Komar Integrals

The Kerr-Newman solution is the unique stationary black hole solution of the Einstein-Maxwell theory. It is characterised by four essential parameters: M, Q, P and J. Of these, the magnetic charge P is naturally given by a surface integral $P(B) = \frac{1}{4\pi}\int_{\partial B} F$ over the boundary ∂B of a spacelike region B. This expression is naturally metric-independent. Eg. for B: $t = \text{constant}$. Now for metric-dependent charges

B: a partial Cauchy surface

$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$dF = 0$

In general relativity, one needs similar expressions for M, Q and J: mass, charge and angular momentum. These all involve the metric and a Killing vector k.

First, a Lemma: (Killing vector Lemma)

For a Killing vector field k, one has

$$\nabla_\rho \nabla_\mu k^\nu = R^\nu{}_{\mu\rho\sigma} k^\sigma .$$

To show this, use $(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) k_\rho = R_{\mu\nu\rho\sigma} k^\sigma$, add the $\nu\rho\mu$ cycled version and subtract the $\rho\mu\nu$ cycled version, using also $\nabla_\rho k_\nu = -\nabla_\nu k_\rho$ for a Killing vector.

Define $K_{\mu\nu} := \nabla_\mu k_\nu = \nabla_{[\mu} k_{\nu]} = \frac{1}{2}(\nabla_\mu k_\nu - \nabla_\nu k_\mu)$ and the associated 2-form $K = \frac{1}{2} K_{\mu\nu} dx^\mu \wedge dx^\nu$.

Taking the $\delta^\rho{}_\nu$ trace of the Killing vector Lemma, find $\nabla_\rho \nabla_\mu k^\rho = R_{\mu\sigma} k^\sigma$, i.e. $\nabla_\rho K_\mu{}^\rho = R_{\mu\sigma} k^\sigma$

Then define $J_\mu = \frac{R_{\mu\sigma} k^\sigma}{8\pi G}$, so $\nabla_\rho K_\mu{}^\rho = 8\pi G J_\mu$

Recall that for an r-form A in n dimensional spacetime, one has $(*d*A)_{\mu_1 \dots \mu_{r-1}} = (-1)^{[t]} (-1)^{r(n-r)} \nabla^\nu A_{\mu_1 \dots \mu_{r-1} \nu}$

$[t]$ = No of time directions

so for the 2-form K, one has $(*d*K)_\mu = -\nabla^\nu K_{\mu\nu} = \nabla^\nu K_{\nu\mu}$

while $*(*A) = (-1)^{[t]} (-1)^{r(n-r)} A$ so $**J = J$ for $J = J_\mu dx^\mu$

So we can write $*d*K = -8\pi G J$, or $d*K = -8\pi G *J$.

From this, one has $d*J = 0$. This is the form way of writing $\nabla_\mu (R^{\mu\sigma} k_\sigma) = \frac{1}{2} \nabla^\sigma R\, k_\sigma = 0$. One way of understanding this is to use adapted coordinates where $k^\rho \partial_\rho = \frac{\partial}{\partial \alpha}$. Then $\frac{\partial}{\partial \alpha} g_{\mu\nu} = 0$ so $\frac{\partial}{\partial \alpha} R = 0$ too.

Check consistency with the Einstein equations with a stress tensor $T_{\mu\nu}$:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G\, T_{\mu\nu}$$

Rewrite this as $R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)$, $T = T^{\rho}{}_{\rho}$

Then for $\mathcal{J}_\mu = \dfrac{R_{\mu\sigma} k^\sigma}{8\pi G} = \left(T_{\mu\sigma} - \frac{1}{2} g_{\mu\sigma} T\right) k^\sigma$ one sees

that $d * \mathcal{J} = 0$ is equivalent to $* d * \mathcal{J} = 0$, or $\nabla^\mu \mathcal{J}_\mu = 0$

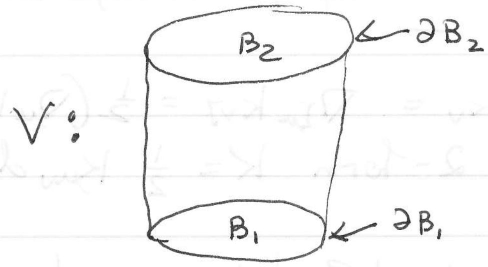
so $\nabla^\mu\left(T_{\mu\sigma} k^\sigma - \frac{1}{2} k_\mu T\right) = T^{\mu\nu} \nabla_\mu k_\nu - \frac{1}{2} T \nabla_\mu k^\mu - \frac{1}{2} k^\mu \nabla_\mu T$

using $\nabla^\mu T_{\mu\sigma} = 0$ (covariant conservation of the matter stress tensor). Then from the Killing equation $\nabla_{(\mu} k_{\nu)} = 0$ one has $-\frac{1}{2} k^\mu \nabla_\mu T = \frac{1}{16\pi G} k^\mu \nabla_\mu R = 0$ as we have seen.

