is t-independent so $k = \partial_t$ is a Killing vector. In IEF coordinates, $k = 2\partial_v$ (similarly obvious, since $g_{\mu\nu}$ is v independent). Noting that $g^{rr} = 0$ on $N_\pm$ and $g^{vr} = 1$, one has normals $l_\pm$ to $N_\pm$:

$l = f g^{\mu\rho} \partial_\rho S \partial_\mu$ with $S = r$

$$l_\pm = f_\pm (g^{rr}\partial_r + g^{vr}\partial_v)|_{N_\pm} = f_\pm \partial_v$$

where $f_\pm$ are some functions to be chosen such that the integral curves to $l_\pm$ are affinely parametrised, i.e. $l_\pm \cdot D l_\pm^\mu = 0$. Thus $k = 2\partial_v = f_\pm^{-1} l_\pm$ so the normals to $N_\pm$ are (proportional to) Killing vectors and consequently $N_\pm$ are Killing horizons.

- Surface gravities

The integral curves of $k = 2\partial_v = f_\pm^{-1} l_\pm$ are not themselves affinely parametrised. One has

$$((k \cdot D)k)^\mu = k^\rho \partial_\rho k^\mu + k^\rho \Gamma^\mu_{\rho\sigma} k^\sigma \quad ; \text{ only } k^v = 1 \text{ is nonzero}$$

So $(k \cdot Dk)^r = 1 \cdot \Gamma^r_{vv} \cdot 1 = -\frac{1}{2} g^{rr} g_{vv,r} = 0$ at $r = r_\pm$, where $g^{rr} = 0$

and $(k \cdot Dk)^v|_{r=r_\pm} = \Gamma^v_{vv} = -\frac{1}{2} g^{vr} g_{vv,r} = \frac{1}{2}\partial_r\left(\frac{\Delta}{r^2}\right)_{|r=r_\pm} = \left(\frac{1}{2r^2}\partial_r \Delta\right)_{|r=r_\pm}$

recall that $\Delta = (r - r_+)(r - r_-)$ so $\partial_r \Delta|_{r=r_\pm} = (r_\pm - r_\mp)$

thus $(k \cdot Dk)^v_{|r=r_\pm} = \frac{1}{2r_\pm^2}(r_\pm - r_\mp) = \kappa_\pm$ on $N_\pm$

So $(k \cdot Dk)^\mu = \kappa_\pm k^\mu$ is the geodesic equation for the integral curves of k^μ.

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Then, noting that $k = \partial_t$ in static coordinates, one has normalization $k^2 \to -1$ as $r \to \infty$, so $\kappa_\pm$ are identified as the surface gravities of the Killing horizons $N_\pm$.

Thus, using the IEF coordinates (v, r, θ, ϕ) we have been able to smoothly extend coverage of the RN spacetime to all $r > 0$, including the region $0 < r < r_+$. As in the Schwarzschild case, however, we have not yet found the maximal extension. One can also use Outgoing EF coordinates (OEF) (u, r, θ, ϕ). For $r > r_+$ the metric in OEF coordinates becomes

$$ds^2 = -\frac{\Delta}{r^2}\,du^2 - 2\,du\,dr + r^2 d\Omega^2 .$$

One can extend the manifold now to $0 < r < r_+$, but this is a <u>different</u> region to that accessed using the IEF coordinates.

As in the Schwarzschild case, one can better explore the extensions using KS type coordinates. For the RN spacetime, however, we have two Killing horizons $N_\pm$ and we can introduce KS type coordinates in the neighborhood of each of these:

$$U^\pm = -e^{-\kappa_\pm u}, \qquad V^\pm = e^{\kappa_\pm v} .$$

Near N_+, using (U^+, V^+, θ, ϕ) one has a smooth metric for $r > r_-$ but still with a coordinate singularity in these coordinates at $r = r_-$

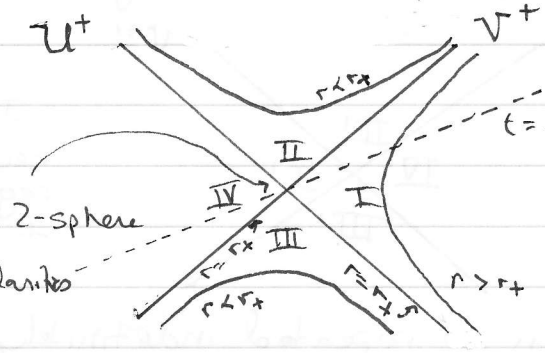
$$ds^2 = -\frac{r_+ r_-}{\kappa_+^2}\frac{e^{-2\kappa_+ r}}{r^2}\left(\frac{r - r_-}{r_-}\right)^{\left(1 + \frac{\kappa_+}{|\kappa_-|}\right)} dU^+ dV^+ + r^2 d\Omega^2$$

where $r(U^+, V^+)$ is determined implicitly by

$$U^+ V^+ = -e^{2\kappa_+ r}\left(\frac{r - r_+}{r_+}\right)\left(\frac{r_-}{r - r_-}\right)^{\kappa_+/|\kappa_-|}$$

Note that $U^+ V^+$ becomes complex for $r < r_-$. Initially, we had $U^+ < 0$ and $V^+ > 0$ for the $r > r_+$ region, but now we can extend to $-\infty < U^+ < \infty$ and $-\infty < V^+ < 0$ to cover down to $r > r_-$.

So far, we have explored the spacetime regions accessible using (U^+, V^+, θ, ϕ) KS-type coordinates, centered on the $U^+ = V^+ = 0$ bifurcation 2-sphere. This covers four regions



Note: no singularities in II + III since $r(U^+, V^+) > r_-$

t = constant subsurfaces are Einstein-Rosen bridges between regions I and IV.

In U^+V^+ coordinates, the coordinate singularity at $r = r_-$ becomes $U^+V^+ = -\infty$. However, we know from the Eddington-Finkelstein type coordinates that radial null geodesics can reach $r = r_-$ in finite affine parameter. So we haven't yet accessed the maximal extension. This is where the (U^-, V^-, θ, ϕ) coordinates are needed. Using them, one has a similar set of four regions for the metric

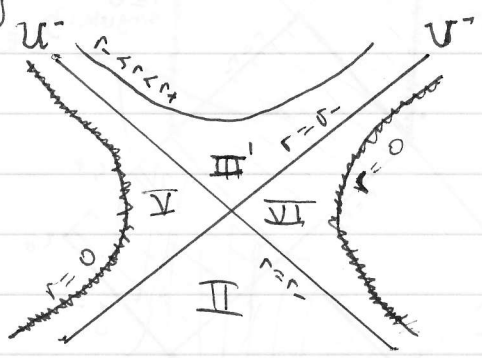
Know from IEF that one can extend to $r < r_-$

$$ds^2 = -\frac{r_+ r_-}{k^2 r^2} e^{2|k_-| r} \left(\frac{r_+ - r}{r_+}\right)^{\left(1 + \frac{|k_-|}{k_+}\right)} dU^- dV^- + r^2 d\Omega^2$$

with $r(U^-, V^-) < r_+$ determined implicitly by

$$U^- V^- = e^{-2|k_-| r} \left(\frac{r - r_-}{r_-}\right)\left(\frac{r_+}{r_+ - r}\right)^{|k_-|/k_+}.$$

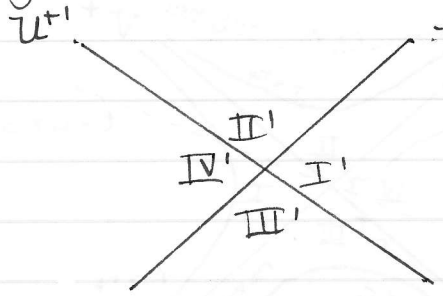
Initially defined in region II with $U^- < 0$ and $V^- < 0$, corresponding to $r_- < r < r_+$, this form of the metric can be analytically continued to $U^- > 0$ or $V^- > 0$:



$U^- V^- = -1$ corresponds to $r = 0$: timelike curvature singularity

There are two new regions V and VI here for which $0 < r < r_-$. Region III' is isometric to region II under the discrete $t \to -t$ time reversal isometry.

