Imperial College London

MSc EXAMINATION May 2019

This paper is also taken for the relevant Examination for the Associateship

BLACK HOLES

For Students in Quantum Fields and Fundamental Forces

Thursday, 2nd May 2019: 14:00–17:00

Answer Question 1 and choose TWO from Questions 2, 3 and 4. Question 1 is worth 40% and Questions 2, 3 and 4 are worth 30% each.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the 3 answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in 3 answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

1. The Schwarzschild metric for a black hole of mass M in Schwarzschild coordinates (t, r, θ, ϕ) is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

(i) (a) Show that on radial, null geodesics we have $dt^2 = \left(1 - \frac{2M}{r}\right)^{-2} dr^2 \equiv dr_*^2$. Define ingoing and outgoing radial null geodesics in terms of (t, r_*).

(b) Explicitly show that the metric can be written in ingoing Eddington-Finkelstein coordinates in the form below

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

(c) Explain why $r = 2m$ is not singular. Also explain, without derivation, how you would argue that $r = 0$ is a spacetime singularity.

(d) Draw an Eddington-Finkelstein diagram with vertical axis $v - r$ and horizontal axis r. Clearly mark several ingoing and outgoing radial null geodesics on the diagram and briefly explain why this indicates that the surface $r = 2m$ is the black hole event horizon.

[15 marks]

(ii) The metric for two-dimensional Minkowski spacetime, $\mathbb{M}^{1,1}$ is

$$ds^2 = -dt^2 + dx^2$$

with $-\infty < t, x < +\infty$. By suitably introducing coordinates $\tilde{u}, \tilde{v}$ with finite range $-\pi/2 < \tilde{u}, \tilde{v} < \pi/2$, clearly describe the conformal compactification of $\mathbb{M}^{1,1}$ explaining the relevance of the metric

$$d\hat{s}^2 = -d\tilde{u}d\tilde{v}$$

Draw the associated Penrose diagram, marking on the diagram $\mathcal{I}^\pm$ ("scri-plus" and "scri-minus"), $i^\pm$, which should be defined including their physical significance, as well as i^0. Lines of constant x and constant t should also be drawn on the diagram. [10 marks]

(iii) Without derivation draw the Penrose diagram for the following spacetimes, marking on $\mathcal{I}^\pm$, $i^\pm$ and i_0.

(a) Four-dimensional Minkowski spacetime.

(b) The spacetime for a radially, collapsing star that settles down to a static configuration and *does not* form a black hole.

(c) The spacetime for a radially, collapsing star that *does* form a black hole.

[6 marks]

(iv) The Lie derivative of the metric with respect to a vector k can be written

$$\mathcal{L}_k g_{\mu\nu} = k^\rho \partial_\rho g_{\mu\nu} + g_{\rho\nu}\partial_\mu k^\rho + g_{\mu\rho}\partial_\nu k^\rho \, .$$

[This question continues on the next page . . .]

If k is a Killing vector, satisfying $\mathcal{L}_k g_{\mu\nu} = 0$, show that $\nabla_{(\mu} k_{\nu)} = 0$. Also prove that

$$\nabla_\nu \nabla_\mu k_\rho = R_{\rho\mu\nu\sigma} k^\sigma$$

and hence $\nabla_\nu \nabla_\mu k^\nu = R_{\mu\nu} k^\nu$. You may use $(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) V^\rho = R_{\mu\nu}{}^\rho{}_\sigma V^\sigma$, valid for an arbitrary vector V^μ. [9 marks]

[Total 40 marks]

2. The electrically charged Reissner-Nordstom (R-N) spacetime has a metric given by

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_2$$

and the vector potential has non-zero component $A_t = -\frac{Q}{r}$ and $d\Omega_2 = d\theta^2 + \sin^2\theta d\phi^2$. Assume that $M > Q$.

(i) The Komar mass is defined to be

$$M_{Komar} = -\frac{1}{8\pi}\int_{S^2_\infty} *dk$$

where k is the one-form dual to the Killing vector ∂_t. Show that the mass parameter entering the R-N metric is the same as the Komar mass. You may assume $\epsilon_{tr\theta\phi} = +\sqrt{-g}$. Also show that Q is the total electric charge of the spacetime by verifying the relation

$$Q = \frac{1}{4\pi}\int_{S^2_\infty} *F$$

[8 marks]

(ii) Define, in general, a Killing horizon, $\mathcal{N}$, associated with a Killing vector ξ. Explain why we can write

$$\nabla_\mu(\xi^2)\big|_{\mathcal{N}} = -2\kappa\xi_\mu|_{\mathcal{N}}$$

where κ is the surface gravity, and hence

$$\xi^\mu\nabla_\mu\xi_\rho|_{\mathcal{N}} = \kappa\xi_\rho|_{\mathcal{N}}$$

[7 marks]

(iii) In ingoing Eddington-Finkelstein coordinates the R-N metric can be written

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dv^2 + 2dvdr + r^2 d\Omega_2$$

Give the definition of a future event horizon and then state its location for the R-N metric. Show that this event horizon is a Killing horizon.