Consequently, Einstein's equations can be written

$$d * K = -8\pi G * \mathcal{J}$$
$$d * \mathcal{J} = 0$$

For a given Killing vector K, and corresponding $\mathcal{J}$, now consider a cylindrical region V in spacetime bounded by bounded spacelike surfaces B_1 and B_2:



choose B_1 and B_2 and V large enough so that the support of $T_{\mu\nu}$ lies entirely inside V. (i.e. $T_{\mu\nu} = 0$ outside V)

For a given bounded spacelike surface B, the Komar integral associated to the Killing vector K is defined as

One may choose ∂B as a 2-surface at spatial infinity

$$Q_K(B) = \frac{c}{8\pi G} \int_{\partial B} * K \qquad c = \text{some constant}$$
$$\left(= -c \int_B * \mathcal{J} \text{ as a } B\text{-volume integral}\right)$$

Then
$$Q_K(B_2) - Q_K(B_1) = \frac{c}{8\pi G}\left(\int_{\partial B_2} * K - \int_{\partial B_1} * K\right)$$
$$= \frac{c}{8\pi G}\left(\int_{B_2} d * K - \int_{B_1} d * K\right) = -c\left(\int_{B_2} * \mathcal{J} - \int_{B_1} * \mathcal{J}\right)$$
$$= -c \int_{\partial V} * \mathcal{J} = -c \int_V d * \mathcal{J} = 0$$

So $Q_K(B)$ is conserved.

dS_μ: volume element on B (non-spacelike)

$dS_{\mu\nu}$: surface element on ∂B

In component notation, the Komar integral is

$$Q_k(B) = \frac{c}{16\pi G} \int_{\partial B} dS_{\mu\nu} \nabla^\mu k^\nu = c \int_B dS_\mu J^\mu(k)$$

where the non-zero $dS_{\mu\nu}$ components are only $dS_{0i} = -dS_{i0} = dS_i$ for ∂B an S^2 at infinity, satisfying $t = \text{const.}$

where $dS_{\mu\nu}$ is the spatial surface element on ∂B.

Examples.

Mass / Energy associated to $k = \partial_t$ (Killing vector for time-translation)

Note sign: $-\int_B * J = \frac{-1}{3!}\int \epsilon_{ijk0} J^0 dx^i dx^j dx^k$ and $\epsilon_{ijk0} = -\epsilon_{0ijk} = -\sqrt{|g|}$ so $-\int_B * J = \int_B dS_\mu J^\mu$

$$E(B) = -\frac{1}{8\pi G}\int_{\partial B} dS_{\mu\nu} \nabla^\mu k^\nu$$

choosing $c = -2$ and ∂B as the 2-sphere at spatial infinity (i.e. large enough that all matter is inside B and so spacetime is a vacuum outside of it)

One can verify that $E(B) = M$ for Schwarzschild for any B in the exterior $(r > 2M)$ spacetime.

Angular Momentum associated to $k = m = \frac{\partial}{\partial\phi}$, with $c = 1$

$$J(B) = \frac{1}{16\pi G}\int_{\partial B} dS_{\mu\nu} \nabla^\mu m^\nu$$

It is instructive to check the coefficient $c = 1$ for this case. Consider a case with a weak field $g_{\mu\nu} \approx \eta_{\mu\nu}$ and use Cartesian coordinates $x^i, i = 1,2,3$. Then $m = x^1 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^1}$ and

$$J(B) = \int_B dS_\mu J^\mu(m) \quad \text{with} \quad J^\mu(m) = T^\mu{}_\nu m^\nu - \tfrac{1}{2} T m^\mu$$

