Imperial College London

MSc EXAMINATION May 2015

BLACK HOLES (ADVANCED GENERAL RELATIVITY)

For MSc students, including QFFF students

Wednesday, 13th May 2015: 14:00–17:00

Answer Question 1 (40%) and TWO out of Questions 2, 3 and 4 (30% each).
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the 3 answer books provided.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of the question on the front cover of its corresponding answer book.

Hand in 3 answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

Go to the next page for questions

1.

(a) (*5 marks*) The Schwarzschild black hole metric is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

in units in which the speed of light $c = 1$ and Newton's constant $G = 1$.

- What the physical interpretation of the parameter M?
- What is the metric in units in which c and G are not equal to 1 and appear explicitly in the metric?
- What symmetries does the spacetime have?

(b) (*12 marks*) Write down the (3+1)-dimensional flat Minkowski metric in spherical polar coordinates, (t, r, θ, ϕ). Define ingoing and outgoing radial null coordinates, transform the metric into these coordinates and state the ranges of these coordinates paying attention to the fact that $r \geq 0$.

Explain in detail how to derive the Penrose diagram for this spacetime by first doing a coordinate transformation to coordinates which have finite ranges, using the trigonometric tan function. Draw the Penrose diagram and label and explain its boundaries. What happens if you want to draw on the diagram the world line of a photon that comes in radially from infinity, goes through the origin of the original radial coordinates on a straight line and goes out to infinity again?

What physical aspect of spacetime does a Penrose diagram capture? Name one type of physical information which cannot be deduced from a Penrose diagram.

(c) (*13 marks*)

Consider a model for a collapsing star which is a spherically symmetric ball of dust of total mass M. Explain how the Schwarzschild metric is relevant to this model.

Sketch the Penrose diagram for the spacetime of the star collapsing to form a black hole. Show on the diagram: $r = 2M$ and $r = 0$, the surface of the star, the interior of the star, the black hole (interior, event horizon and singularity), the exterior region, $\mathcal{I}^{\pm}$ ("scri-plus" and "scri-minus"), $i_{\pm}$ and i_0, and the singularity. Explain the *physical meaning* of all these features.

Explain, using your diagram, why an astronaut who falls across the event horizon cannot avoid the singularity.

(d) (*10 marks*)

In a thought experiment, an astronaut falls radially and feet first to earth from the top floor of the Imperial College Blackett Laboratory. In a second thought experiment, the same astronaut falls radially and feet first into a Schwarzschild black hole. Assuming that the tidal force, between head and feet, on the astronaut falling from the Blackett lab would be equal to the tidal force on the astronaut as

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he/she falls across the horizon of the black hole, estimate the order of magnitude of the mass of the black hole. Use Newtonian gravity (not General Relativity) in your estimation of the tidal forces in both cases.

What is the approximate Schwarzschild radius of this black hole in metres?

You may use these approximate values:

$$\begin{aligned} G &= 6.7 \times 10^{-11} N\, m^2 kg^{-2} \\ c &= 3 \times 10^8 ms^{-1} \\ \text{mass of earth} &= 6 \times 10^{24} kg \\ \text{radius of earth} &= 6 \times 10^6 m \,. \end{aligned}$$

2.

(a) (*10 marks*)

A free, massive, real scalar field, ϕ, in a globally hyperbolic spacetime with metric $g_{\mu\nu}$ satisfies the Klein Gordon equation

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi - m^2\phi = 0\,.$$

Describe how this free field theory may be quantised using the fact that there is a Hilbert space of complex solutions of this equation with Klein Gordon inner product

$$(f,g)_{KG} = i\int_\Sigma dS^\mu\left(f^*\frac{\partial}{\partial x^\mu}g - g\frac{\partial}{\partial x^\mu}f^*\right),$$

where Σ is a Cauchy surface. Explain why the notions of "vacuum state" and "particle" are in general ambiguous in a curved spacetime.

Now suppose the spacetime is stationary. Explain how this allows the identification of physically meaningful positive frequency modes and show how one can now quantise the field and define a quantum state which is a physically meaningful vacuum state of the field.

(b) (*8 marks*) Suppose a spacetime (M,g) has the form of a "sandwich": there are two non-intersecting Cauchy surfaces, Σ_1 and Σ_2, such that the spacetime is stationary to the past of Σ_1 and to the future of Σ_2, and in between it is time dependent.

Describe how this may lead to particle production. Derive the formula, in terms of Bogoliubov coefficients, for the expectation value of the number of particles in a certain mode as measured by a stationary observer in the far future, in the vacuum state as defined by a stationary observer in the far past?

(c) (*4 marks*)

With reference to a relevant Penrose diagram, explain why the sandwich spacetime example in part (b) is relevant to to the process in which a black hole is formed from gravitational collapse if there is a free scalar quantum field in the spacetime.

(d) (*8 marks*)

A finite reservoir of radiation at temperature T and fixed volume V has energy

$$E_{\rm res} = \sigma V T^4$$

where σ is a constant.

Consider an uncharged, non rotating black hole in a reservoir of fixed volume V. Initially, let the mass of the black hole be M, and let the reservoir have the same initial temperature as the black hole, *i.e.* the Hawking temperature $T = \frac{1}{8\pi M}$ (in units in which $G = c = \hbar = 1$ and in which Boltzmann's constant is equal to one).

Let the total energy of the system $E_{\rm tot} = E_{\rm res} + M$ be constant. Find the critical value, V_c, of the volume of the reservoir below which the black hole is in stable thermal equilibrium with the reservoir and above which the black hole is in unstable equilibrium. Express V_c in terms of $E_{\rm tot}$.