Calculate the surface gravity κ as a function of M,Q. Also, calculate the area, A, of the event horizon as well as the potential difference between infinity and the horizon, Φ_H, expressing both as a function of M,Q.

Hence prove the First Law of black hole mechanics

$$dM = \frac{1}{8\pi}\kappa dA + \Phi_H dQ$$

for charged, non-rotating black holes, carefully stating what result you are using. [15 marks]

[Total 30 marks]

3. Let l^μ be the tangent vector field for a congruence of affinely parametrised null geodesics and let n^μ be a null vector field such that $n^\mu l_\mu = -1$ and $l^\mu \nabla_\mu n^\nu = 0$. Let η_1^μ and η_2^μ be two spacelike connecting vector fields for the congruence which are orthogonal to each other and also to l^μ and n^μ, and satisfy

$$l^\mu \nabla_\mu \eta_i^\nu = \eta_i^\mu \nabla_\mu l^\nu, \quad i = 1, 2$$

Let $P^\mu{}_\nu = \delta^\mu{}_\nu + l^\mu n_\nu + n^\mu l_\nu$.

(i) Show that $P^\mu{}_\nu$ projects onto the subspace of the tangent space spanned by the η_i^μ. Also prove that

$$l^\mu \nabla_\mu \eta_i^\nu = \hat{B}^\nu{}_\rho \eta_i^\rho, \quad i = 1, 2$$

where $\hat{B}^\nu{}_\rho = P^\nu{}_\alpha P^\beta{}_\rho \nabla_\beta l^\alpha$.

[6 marks]

(ii) Raychaudhuri's equation for the congruence is

$$\frac{d\theta}{d\lambda} = -\tfrac{1}{2}\theta^2 - \hat{\sigma}_{\mu\nu}\hat{\sigma}^{\mu\nu} + \hat{\omega}_{\mu\nu}\hat{\omega}^{\mu\nu} - R_{\mu\nu} l^\mu l^\nu$$

What is λ? Give the definitions of θ, $\hat{\sigma}_{\mu\nu}$ and $\hat{\omega}_{\mu\nu}$ and briefly state the geometric interpretation of these quantities. [6 marks]

(iii) If the congruence contains the generators of a null hypersurface $\mathcal{N}$, show that $\hat{B}_{\mu\nu} = \hat{B}_{(\mu\nu)}$ on $\mathcal{N}$. You may use Frobenius's theorem. [6 marks]

(iv) Now assume the congruence contains the generators of a Killing horizon $\mathcal{N}$, with Killing vector ξ^μ. Show that $\hat{B}_{\mu\nu} = 0$ on $\mathcal{N}$. Hence show that $R_{\mu\nu}\xi^\mu\xi^\nu = 0$ on $\mathcal{N}$. [6 marks]

(v) Consider a spacetime solving Einstein's equations. Assuming the null energy condition show that if at some point on a null generator of a null hypersurface we have $\theta < 0$ then $\theta \to -\infty$ within finite affine parameter. [6 marks]

[Total 30 marks]

4. (i) Consider the Klein-Gordon equation $(\nabla^2 - m^2)\phi = 0$ for a real scalar field in a globally hyperbolic spacetime with a Cauchy surface Σ. Define the Klein-Gordon bracket

$$(f, g) = i \int_\Sigma dS n^\mu f^* \overleftrightarrow{\nabla}_\mu g$$

(a) Show that on the space of solutions to the Klein-Gordon equation the bracket is the same for another Cauchy surface Σ' that asymptotically coincides with Σ.

(b) Describe the quantisation of this theory. Clearly explain why the notions of vacuum and particle states are, in general, ambiguous in a curved spacetime.

(c) For a stationary spacetime explain why there is a preferred notion of a vacuum state and particle states.

[15 marks]

(ii) Sketch out the main arguments in the derivation of Hawking radiation for a real scalar field satisfying the massless Klein-Gordon equation. You should clearly summarise the steps just leading up to using the geometric optics approximation and then conclude by briefly stating how one uses this to obtain the final result. [15 marks]

[Total 30 marks]

 End of examination paper