Choose B to be on a $t =$ constant hypersurface so the only component of dS_μ is dS_0, so $dS_\mu m^\mu = 0$ and $J(B) = \int_B \sqrt{{}^3g}\, d^3x\, T^0{}_\nu m^\nu = \int_B \sqrt{{}^3g}\, d^3x \left(T^0{}_2 x^1 - T^0{}_1 x^2\right)$

$$\approx \epsilon_{3jk} \int_B d^3x\, x^j T^{k0} \quad \begin{cases} \text{3rd component of } J \\ \text{in Minkowski space} \\ \text{with stress tensor } T_{\mu\nu} \end{cases}$$

Electric charge

The Gauss' Law expression for electric charge is a standard general covariant generalization of the Minkowski space expression. However, it now provides a useful analogy to the Komar integrals:
$F \longleftrightarrow K$ and $j \longleftrightarrow \xi$.
In a gravitational background,
$$\nabla_\nu F^{\nu\mu} = \frac{1}{\sqrt{-g^{(4)}}}\, \partial_\nu \left(\sqrt{-g^{(4)}}\, F^{\nu\mu}\right)$$
and $(\ast d \ast F)_\mu = \nabla^\nu F_{\nu\mu}$
The current 4-vector j^μ has time component $j^0 = \rho$, the electric charge density. The corresponding total electric charge is, on a spacelike manifold B,
$$Q = \frac{1}{4\pi}\int_B dS_\mu j^\mu .$$

Choosing B to be determined by $t =$ constant, the only component of dS_μ is dS_0. Using Maxwell's equations $\nabla_\nu F^{\mu\nu} = 4\pi j^\mu$, one can rewrite Q as
$$Q = \frac{1}{4\pi}\int_B dS_\mu \nabla_\nu F^{\mu\nu} = \frac{1}{8\pi}\int_{\partial B} dS_{\mu\nu} F^{\mu\nu} \text{ by Gauss' Law}$$

Then, if B is determined by $t =$ constant, the only non-vanishing components of $dS_{\mu\nu}$ are
$$dS_{0i} = -dS_{i0} = dS_i$$
so $Q = \frac{1}{4\pi}\int_{\partial B} dS_i F^{0i} = \frac{1}{4\pi}\int_{\partial B} dS_i E^i$ since $F^{0i} = E^i$

the usual Gauss' Law expression for Q.
Using differential forms, Q may be rewritten:
$(\ast d \ast F)_\mu = -\nabla^\nu F_{\mu\nu} = -4\pi j_\nu$ so $d \ast F = -4\pi \ast j$
Note that current conservation $\nabla_\mu j^\mu = 0$ becomes $d \ast j = 0$.
Electric charge is $Q(B) = -\int_B \ast j = \frac{1}{4\pi}\int_B d\ast F = \frac{1}{4\pi}\int_{\partial B} \ast F$.
Note that in a gravitational background, $\int_{\partial B} \ast F$ involves the metric.
<u>Magnetic charge</u> $P = \frac{1}{4\pi}\int F$, on the other hand, is independent of the metric.

Black Hole Uniqueness : "No-hair theorems"

The Kerr-Newman solution is the general 4-parameter stationary, axisymmetric, asymptotically flat solution to Einstein-Maxwell theory that is suitably regular on and outside a connected event horizon. Since astrophysical black holes will generally have vanishing electromagnetic charges, this leads to the expectation that the final state of gravitational collapse will generically be a Kerr black hole, characterized by just two parameters: M and J. This is in contrast to the initial state of the collapse, which may be arbitrarily complicated.

There are more fields to consider aside from the electromagnetic field of Einstein-Maxwell theory, however. Nonetheless, there are a series of "no-hair" theorems which establish that black holes that have settled down to a stationary final state (a generally fast process on a time scale set by the mass/or radius/ : microseconds for a solar mass black hole) - such black holes have no other properties than mass, angular momentum and the EM charges.

Let's illustrate this sort of assertion for GR minimally coupled to a real scalar field.

Scalar no-hair theorem

A static black hole cannot be the source of a real minimally coupled scalar field.

To prove this, let ϕ be a real scalar field satisfying the Klein-Gordon equation in a gravitational background

$$\nabla^\mu \nabla_\mu \phi - m^2 \phi = 0$$

fancy way of saying $\dot{\phi} = 0$

The "static" condition means that there exists a timelike Killing vector k of the metric such that $k^\mu \nabla_\mu \phi = 0$.

Given k, one can foliate the spacetime by $t=$ constant hypersurfaces with unit normals n that are proportional to k, i.e. $n \propto k$. The black hole horizon is a Killing horizon of k.