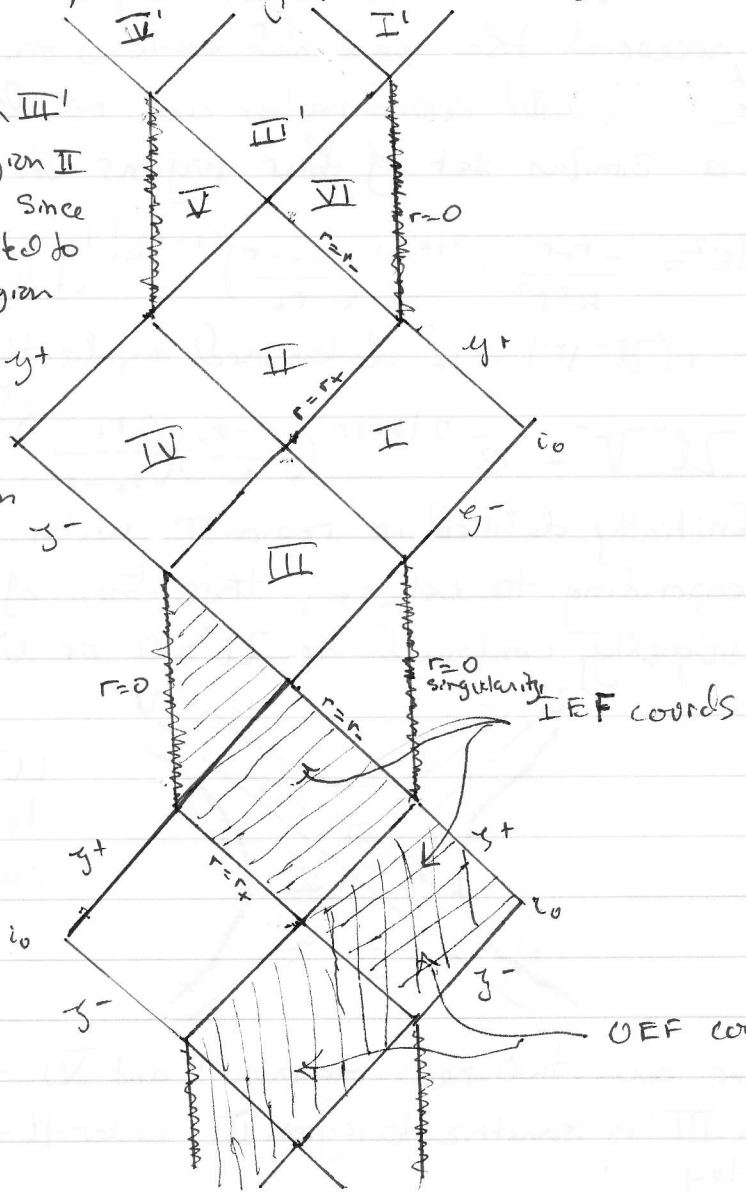
From region III', one can introduce yet more new coordinates $(U^{+\prime}, V^{+\prime}, \theta, \phi)$ which can be analytically continued into the future giving further new regions I' II' and IV'



I' and IV' are new asymptotically flat regions isometric to I and IV

This process can be repeated indefinitely ∴ the maximally extended RN solution comprises infinitely many regions. The corresponding CP diagram extends forwards and backwards in time indefinitely. By an appropriate choice of conformal factor, the singularities can be made into straight lines.

Note that region III' is related to region II by time reversal. Since region II is connected to I, IV and III, region III' must similarly be connected to similar regions to the future. So the CP diagram extends indefinitely.



Cauchy horizons

The timelike nature of the $r=0$ singularity in RN spacetime means that one does not necessarily have to fall into it after crossing a Killing horizon. One could imagine starting in region I, crossing the $r=r_+$ horizon into region II then crossing the $r=r_-$ horizon into region VI (or V) then accelerating to avoid the $r=0$ singularity there, then into region III' (white hole) and then into region I', and so on forward in time. Is there something wrong with this?

The problem is that there is no way to ensure in advance that the desired trajectory will be achieved. In order to do so, one would have to cross a Cauchy horizon between regions II and V or VI.

Some definitions:

- A partial Cauchy surface Σ for a timelike orientable spacetime (M,g) is a hypersurface which no causal curve intersects more than once, ie no two points on Σ can be connected by a causal curve in M.

- A causal curve is past-inextendable if it has no past endpoint in M.

- The future domain of dependence of Σ, denoted $D^+(\Sigma)$ is the set of points $p \in M$ for which every past-inextendable causal curve through p intersects Σ.
  - The significance of $D^+(\Sigma)$ is that the behaviour of solutions to partial differential equations outside $D^+(\Sigma)$ is not determined by initial data on Σ.

- The past domain of dependence of Σ, denoted $D^-(\Sigma)$, is defined similarly: the set of points $p \in M$ for which every future-inextendable causal curve through p intersects Σ.

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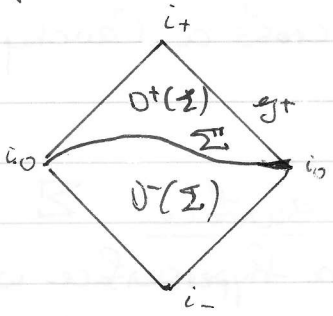
- The <u>domain of dependence</u> $D(\Sigma) = D^+(\Sigma) \cup D^-(\Sigma)$.
- <u>Cauchy Surface</u> : A partial Cauchy Surface Σ with $D(\Sigma) = M$ is a Cauchy Surface.

- If M has a Cauchy Surface, it is said to be <u>globally hyperbolic</u>.

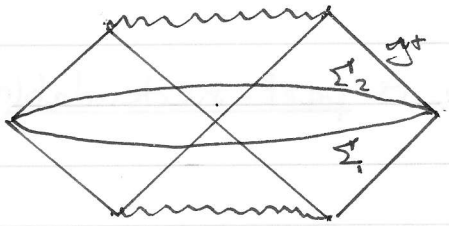
$D(\Sigma)$ is the complete region of spacetime where one can determine the solutions of partial differential equations uniquely from data on Σ.