3.

(a) (*8 marks*)

Let t^μ be the tangent vector field of an affinely parametrised null geodesic congruence and let n^μ be a null vector field such that $n^\mu t_\mu = -1$ and $t^\mu \nabla_\mu n^\nu = 0$ *i.e.* n^μ is parallely propagated along the congruence. Let η_1^μ and η_2^μ be two spacelike connecting vector fields for the congruence which are orthogonal to each other and to t^μ and n^μ and which satisfy

$$t^\mu \nabla_\mu \eta_i^\nu = \eta_i^\mu \nabla_\mu t^\nu \,, \quad i = 1, 2 \,.$$

Let $P^\mu{}_\nu = \delta^\mu{}_\nu + t^\mu n_\nu + n^\mu t_\nu$.

Prove that

$$t^\mu \nabla_\mu \eta_i^\nu = \hat{B}^\nu{}_\rho \eta_i^\rho \,, \quad i = 1, 2 \,,$$

where $\hat{B}^\mu{}_\rho = P^\mu{}_\alpha P^\beta{}_\rho \nabla_\beta t^\alpha$.

Raychaudhuri's equation for the congruence is

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \hat{\sigma}^{\mu\nu}\hat{\sigma}_{\mu\nu} + \hat{\omega}^{\mu\nu}\hat{\omega}_{\mu\nu} - R_{\mu\nu} t^\mu t^\nu \,.$$

What is λ? Give the definitions of θ, $\hat{\sigma}_{\mu\nu}$ and $\hat{\omega}_{\mu\nu}$ in terms of $\hat{B}_{\mu\nu}$ and interpret these quantities geometrically.

(b) (*10 marks*)

The future event horizon, H^+, of a stationary black hole is the Killing horizon of a future pointing Killing vector ξ^μ and is a null geodesic congruence. Using part (a) and assuming Frobenius' theorem, show that $\hat{B}_{\mu\nu} = \hat{B}_{(\mu\nu)}$ on H^+ (where the braces ( ) around the indices denote symmetrisation). Hence show that $\hat{B}_{\mu\nu} = 0$ and $\theta = 0$ on H^+. Hence show that $R_{\mu\nu}\xi^\mu\xi^\nu = 0$ on H^+.

(c) (*4 marks*)

Prove that if a vector field v^μ is tangent to a Killing horizon of Killing vector ξ^μ, then v^μ cannot be timelike. Hint: for each point p on the horizon, work in an orthonormal basis of the tangent space at p in which the metric is the Minkowski metric, $g_{\mu\nu}(p) = \mathrm{diagonal}(-1, 1, 1, 1)$, and in which the Killing vector has components $\xi^\mu = (1, 1, 0, 0)$ at p.

(d) (*4 marks*) The Dominant Energy Condition on a non-zero energy momentum tensor, $T_{\mu\nu}$, states that the vector $w^\mu = -T^\mu{}_\nu v^\nu$ is future pointing and non-spacelike for any vector v^μ which is future pointing and non-spacelike. (Non-spacelike means timelike or null.)

Assuming that the stationary black hole spacetime satisfies the Einstein equations and $T_{\mu\nu}$ satisfies the Dominant Energy Condition, use parts (b) and (c) to show that $-T^\mu{}_\nu \xi^\nu$ is proportional to ξ^μ on H^+.

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(e) (*4 marks*) The zeroth law of black hole mechanics states that the surface gravity κ is constant on the future event horizon of a stationary black hole satisfying the Einstein equations and the Dominant Energy Condition.

Given that $\xi_{[\mu} R_{\nu]}{}^{\rho} \xi_{\rho} = -2\xi_{[\mu} \partial_{\nu]} \kappa$ (where the square braces [] around indices denote antisymmetrisation) on H^{+}, and using the result of part (d), prove the zeroth law of black hole mechanics.

You may assume that if v and u are vectors and $v_{[\alpha} u_{\beta]} = 0$ and v is nonzero then u is proportional to v.

4. The Reissner-Nordstrom solution of the Einstein-Maxwell equations has metric

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_2^2,$$

and electric potential $A_t = \frac{Q}{r}$, where $Q > 0$, and all other components of A_μ are equal to zero.

(a) (*4 marks*) With reference to the metric, explain briefly why there are different physical interpretations of the solution depending on the relative sizes of M and Q.

(b) (*18 marks*) The First Law of Black Hole Mechanics for a charged, non-rotating black hole is

$$dM = \frac{1}{8\pi}\kappa dA + \Phi_H dQ,$$

where A is the area of the horizon, κ is the surface gravity of the horizon and Φ_H is the horizon potential, which is the difference between the electric potential at the horizon and at infinity.

By transforming to ingoing null radial Eddington-Finkelstein coordinates, calculate the surface gravity κ as a function of M and Q.

Calculate A and Φ_H for the Reissner-Nordstrom black hole as functions of M and Q.

Hence prove the First Law of Black Hole Mechanics, carefully stating what result you are assuming.

(c) (*8 marks*) What is the "Generalised Second Law" of Black Hole Thermodynamics?

Consider a charged, non-rotating Black Hole of initial mass M and assume it evaporates completely by Hawking radiation into massless particles. Working in units in which $G = c = \hbar = 1$ and Boltzmann's constant is also equal to 1, what is the initial entropy of the system? By considering the typical energy of a particle emitted by a black hole of Hawking temperature $T = \frac{1}{8\pi M}$, estimate of the total number of Hawking particles emitted by the black hole over its lifetime and hence estimate the total entropy of the resultant Hawking radiation.

 End of examination paper