Now consider the integral

$$I = \int_R \phi\,(\nabla^2\phi - m^2\phi)\sqrt{g}\,d^4x$$

where $R \subset M$ is the spacetime region outside the horizon. From the Klein-Gordon equation, one has $I=0$. However, using Stoke's theorem after integrating by parts the derivative term, one gets

$$I = \int_R \left(-g^{\mu\nu}\nabla_\mu\phi\,\nabla_\nu\phi - m^2\phi^2\right)\sqrt{g}\,d^4x + \int_{\partial R}\phi\,\nabla_\mu\phi\,dS^\mu$$

where ∂R is the boundary of the region R and dS^μ is the normal surface element on ∂R.

There are four contributions to ∂R: the horizon at $r=2M$, the surfaces $t=\pm\infty$ and the "sphere at infinity" surface as $r\to\infty$. On the horizon and on the $t=\pm\infty$ surfaces, dS^μ is proportional to k^μ, so $\nabla_\mu\phi\,dS^\mu \propto k^\mu\nabla_\mu\phi = 0$. As for the $r\to\infty$ surface, observe that the metric for a static and asymptotically flat spacetime is of the form

$$ds^2 = f(r)\,dt^2 + g(r)\,dr^2 + r^2 d\Omega_2^2 \quad \text{with}$$

$f(r) = -1 + f_1 r^{-1} + O(r^{-2})$ and $g(r) = 1 + g_1 r^{-1} + O(r^{-2})$.

Use $\nabla^2\phi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\phi\right)$ and expand to obtain the leading Klein-Gordon equation terms as $r\to\infty$:

$$\partial_r^2\phi + 2r^{-1}\partial_r\phi - m^2\phi + \text{lower order} = 0$$

Dropping the lower order terms, the solutions to this are $\phi \propto \frac{e^{\pm mr}}{r}$. Requiring ϕ not to diverge as $r\to\infty$, one has $\phi \propto e^{-mr}/r$.

Actually, all one needs to know is that $\phi \sim \frac{1}{r}$ or faster at ∞.

eg. $S(r) = 4\pi r^2$
$dS \sim r\,dr$

For a sphere of radius r, $dS^r \propto r\,dr$, so $\phi\nabla_\mu\phi\,dS^\mu \propto r\,dr\,\phi\,\partial_r\phi \sim dr/r^2$.

Thus, the integral $\int_{r\to\infty}\phi\nabla_\mu\phi\,dS^\mu = 0$ vanishes. So all the terms $\int_{\partial R}\phi\nabla_\mu\phi\,dS^\mu$ vanish.

Consequently, one has

$$I = \int_R \left(-g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - m^2 \phi^2\right) = 0$$

Minkowski: flat space: $n_\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$h_{\mu\nu} =$ diag$(-1,1,1,1)$ + diag$(1,0,0,0)$ = diag$(0,1,1,1)$ becomes diag$(1,1,1)$ on B

On spacelike hypersurfaces $t =$ constant, with normal vector n^μ, the induced metric is $h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$. This metric is the restriction of g onto such a spacelike hypersurface $B \subset M$. Although h is degenerate on the full manifold M, it becomes positive definite for spacelike $B \subset M$ on the subspace of the tangent space spanned by vectors tangent to curves in B (ie. vectors satisfying $v \cdot n = 0$).

indices on $h^{\mu\nu}$ and n^μ raised with $g^{\mu\nu}$

In terms of the induced metric $h_{\mu\nu}$, the integrand of I becomes $g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi = h^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - (n^\mu \nabla_\mu \phi)^2$ noting that $g^{\mu\nu} = h^{\mu\nu} - n^\mu n^\nu$ (check using $n^2 = -1$ & finding $h_{\mu\nu} n^\nu = 0$)

On R tangent space, $h^{\mu\nu}$ is the inverse of $h_{\mu\nu}$.

Then use $n^\mu \partial_\mu \phi = 0$ so

$$\underline{I} = \int_R \left(-h^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - m^2 \phi^2\right) = 0$$

However, the positive definiteness of $h_{\mu\nu}$ on the tangent space to R then implies that the integrand is negative semidefinite, with the integral vanishing only if the integrand vanishing everywhere. For $m \neq 0$, this implies that $\phi = 0$ everywhere in R. For $m = 0$, it implies that $\nabla_\mu \phi = 0$, so $\phi =$ const everywhere. In either case, ϕ is in a vacuum configuration everywhere in R, ie. everywhere outside the horizon.