Some examples of globally hyperbolic spacetimes:

1) 1+1 Minkowski spacetime

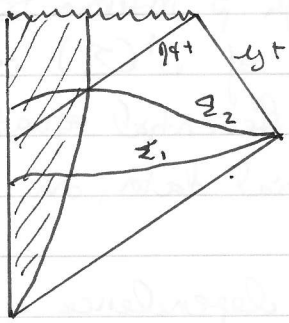


2) 3+1 Kruskal



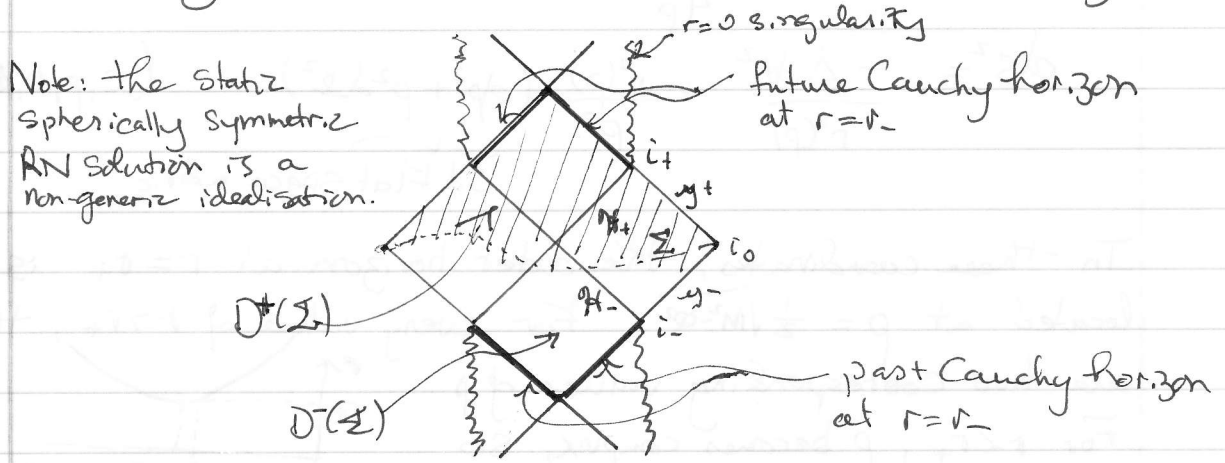
Σ_1 and Σ_2 are both Cauchy Surfaces

3) Spherical collapse of a matter distribution without charge



If a spacetime (M, g) is not globally hyperbolic, then $D^+(\Sigma)$ or $D^-(\Sigma)$ will have a boundary in M, called the future or past Cauchy horizon.

- The maximal analytic extension of RN spacetime is an example of a spacetime with future and past Cauchy horizons



The existence of the Cauchy horizons makes most of the infinitely extended CP diagram unphysical. One should take seriously only the part of the CP diagram corresponding to $D(\Sigma) = D^+(\Sigma) \cap D^-(\Sigma)$ since only this region has PDE solutions that are uniquely determined by initial data on Σ. future + past

In order to obtain the infinitely extended CP diagram, one has to assume analyticity and spherical symmetry. If one just assumes a smooth spacetime once it is perturbed, there are an infinite number of ways of extending $D(\Sigma)$. Such perturbations could break spherical symmetry, for example.

Moreover, the future Cauchy horizon typically becomes singular when the RN spacetime is perturbed, no matter how small the perturbation. An example of this can be constructed when a massless scalar field is coupled in: region II becomes unstable.

In physically realistic gravitational collapse situations, the future Cauchy horizon, generally becomes a singular null hypersurface as perturbations pile up on the $r=r_-$ surface. Accordingly, regions to the future of that hypersurface typically become unphysical.

$r=r_-$

Einstein-Rosen Bridge for RN

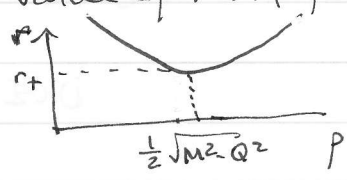
- Isotropic coordinates for RN

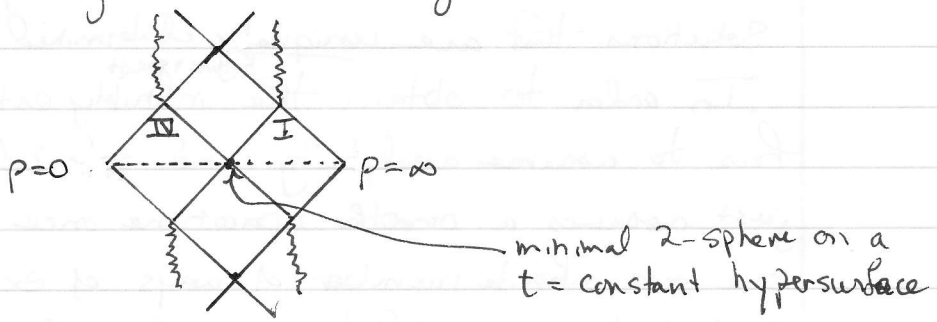
Let $r = \rho + M + \frac{(M^2 - Q^2)}{4\rho}$

Then $\Delta = \left[\rho - \frac{(M^2-Q^2)}{4\rho}\right]^2$ and the metric becomes

$$ds^2 = -\frac{\Delta}{r^2(\rho)}dt^2 + \frac{r^2(\rho)}{\rho^2}\underbrace{(d\rho^2 + \rho^2 d\Omega^2)}_{\text{3d Flat space metric}} \qquad (t, \rho, \theta, \phi) \text{ coords}$$

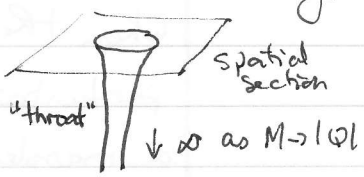
In these coordinates, the outer horizon at $r = r_+$ is located at $\rho = \frac{1}{2}\sqrt{M^2-Q^2}$. For every value of $r > r_+$, there are two corresponding values of ρ. For $r < r_+$, ρ becomes complex, so these isotropic coordinates don't cover that region. Only regions I and IV are covered, and they are isometrically related by $\rho \to \frac{M^2-Q^2}{4\rho}$. At the fixed point of this isometric map, $\rho = \frac{1}{2}(M^2-Q^2)^{1/2}$, one has the minimal 2-sphere of an ER bridge





Calculate the distance to the $r = r_+$ horizon from $r = R$ along a curve at constant t, θ and ϕ:

$$S = \int_{r_+}^{R} \frac{dr}{\left[\left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right)\right]^{1/2}}$$



As $M - |Q| \to 0$, $r_+ - r_- \to 0$ and $S \to -r_+ \ln \epsilon \to \infty$ as $\epsilon \to 0$

So the ER bridge becomes <u>infinitely long</u> in the $M - |Q| \to 0$ limit.