Elements of Black Hole Thermodynamics

Return to the notion of surface gravity κ, to derive a general formula for it. κ is associated to a Killing horizon N of a Killing vector ξ.

In general, consider a hypersurface Σ in a spacetime M, specified by a condition $f(x) = 0$ where $f: M \to \mathbb{R}$ is a smooth function with $df \neq 0$ on Σ. Then df is normal to Σ. Any other 1-form n normal to Σ can be written as $n = g\,df + f n'$, where g is a smooth function with $g \neq 0$ on Σ and n' is a smooth 1-form.

Hence $dn = dg \wedge df + df \wedge n' + f\,dn'$, so $(dn)|_\Sigma = (dg - n') \wedge df$, and consequently for any n normal to Σ one must have $(n \wedge dn)_\Sigma = 0$.

[Conversely, Frobenius' Theorem says that if n is a non-zero 1-form such that $n \wedge dn = 0$ <u>everywhere</u>, then there exist functions f and g such that $n = g\,df$ so n is normal to surfaces of constant f. In that case, one says that n is <u>hypersurface-orthogonal</u>.]

Now, since ξ is normal to the null hypersurface N, one has $\xi_{[\mu}\nabla_\nu \xi_{\rho]}|_N = 0$ by the above discussion. Moreover, $\nabla_\mu \xi_\nu = \nabla_{[\mu}\xi_{\nu]}$ for a Killing vector, so

$$\xi_\rho \nabla_\mu \xi_\nu|_N + (\xi_\mu \nabla_\nu \xi_\rho - \xi_\nu \nabla_\mu \xi_\rho)|_N = 0$$

multiply by $\nabla^\mu \xi^\nu$:

$$\xi_\rho (\nabla^\mu \xi^\nu)(\nabla_\mu \xi_\nu)|_N = -2(\nabla^\mu \xi^\nu)\xi_\mu (\nabla_\nu \xi_\rho)|_N \quad \left(\text{noting } \nabla^\mu\xi^\nu = \nabla^{[\mu}\xi^{\nu]}\right)$$

on Killing horizon, $\xi^\nu \nabla_\nu \xi_\rho|_N = \kappa \xi_\rho$

$$= -2(\xi \cdot \nabla \xi^\nu)\nabla_\nu \xi_\rho|_N$$

$$= -2\kappa\, \xi \cdot \nabla \xi_\rho|_N$$

$$= -2\kappa^2 \xi_\rho|_N$$

So, except on points where $\xi_\rho = 0$, one has the <u>Killing vector formula</u>

$$\kappa^2 = -\frac{1}{2}(\nabla^\mu \xi^\nu)(\nabla_\mu \xi_\nu)|_N$$

Using this, one easily proves that κ is constant on orbits of ξ.

Hawking (1974): interpretation in terms of <u>temperature</u> $T_H = \frac{\hbar \kappa}{2\pi k_B c}$.

Area of the horizon : entropy

The area of the event horizon of a black hole can be defined as the area of the intersection of $\mathcal{H}^+$ with a partial Cauchy surface (i.e. a hypersurface which no causal curve intersects more than once).

For the Kerr metric in Kerr coordinates, this will be the area of the intersection of $\mathcal{H}^+$ at $r=r_+$ and B, the partial Cauchy surface $v=$ constant. Setting $dr=dv=0$, one has the induced line element

$$ds^2_{\mathcal{H}^+\cap B} = \Sigma d\theta^2 + \left(\frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\right)\sin^2\theta\, d\chi^2$$

while on $\mathcal{H}^+$ one has $\Delta = 0$, so the volume element on $\mathcal{H}^+\cap B$ is $\sqrt{h} = (r_+^2+a^2)\sin\theta$ and consequently

$$A = \int_{\mathcal{H}^+\cap B} \sqrt{h}\, d\theta d\phi = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, (r_+^2+a^2)\sin\theta = 4\pi(r_+^2+a^2).$$

Recalling that $r_+ = M+[M^2-a^2]^{1/2}$ and $a = J/M$, the area can be re-expressed as $A = 8\pi(M^2+[M^4-J^2]^{1/2})$. Even though BL coordinates are singular at $\mathcal{H}^+$, the same result is obtained for them since the singularity doesn't affect the $dt=dr=0$ calculation.