Case iii) Extreme Reissner-Nordstrom black holes

In the extreme case where $M = |Q|$, one has $r_+ = r_- = M$ and the static coordinate (t, r, θ, ϕ) metric becomes

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dt^2 + \frac{dr^2}{\left(1 - \frac{M}{r}\right)^2} + r^2 d\Omega^2$$

This is singular at $r = r_+ = r_- = M$. Define the corresponding tortoise coordinate by

$$dr^* = \frac{dr}{1 - M/r} \qquad \text{integrating to}$$

$$r^* = r + 2M \ln\left|\frac{r - M}{M}\right| - \frac{M^2}{r - M} .$$

One can then introduce IEF coordinates as before: $v = t + r^*$, in terms of which

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dv^2 + 2 dv\, dr + r^2 d\Omega^2$$

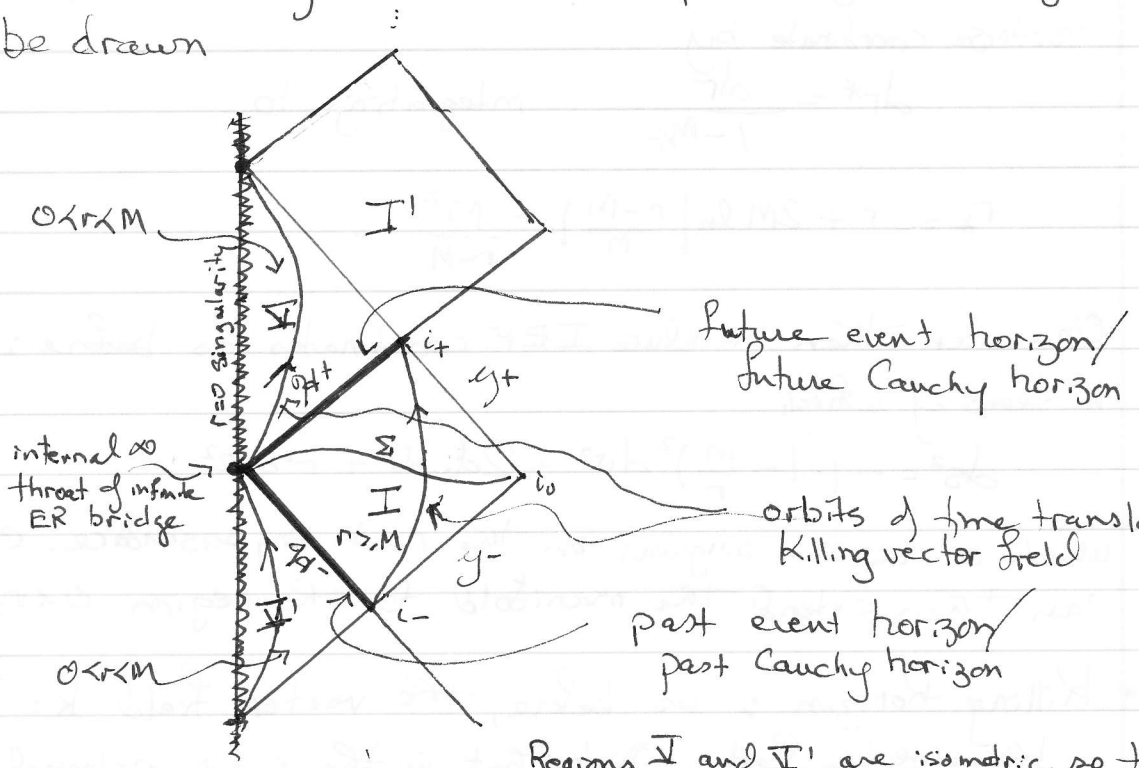
which is now non-singular on the $r = M$ hypersurface. One can then extend the manifold to the region $0 < r \leq M$.

- Killing horizon: as before, the vector field $k = \partial_v$ is a Killing vector field. Note that in the $r_+ = r_-$ extremal case $\kappa_+ = (r_+ - r_-)/2r_+^2 = 0 = (r_- - r_+)/2r_-^2 = \kappa_-$ So the surface gravity vanishes. (In the thermodynamic analysis of black holes, this will correspond to extremal RN black holes having zero temperature.)

Since $\kappa_+ = \kappa_- = 0$, the orbits (ie integral curves) of $k = \partial_v$ are affinely parametrised already, without the need for an f factor in $l = f\partial_v$. One can go to infinite affine parameter in either direction in this case: heading out towards $\mathcal{I}^+$ in one direction, the other direction inwards. So this solution has an internal infinity down the "throat" of the infinitely long ER bridge.

Note that the Killing vector k is null at the surface $r=M$, but it is <u>timelike everywhere else</u> : $k^\mu k^\rho g_{\mu\rho} = g_{tt} = -(1-\frac{M}{r})^2$.
Accordingly, region II has disappeared from the CP diagram and region I abuts directly onto region IV. Similarly, region III has disappeared, as well as all the other regions isometric to regions II and III.

The resulting extremal RN spacetime's CP diagram may be drawn



Regions IV and IV' are isometric, so there are new asymptotically flat regions like I' to future of IV, past of IV'.

- <u>Interpolating Structure</u>

extremal RN
The asymptotic metric as $r\to\infty$ is just 4 dimensional Minkowski and $F=dA=0$
Space, which may be considered a "vacuum" solution. Now consider the asymptotic form of the extremal RN metric as $r\to M$.
Let $r = M(1+\lambda)$ and keep only the leading terms in λ:

$Q=M$ $\quad F = dA = \frac{Q}{r^2}\, dr\wedge dt = \frac{Q}{M(1+\lambda)^2}\, d\lambda\wedge dt \to d\lambda\wedge dt$ constant Maxwell field strength

$$ds^2 = -\frac{\lambda^2 dt^2}{(1+\lambda)^2} + \frac{M^2(1+\lambda)^2}{\lambda^2} d\lambda^2 + M^2(1+\lambda)^2 d\Omega^2$$

$$\to \underbrace{\left(-\lambda^2 dt^2 + M^2 \frac{d\lambda^2}{\lambda^2}\right)}_{AdS_2} + \underbrace{M^2 d\Omega^2}_{\text{2-sphere of radius } M}$$

the metric for
ie. $AdS_2 \times S^2$.
<u>Robinson-Bertotti metric</u>

Accordingly, the extremal RN spacetime interpolates between two "vacuum" solutions of the Einstein-Maxwell theory: flat Minkowski spacetime with constant Maxwell field strength F as $r \to \infty$ and the $AdS_2 \times S^2$ Robinson-Bertotti spacetime as $r \to M$. Both Minkowski spacetime and the Robinson-Bertotti spacetime are separately valid solutions in their own right for Einstein-Maxwell theory. The extremal RN solution interpolates between these two different vacua.

AdS_2: d=2 spacetime of constant negative curvature

- Zero force condition

The coordinate λ used above is related to the standard isotropic radial coordinate $\rho = M\lambda$, ie $\rho = r - M$. In isotropic coordinates, one may write the extremal RN metric ansatz as

$$ds^2 = -e^{-2B} dt^2 + e^{2B}(dx^i dx^i)$$

and for spherical symmetry this becomes

$$ds^2 = -e^{-2B(\rho)} dt^2 + e^{2B(\rho)}(d\rho^2 + \rho^2 d\Omega^2)$$

with corresponding Em potential $A_0(\rho)$

Analysis of the coupled Einstein-Maxwell equations leads to the equations

$$\nabla^2 B = -\frac{1}{4} S^2$$

$$(B')^2 = \frac{1}{4} S^2 \quad \text{where } S = e^B A_0'$$

$B' = \frac{dB}{d\rho}$, etc

and $\nabla^2 B = B'' + \frac{2}{\rho} B'$ is the 3d Laplacian in polar coordinates for a spherically symmetric function (so $\partial_\theta B = \partial_\phi B = 0$)