For the Kerr-Newman metric, $r_+ = M+[M^2-(a^2+e^2)]^{1/2}$ with $e^2 = Q^2+P^2$. For simplicity, consider the pure electric case $P=0$, so $e^2=Q^2$. Then for electric Kerr-Newman,

$$A = A(M,Q,J) = 4\pi(2M^2-Q^2+2M[M^2-Q^2-a^2]^{1/2})$$

Hawking (1974): Black hole entropy : $S = \frac{Ac^3}{4G\hbar}$

Electric potential Φ_H at the Kerr-Newman horizon:

The horizon $\mathcal{H}^+$ is the Killing horizon for the Killing vector $\xi = \partial_t + \Omega_H\partial_\phi$ with $\Omega_H = \frac{a}{r_+^2+a^2}$.

Bring a unit charge of the same sign as the Q of the KN black hole in along an orbit of ξ to the horizon at $r=r_+$, starting from $r=\infty$.

The work required to move this unit charge in to $\mathcal{H}^+$ is

$$\Phi_H = \xi^\mu A_\mu|_\infty - \xi^\mu A_\mu|_{r=r_+}$$

Now $\xi^\mu A_\mu = A_t + \Omega_H A_\phi = -\dfrac{Qr}{\Sigma}(1 - a\sin^2\theta\,\Omega_H)$ "co-rotating potential"

$\Sigma = r^2 + a^2\cos^2\theta$

$$= \frac{-Qr}{\Sigma}\frac{(r_+^2 + a^2\cos^2\theta)}{r_+^2+a^2} \quad \text{since } \Omega_H = \frac{a}{r_+^2+a^2}$$

Then $\xi^\mu A_\mu|_\infty = 0$ and $\xi^\mu A_\mu|_{r_+} = \dfrac{-Qr_+}{r_+^2+a^2}$

so $\Phi_H = \dfrac{Qr_+}{r_+^2+a^2} = \dfrac{Q}{M}\dfrac{M^2 + [M^4 - J^2 - M^2Q^2]^{1/2}}{2M^2 - Q^2 + 2[M^4 - J^2 - M^2Q^2]^{1/2}}$

One may combine these expressions into <u>Smarr's formula</u>: $M = \frac{\kappa}{4\pi}A + 2\Omega_H J + \Phi_H Q$

<u>Variation of the area</u>

$A(M,J,Q)$ is a function of the independent parameters M, J and Q, so $dA = \dfrac{\partial A}{\partial M}dM + \dfrac{\partial A}{\partial J}dJ + \dfrac{\partial A}{\partial Q}dQ$.

Recalling that for the Kerr-Newman black hole, on $\mathcal{H}^+$

$$\kappa = \frac{r_+ - r_-}{2(r_+^2+a^2)} = \frac{1}{M}\frac{[M^4 - J^2 - M^2Q^2]^{1/2}}{(2M^2 - Q^2 + 2[M^4 - J^2 - M^2Q^2]^{1/2})}$$

after a bit of work, one finds

$$\frac{\kappa\, dA}{8\pi} = dM - \Omega_H\, dJ - \Phi_H\, dQ, \quad \text{or}$$

$$dM = \frac{\kappa\, dA}{8\pi} + \Omega_H\, dJ + \Phi_H\, dQ$$

This is the <u>First Law of Black Hole Mechanics</u>

As presented, this derivation is deceptively simple. The more involved part of the first law resides in the uniqueness theorem. One needs to know that after a perturbation, a Kerr-Newman black hole settles down to another Kerr-Newman black hole.

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<u>Simplified argument for 1st laws when $Q=0$:</u>
The uniqueness theorems for the Kerr family imply $M = M(A, J)$.

gravitational units, different from particle physics

Now, in units where $G=c=1$, both A and J have dimensions of M^2 [note: $[GM] \sim l$, so $[M] \sim l$]. Consequently, $M[A,J]$ must be a <u>homogeneous function</u> of degree 1/2. Then, by Euler's Theorem for homogeneous functions, $A\frac{\partial M}{\partial A} + J\frac{\partial M}{\partial J} = \frac{1}{2}M$,

and by the Smarr formula with $Q=0$, $M = \frac{\kappa}{4\pi}A + 2\Omega_H J$.
Thus, $A\left(\frac{\partial M}{\partial A} - \frac{\kappa}{8\pi}\right) + J\left(\frac{\partial M}{\partial J} - \Omega_H\right) = 0$

now, A and J are free parameters, so
$$\frac{\partial M}{\partial A} = \frac{\kappa}{8\pi} \quad \text{and} \quad \frac{\partial M}{\partial J} = \Omega_H .$$
Establishing the relation between the 1st law and the Smarr formula.