Thus one needs to solve

$$\nabla^2 B + (B')^2 = 0, \text{ ie. } B'' + \frac{2}{\rho} B' + (B')^2 = 0.$$

Let $B = \ln H$, so $B' = \frac{H'}{H}$ and $B'' = \frac{H''}{H} - \frac{(H')^2}{H^2}$

so one has

$$\frac{H''}{H} - \frac{(H')^2}{H^2} + \frac{2H'}{\rho H} + \frac{(H')^2}{H^2} = 0$$

$$\text{ie. } H'' + \frac{2}{\rho} H' = 0 \quad \text{or} \quad \nabla^2 H = 0$$

constant 1 to make $ds^2 \to$ Minkowski as $\rho \to \infty$

This is solved by $H = 1 + \frac{M}{\rho}$, giving

$$ds^2 = -H^{-2} dt^2 + H^2(d\rho^2 + \rho^2 d\Omega^2) \quad \text{and} \quad A_0 = H^{-1}$$

The occurrence of a 3d flat-space Laplace equation here, $\nabla^2 H = 0$, leads to a more general family of solutions:

- Majumdar–Papapetrou solutions

$$ds^2 = -H(x)^{-2} dt^2 + H(x)^2 (dx^i dx^i)$$
$$A = H^{-1}(x)\, dt$$

where $H(x)$ is a multi-centre solution to $\nabla^2 H = 0$:

$$H(x) = 1 + \sum_{a=1}^{N} \frac{M_a}{|x - x_a|}$$

This is a static solution of N extreme RN black holes of masses M_a at positions x_a^i. (Note that each of these is an S^2, not just a point, however)

Physically, this solution can remain static because $M_a = Q_a$ for all $a = 1 \ldots N$ and there is an exact cancellation between gravitational attraction and electromagnetic repulsion among all the extreme RN black holes.

Rotating Black Holes

In order to discuss black hole solutions with nonzero angular momentum, we need first to make some more precise definitions.

- Stationary spacetime: a spacetime which is asymptotically flat at $\mathcal{I}^+$ null infinity is stationary if it admits a Killing vector field k that is timelike in a neighborhood of $\mathcal{I}^+$.

It is conventional to normalize k such that $k \cdot k = k^\mu k_\mu \to -1$ at $\mathcal{I}^+$. Outside a possible horizon, one may write $k = \partial/\partial t$, where t is a time coordinate. The general stationary metric can then be written

$x^\mu = (t, x^i)$

$$ds^2 = g_{00}(\vec{x})\, dt^2 + 2 g_{0i}(\vec{x})\, dt\, dx^i + g_{ij}(\vec{x})\, dx^i dx^j.$$

- Static spacetime: a Stationary spacetime is static if it is also invariant under time-reversal. This requires having $g_{0i}=0$, so the general static metric can be written

$$ds^2 = g_{00}(\vec{x})\,dt^2 + g_{ij}(\vec{x})\,dx^i dx^j .$$

[Sometimes the terms "strictly stationary/static" are used if k is timelike everywhere, not just near $\mathcal{I}^+$. Thus, Minkowski spacetime is strictly static. Kruskal spacetime is static, but not strictly static because k is spacelike in regions II and III.]

Up until now, we have discussed spherically symmetric black holes. Rotating black holes, however, cannot be spherically symmetric. They can, however, be axisymmetric.

- Axisymmetric spacetime: an asymptotically flat spacetime is axisymmetric if there exists a Killing vector field m (known as an 'axial' Killing vector field) that is spacelike near $\mathcal{I}^+$ and that generates a one-parameter group of isometries isomorphic to $U(1)$. Accordingly, all orbits of m are closed.

- Stationary and axisymmetric: a spacetime which is asymptotically flat at null infinity is stationary and axisymmetric if it is
  i) Stationary ii) axisymmetric with axial Killing vector m
  iii) $[k,m]=0$.

For such a spacetime, one can choose adapted coordinates such that $k=\partial/\partial t$ and $m=\partial/\partial\phi$ with ϕ identified modulo 2π. Using polar asymptotic coordinates (t,r,θ,ϕ) one can normalise m so that $\underbrace{m\cdot m}_{r^2} = \underbrace{m^a m_a}_{r^2} \to 1$ as $r\to\infty$. As for k, in an asymptotically flat spacetime there is a natural choice of normalisation for the axial Killing vector field.

We have previously seen Birkhoff's theorem, which says that any spherically symmetric vacuum solution to Einstein's theory must be static, and hence isometric to Schwarzschild. A generalisation of this theorem to the Einstein-Maxwell system says that the only spherically symmetric solution in that case must be Reissner-Nordstrom.

The converse of this is not true, however: a static spacetime need not be spherically symmetric. However, if the only object in the spacetime is a black hole, one has a number of theorems:

Israel's Theorem (Israel 1967; Bunting + Masood 1987):
If (M,g) is an asymptotically flat, static, vacuum spacetime that is non-singular on and outside an event horizon, then (M,g) is Schwarzschild.

Note: this implies that static vacuum multi-black holes don't exist

(Hawking 1973; Wald 1992) Theorem:
If (M,g) is a stationary and asymptotically flat analytic solution of the Einstein-Maxwell equations that is non-singular on and outside an event horizon, then (M,g) is stationary and axisymmetric.

This theorem makes the unsatisfactory assumption of analyticity, which implies that the full spacetime structure is determined by its behaviour at a single point. However, if one assumes also stationarity and axisymmetry, one has:

(Carter 1971; Robinson 1975) Theorem:
If (M,g) is a stationary, axisymmetric, asymptotically flat spacetime that is non-singular on and outside an event horizon, then (M,g) is member of the two-parameter Kerr (1963) family of black holes. These two parameters are the mass M and angular momentum J. For the Einstein-Maxwell theory, the theorem is generalised to state that (M,g) is a member of the 4-parameter Kerr-Newman solution family.

The Kerr-Newman Solution (KN)

This is a rotating, charged (electrically and magnetically) solution of Einstein-Maxwell theory. In Boyer-Lindquist coordinates, one has

$$ds^2 = -\frac{(\Delta - a^2\sin^2\theta)}{\Sigma}dt^2 - 2a\frac{\sin^2\theta(r^2+a^2-\Delta)}{\Sigma}dtd\phi + \left(\frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\right)\sin^2\theta\, d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2$$

$$A = \frac{-Qr(dt - a\sin^2\theta d\phi) + P\cos\theta(a dt - (r^2+a^2)d\phi)}{\Sigma}$$

where $\Sigma = r^2 + a^2\cos^2\theta$ $\quad \Delta = r^2 - 2Mr + a^2 + e^2$ $\quad e = [Q^2+P^2]^{1/2}$

At large r, the coordinates (t, r, θ, ϕ) become spherical polar coordinates in the asymptotic Minkowski spacetime. The coordinates (θ, ϕ) have the usual interpretation as angles on the 2-sphere S^2, so $0 \le \theta < \pi$ and $\phi \sim \phi + 2\pi$. The KN solution is asymptotically flat at null infinity.

This solution is stationary and axisymmetric with two commuting Killing vectors

$$k = \partial/\partial t \qquad m = \partial/\partial\phi .$$

Near $\mathcal{I}^+$ null infinity, k is timelike but, as for Schwarzschild, it is not globally timelike. The KN solution has in addition a discrete isometry $t \to -t$ and $\phi \to -\phi$. This simultaneously reverses the direction of time and the sense of rotation.