<u>Analogies between thermodynamic and black hole laws</u>

Law	Thermodynamics	Black holes
0th	Temperature T is constant throughout a system in thermal equilibrium	Surface gravity κ is constant over the event horizon of a black hole
1st	$dE = T\,dS + \sum_i \mu_i dN_i$ μ_i: chemical potentials	$dM = \frac{1}{8\pi}\kappa dA + \Omega_H dJ + \Phi_H dQ$
2nd	$dS \geq 0$	$dA \geq 0$
3rd	T cannot be reduced to zero by a finite number of operations	κ cannot be reduced to zero by a finite number of operations

The 2nd law is the important theorem [Hawking 1972]: The area A of $\mathcal{H}^+$ cannot decrease if the weak energy condition holds and cosmic censorship holds (i.e. there are no singularities on or outside the horizon $\mathcal{H}^+$).
<u>Weak energy condition</u> (WEC): $T_{\mu\nu}t^\mu t^\nu \geq 0$ for all timelike t.

Look at the zeroth law in some more detail:

Theorem (Hawking 1972): In a stationary, analytic, asymptotically flat vacuum black hole spacetime, $\mathcal{H}^+$ is a Killing horizon.

The theorem extends also to Einstein-Maxwell theory or theories where matter fields obey hyperbolic equations.

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The full statement of the zeroth law is:
κ is constant on the future event horizon $\mathcal{H}^+$ of a stationary black hole spacetime obeying the dominant energy condition.

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The proof starts from Hawking's theorem's implication that $\mathcal{H}^+$ is a Killing horizon with respect to some Killing vector ξ^μ. One may also derive (using Raychaudhuri's equation) that $R_{\mu\nu}\xi^\mu\xi^\nu|_{\mathcal{H}^+} = T_{\mu\nu}\xi^\mu\xi^\nu|_{\mathcal{H}^+} = 0$.

$\xi_\mu\xi^\mu \le 0$ (in fact $\xi_\mu\xi^\mu = 0$ and $\xi^0 > 0$)

Now, ξ^μ is a future-directed causal vector field. Consequently, $b_\mu = -T_{\mu\nu}\xi^\nu$ must also be a future-directed causal vector field. Moreover by the above, $b_\mu\xi^\mu|_{\mathcal{H}^+} = 0$. Now expand b_μ in a basis of tangent vectors to $\mathcal{H}^+$: $b = a\xi + c_1\eta_1 + c_2\eta_2$ with $\xi\cdot\eta_i = 0$.
Such a combination of tangents to $\mathcal{H}^+$ must be spacelike or null. But b_μ must be causal, hence timelike or null. Consequently, $c_1 = c_2 = 0$, so $b_\mu \propto \xi_\mu$ on $\mathcal{H}^+$, and then $0 = \xi_{[\mu}b_{\nu]}|_{\mathcal{H}^+} = -\xi_{[\mu}T_{\nu]\rho}\xi^\rho|_{\mathcal{H}^+} = -\frac{1}{8\pi}\xi_{[\mu}R_{\nu]\rho}\xi^\rho|_{\mathcal{H}^+}$.

Then, using the Killing vector Lemma $\nabla_\mu\nabla_\nu\xi_\rho = R^\sigma{}_{\mu\nu\rho}\xi_\sigma$ and $\xi_\rho\nabla_\mu\xi_\nu|_{\mathcal{H}^+} = -2\xi_{[\mu}\nabla_{\nu]}\xi_\rho|_{\mathcal{H}^+}$ as in the derivation of the Killing vector formula for κ, one can derive $\xi_{[\sigma}\nabla_{\rho]}\kappa|_{\mathcal{H}^+} = -\xi_{[\sigma}R_{\rho]}{}^{\sigma}\xi_\sigma|_{\mathcal{H}^+}$
so $\xi_{[\mu}\nabla_{\nu]}\kappa|_{\mathcal{H}^+} = 0$, and consequently $\nabla_\nu\kappa \propto \xi_\nu$ on $\mathcal{H}^+$.
But then $t^\nu\nabla_\nu\kappa \propto t^\nu\xi_\nu = 0$ for any tangent vector to $\mathcal{H}^+$
so κ is constant on $\mathcal{H}^+$.