The 4 parameters of the KN solution are: M, a, Q, P. M will be seen to be the mass; Q the electric charge; P the magnetic charge and $a = J/M$, where J is the angular momentum. When $a = 0$, the KN solution reduces to the RN solution (now including magnetic charge). Sending just $\phi \to -\phi$ produces a solution with $a \to -a$. So one may assume $a \ge 0$.

The Kerr Solution i.e. $e=0$

Setting $Q=P=0$, one obtains the Kerr solution to the vacuum Einstein equations. As for RN, write

$$\Delta = (r-r_+)(r-r_-) \quad \text{with} \quad r_\pm = M \pm [M^2-a^2]^{1/2}.$$

Solutions with $M^2<a^2$ have a naked singularity. Focus instead first on the case $M^2>a^2$.

The metric has coordinate singularities at $\theta=0, \pi$ (the usual spherical polar coordinate singularities) and at $r=r_+$ and $r=r_-$. More significantly, it has a curvature singularity where $\Sigma=0$. This condition requires $r=0$ and $\theta=\pi/2$, i.e. the curvature singularity occurs only when $r=0$ is approached along the equator of S^2. When approached from any other angle, there is no $r=0$ singularity: ring singularity where $\Sigma=0$.

In order to remove the $r=r_\pm$ coordinate singularities, one transforms to ingoing Kerr coordinates (v, r, θ, χ) where $v=t+r_*$ and r_* and χ are defined by

$$dr_* = \frac{r^2+a^2}{\Delta}dr \qquad d\chi = d\phi + \frac{a}{\Delta}dr$$

One first makes this coordinate transformation in the region where $r>r_+$, so $\Delta>0$. Note that the definition of χ implies that $\phi=$ const does not correspond to $\chi=$ const. In order to stay at $\chi=$ const, one needs to rotate as well: $d\phi = -\frac{a}{\Delta}dr$.

In the ingoing Kerr coordinates, the metric is

$$ds^2 = -\left(\frac{(\Delta - a^2\sin^2\theta)}{\Sigma}\right)dv^2 + 2dvdr - \frac{2a\sin^2\theta}{\Sigma}(r^2+a^2-\Delta)\,dv\,d\chi$$
$$-2a\sin^2\theta\, d\chi\, dr + \left[\frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\right]\sin^2\theta\, d\chi^2 + \Sigma d\theta^2$$

$\Sigma = r^2+a^2\cos^2\theta$ as before

In these coordinates, there are no more factors of Δ in the denominator, so the metric is now regular at $r=r_+$ and $r=r_-$. The remaining singularity is the curvature singularity where $\Sigma=0$.

• Killing horizons for the Kerr solution

On the hypersurface N_+ defined by $S_+ = r - r_+ = 0$ one has a normal vector

$$l_+ = f_+ g^{\mu\rho}|_{N_+} \partial_\rho S_+ \partial_\mu = f_+ g^{\mu r} \partial_\mu \quad \text{where } f_+$$

is as usual some non-zero function which can be chosen to give an affinely parametrised orbit.

On the N_+ surface, $r = r_+$ so $\Delta = 0$, and the needed inverse metric components are $g^{vr} = \frac{r_+^2 + a^2}{\Sigma_+}$ and $g^{\chi r} = \frac{a}{\Sigma_+}$ where $\Sigma_+ = r_+^2 + a^2\cos^2\theta$; also note $g^{rr} = 0$.

Consequently,

$$l_+ = f_+ \left(\frac{r_+^2 + a^2}{\Sigma_+}\right)\left(\partial_v + \frac{a}{r_+^2 + a^2}\partial_\chi\right).$$

Moreover, the norm of l_+ is thus proportional to

$$\left(\partial_v + \frac{a}{r_+^2+a^2}\partial_\chi\right)^2 = \left(g_{vv} + \frac{2a}{r_+^2+a^2} g_{v\chi} + \frac{a^2}{(r_+^2+a^2)^2} g_{\chi\chi}\right)_{|N_+}$$

$$= \frac{a^2}{\Sigma_+}\left(\sin^2\theta - 2\sin^2\theta + \sin^2\theta\right) = 0$$

So N_+ is a null hypersurface whose normal l_+^μ is proportional to a linear combination ξ_+^μ of Killing vectors, ie. N_+ is a Killing horizon.

$$\xi_+^\mu = k^\mu + \Omega_H m^\mu \quad \text{where } k = \partial_v \text{ and } m = \partial_\chi$$

are the temporal and rotational Killing vectors in ingoing Kerr coordinates. The coefficient $\Omega_H = \frac{a}{r_+^2 + a^2}$ is a rotational velocity. In BL coordinates, one has

$\xi_+ = \partial_t + \Omega_H \partial_\phi$, so $\xi_+^\mu \partial_\mu (\phi - \Omega_H t) = 0$, ie. $\phi = \Omega_H t$ on orbits (ie. integral curves) of ξ_+. On orbits of $k = \partial_t$, one has $\phi =$ constant, however. Thus, particles moving on orbits of ξ_+ rotate with angular velocity Ω_H with respect to a stationary observer (ie. someone on an orbit of k). In particular, such particles rotate with this angular velocity with respect to a stationary observer at infinity.

Now, ξ_+, being a null vector, is tangent to the generators of $\mathcal{H}^+$, so these generators also rotate with angular velocity Ω_+ with respect to a stationary observer at infinity. Consequently, one interprets Ω_H as the angular velocity of the black hole.

- Surface gravity

The function f_+ in $l_+ = f_+ g^{\mu\nu}\partial_\mu$ can be chosen to make the orbits of l_+ affinely parametrised, but the orbits of $\xi_+ = k + \Omega_H m$ are not affinely parametrised. Calculating $\xi_+^\rho D_\rho \xi_+^\mu = k_+ \xi_+^\mu$ gives the surface gravity

$$k_+ = \frac{r_+ - r_-}{2(r_+^2 + a^2)} .$$

- Inner horizon $\mathcal{N}_-$

One may repeat the above discussion for the $\mathcal{N}_-$ hypersurface $S_- = r - r_- = 0$. One then has a normal

$$l_- = f_- \left(\frac{r_-^2 + a^2}{\Sigma_-}\right)\left(\partial_v + \frac{a}{r_-^2 + a^2}\partial_\chi\right) \quad ; \quad \Sigma_- = r_-^2 + a^2\cos^2\theta$$

which is null on $\mathcal{N}_-$ and proportional to the ξ_- Killing vector $\xi_- = \partial_r + \frac{a}{r_-^2 + a^2}\partial_\chi$. The surface gravity on $\mathcal{N}_-$ is

$$k_- = \frac{r_- - r_+}{2(r_-^2 + a^2)} .$$

- Ring singularity

Consider in some more detail now the curvature singularity at $r = 0$, $\theta = \pi/2$. This singular locus has the topology of a ring. For fixed r, v and θ, the Kerr metric becomes

$$ds^2 = \frac{\left[(r^2 + a^2)^2 - \sin^2\theta\,(r^2 - 2Mr + a^2)a^2\right]}{a^2\cos^2\theta}\sin^2\theta\, d\chi^2 .$$

and as $r \to 0$, this tends to $ds^2 = a^2\sin^2\theta\, d\chi^2$

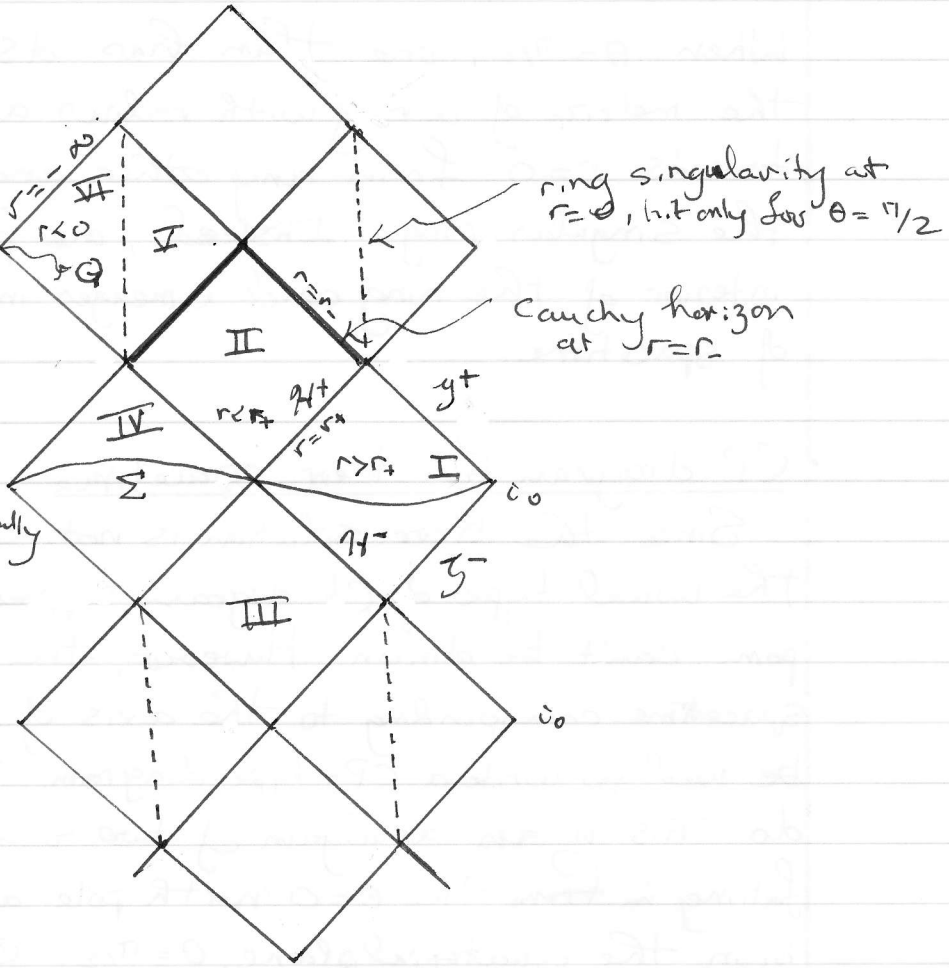
When $\theta = \pi/2$, one then has $ds^2 = a^2 d\chi^2$, ie. the metric of a ring with radius a. If one heads towards $r=0$ from any other angle, one will not hit the singular ring. Instead, one passes through the interior of the ring and emerges into a new region of spacetime.

- CP diagram for Kerr spacetime

Since the Kerr solution is not spherically symmetric, the usual type of CP digram suppressing 2-spheres at each point can't be drawn. However, the submanifold of this spacetime corresponding to the axis of symmetry $\theta = 0$ or π may still be used to write a CP-type diagram. The customary way to do this is an amalgam of two situations: 1) an observer falling in from the $\theta = 0$ north pole and 2) an observer falling in along the equatorial plane, $\theta = \pi/2$. Only in situation 2) does the observer encounter the singularity at $r=0$; accordingly the singularity is denoted by a dashed line.

Both the $\theta = 0$ or π and the $\theta = \pi/2$ submanifolds are <u>totally geodesic</u>. That means that a geodesic initially tangent to such a submanifold will remain tangent to it.

As for the RN spacetime, the $r = r_-$ null hypersurfaces are also Cauchy horizons for a geodesically complete (with two ends!) and asymptotically flat Cauchy surface Σ. Thus, the spacetime beyond $r = r_-$ is not determined uniquely by data on Σ, unless one makes the physically unwarranted additional assumption of analyticity. As for the RN case, these Cauchy horizons are expected to be unstable against initially small perturbations in region I or IV, with the perturbed spacetime developing null or spacelike singularities instead of the Cauchy horizon.



Assuming perturbations build up on the $r=r_-$ horizon/Cauchy horizon N_-, only regions I and II are relevant for a gravitationally collapsed situation.

Unlike the spherically symmetric Schwarzschild solution, where Birkhoff's theorem applied, the Kerr solution does not in general describe correctly the spacetime outside a rotating star, because such a spacetime will in general be time dependent. It is, however, expected to describe the final state of gravitational collapse.

- <u>Closed Timelike Curves</u>

Consider an orbit of the Killing vector m in region VI, where $r<0$ but close to the $r=0$, $\theta=\pi/2$ singularity. At fixed values of v and r and for $\theta=\pi/2$, one has

$$ds^2=\left(r^2+a^2+\frac{2Ma^2}{r}\right)d\chi^2$$

So for $|r|<\frac{2Ma^2}{r^2+a^2}$ and $r<0$, one sees that χ becomes a <u>timelike</u> coordinate. Another way of putting this is for $m=\partial_\chi$, one has $m^2=g_{\chi\chi}<0$ so m is a timelike vector.

However, in all regions the orbits of m are closed, since $\chi \sim \chi + 2\pi$. So when $r<0$ and $|r| < \frac{2Ma^2}{r^2+a^2}$, one has closed timelike curves in region VI.

Being on the far side of the Cauchy horizon N_- at $r=r_-$, however, region VI is expected to be unphysical on account of the build-up of unstable perturbations on N_- leading to a curvature singularity there.

- **Extreme Kerr solution $M^2=a^2$**

In this case one has $r_+ = r_- = M$ and there is a degenerate horizon with $\kappa = 0$: vanishing surface gravity. This solution has some properties similar to the extremal RN solution. In particular, surfaces of constant t extending between regions I and IV exhibit an infinite "throat" instead of leading into region IV at finite affine parameter. Moreover, the coalescence of N_- and N_+ brings the future Cauchy horizon right up to the edge of region I, with a corresponding expectation of unstable perturbations leading to a curvature singularity on $N = N_+ = N_-$.

- **The Ergosphere**

$\Delta = r^2 - 2Mr + a^2$
$\Sigma = r^2 + a^2\cos^2\theta$

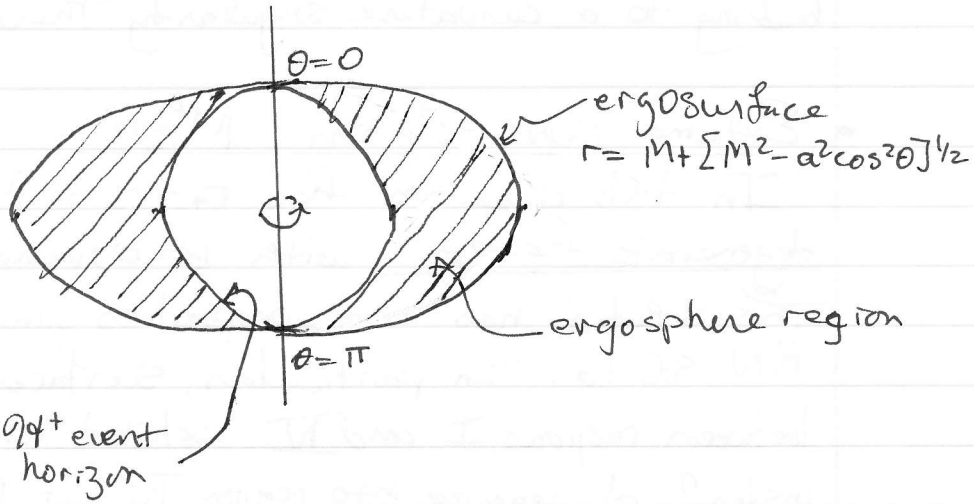
In BL coordinates, the norm of the Killing vector k is $k^2 = g_{tt} = -\frac{(\Delta - a^2\sin^2\theta)}{\Sigma} = -\left(1 - \frac{2Mr}{r^2+a^2\cos^2\theta}\right)$.

Thus k^μ is timelike provided that

$$r^2 + a^2\cos^2\theta - 2Mr > 0.$$

Consequently, in region I where $r > r_+ = M + (M^2-a^2)^{1/2}$, there is a subregion of region I where k becomes spacelike: $r_+ = M + [M^2-a^2]^{1/2} < r < M + [M^2 - a^2\cos^2\theta]^{1/2}$. This region is called the ergosphere; its boundary where $r = M + [M^2 - a^2\cos^2\theta]^{1/2}$ is called the ergosurface. The ergosurface coincides with the N_+ event horizon when $\theta = 0$ or π, but it lies outside the horizon for other θ values.

Stationary observers are observers whose 4-velocities are parallel to k^μ. Because k becomes spacelike within the ergosphere region, such observers do not exist there. Any causal curve in the ergosphere region must rotate relative to observers out at infinity, in the same direction as the black-hole.



To see that a causal curve inside the ergosphere region must be rotating, consider its 4-velocity $\frac{dx^\mu}{d\lambda} = u^\mu$. For a causal curve, one must have

$u^2 = g_{\mu\nu} u^\mu u^\nu < 0$. However, within the ergosphere region all the products in $g_{\mu\nu}u^\mu u^\nu$ are positive except for $g_{t\phi} u^t u^\phi$. Moreover, for a future-directed causal curve one has $u^t > 0$. Hence, one must have $u^\phi > 0$ and a causal curve within the ergosphere region must be rotating. This is an example of <u>frame dragging</u> in the rotating black-hole spacetime.

The Penrose Process

Consider a particle approaching a Kerr black-hole along a geodesic. Let it be affinely parametrised by $\lambda = a\tau + b$, where τ is the proper time evolved along the geodesic; then the einbein $e = \frac{1}{m}\frac{d\tau}{d\lambda}$ is a constant function of λ. We saw earlier that $Q = e^{-1} V_\mu \dot{x}^\mu$

$\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$

is a conserved quantity along the geodesic if V_ν is a Killing vector. Take $V_\nu = k_\nu$; then $E = -k \cdot P$ is conserved where $P^\mu = m_0 u^\mu = m_0 \frac{dx^\mu}{d\tau}$. At asymptotically flat infinity one may identify $p^0 = E$ as the energy.

Suppose then that the particle decays at some point X inside the ergosphere into two other particles with 4-momenta P_1^μ and P_2^μ: $P \rightsquigarrow P_1 + P_2$.

Consequence of special relativity

Considering this decay in a local inertial reference frame at point X, we know that the decay must conserve 4-momentum: $P^\mu = P_1^\mu + P_2^\mu$. So $E = E_1 + E_2$ where $E_i = -k \cdot P_i$.

Eg. of $E_1 < 0$ for k spacelike: let $k = (0,1,0,0)$ take $P_1^\mu = (a, c, 0, 0)$ with c small so P_1^μ is timelike. Then $P_1 \cdot k = c$ So $E_1 = -c$

Normally one would have $E_1 > 0$ so $E_2 < E$. However, in the ergosphere region $E_1 = -P_1 \cdot k$ and k is spacelike in the ergosphere, it can happen that $E_1 < 0$. In that case one would have $E_2 = E - E_1 = E + |E_1| > E$. It can be shown that the negative energy particle must fall into the black hole, while the other one with $E_2 > E$ can escape to infinity. It is thus possible to extract energy from the Kerr black hole by this Penrose process.

• Limit to Energy extraction

For a particle crossing the $\mathcal{H}^+$ horizon at $r = r_+$, one has $-p \cdot \xi > 0$ since both p and ξ are future-directed (ξ is future-directed null and p is future-directed timelike or null). Since $\xi = k + \Omega_H m$, one consequently has

$E_1 - \Omega_H L_1 \geq 0$, where $L_1 = p_1 \cdot m$ is the component of the angular momentum of particle 1 in the direction given by m. (Note that only this component of angular momentum is a constant of the motion.)

Accordingly, $L_1 \leq \frac{E_1}{\Omega_H}$. If $E_1 < 0$ as for particle 1 in the Penrose process, then one has $L_1 < 0$ also. Thus, the black hole's energy and angular momentum are both reduced.

One thus ends up, after the black hole settles down after swallowing particle 1 and absorbing (E_1, L_1), with a black hole of mass $M + \delta M < M$ and $J + \delta J < J$ angular momentum, where $\delta M = E_1$ and $\delta J = L_1$ from particle 1.

Thus $\delta J \leq \frac{\delta M}{\Omega_H}$. Then recall that $\Omega_H = \frac{a}{r_+^2 + a^2}$ and $a = \frac{J}{M}$

and $r_+ = M + [M^2 - a^2]^{1/2}$, so $\frac{\delta M}{\Omega_H} = \frac{2M}{J}(M^2 + (M^4 - J^2)^{1/2})\delta M$

Consequently $\delta J \leq \frac{\delta M}{\Omega_H}$ is equivalent to

$$\delta(M^2 + [M^4 - J^2]^{1/2}) \geq 0$$

Define the irreducible mass

$$M_{irr} = \left[\tfrac{1}{2}(M^2 + [M^4 - J^2]^{1/2})\right]^{1/2} \qquad \text{so } \delta M_{irr} \geq 0$$

Invert this: $M^2 = M_{irr}^2 + \frac{J^2}{4M_{irr}^2} \geq M_{irr}^2$

So in the Penrose process, the mass M of the black hole cannot be reduced below the initial value of M_{irr}: there is a limit to the amount of energy that can be extracted.

- Area of the event horizon

The quantity $A = 8\pi(M^2 + [M^4 - J^2]^{1/2}) = 16\pi M_{irr}^2$ is the area of the event horizon, i.e. the area of the intersection of $\mathcal{H}^+$ with the partial Cauchy surface v = constant, $r = r_+$ in Kerr coordinates. So the process of energy extraction from a spinning black hole is limited by the requirement $\delta A \geq 0$.

This is an illustration of the "second law of black hole mechanics", i.e. Hawking's area theorem.