Imperial College London

MSc EXAMINATION May 2016

BLACK HOLES (ADVANCED GENERAL RELATIVITY)

For MSc students, including QFFF students

Monday, 9th May 2016: 14:30–17:30

Answer Question 1 (40%) and TWO out of Questions 2, 3 and 4 (30% each).
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the 3 answer books provided.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of the question on the front cover of its corresponding answer book.

Hand in 3 answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

Go to the next page for questions

1. (i) (*5 marks*)

The Schwarzschild metric for a black hole of mass M in Schwarzschild coordinates (t, r, θ, ϕ) is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

in units in which the speed of light $c = 1$ and Newton's constant $G = 1$.

Write down two Killing vectors of the metric and describe the symmetries that these correspond to. Does the metric have any other symmetries?

(ii) (*6 marks*)

Define the *ingoing* radial null Eddington-Finkelstein (EF) coordinate, v and calculate the Schwarzschild metric in terms of (v, r, θ, ϕ). Find the inverse metric. Hence show that $r = 2M$ is a coordinate singularity.

How do we know that the singularity at $r = 0$ is not a coordinate singularity?

(iii) (*16 marks*)

(a) Explain briefly what a Penrose diagram is, what physical information it captures and what physical information it does not convey.

(b) Draw the Penrose diagram for the maximally extended Schwarzschild spacetime, also known as the Kruskal spacetime. Show on the diagram: the black hole (interior, horizon and singularity) the white hole (interior, horizon and singularity), the exterior regions, $\mathcal{I}^{\pm}$ ("scri-plus and scri-minus"), $i_{\pm}$ and i_0, $r = 2M$ and $r = 0$.

(c) Explain the physical meaning of *all* the features you have labelled on the diagram.

(d) Where in the Kruskal spacetime are Schwarzschild coordinates defined? Where in the Kruskal spacetime are ingoing EF coordinates defined?

(e) Which features of this spacetime will *not* be present in the spacetime of a black hole formed by gravitational collapse of a star?

(iv) (*7 marks*)

Draw a copy of the Penrose diagram of the Kruskal spacetime (you don't need to copy all the labels).

(a) Explain, using this diagram, why a massless or massive particle inside the black hole cannot get out and why it must inevitably reach the singularity. Explain, using this diagram, why a particle inside the white hole must leave the white hole and why a particle outside it cannot get inside.

(b) Explain qualitatively, using another copy of the relevant part of this diagram, why an observer far from the black hole sees light from a object falling into the black hole become arbitrarily highly redshifted as the object nears the horizon.

(v) (*6 marks*)

[This question continues on the next page . . .]

Two astronauts, Aliya and Biff, both fall freely from rest, radially into a very large Schwarzschild black hole of mass M.

Intially, they are both together, following a stationary trajectory at constant radial coordinate $r = r_0 > 2M$ and at the same angular position $\theta = \theta_0, \phi = \phi_0$. Aliya starts to fall first. Biff waits at $r = r_0$ for some time after Aliya starts falling and then falls in after Aliya.

Sketch a Penrose diagram of the relevant portion of the black hole spacetime showing the worldlines of Aliya and Biff. Does Biff lose sight of Aliya as Aliya crosses the event horizon? Justify your answer using the diagram.

2. The vacuum solution for a 5-dimensional spherically symmetric black hole with mass parameter μ is the Schwarzschild-Tangherlini metric:

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega_3^2$$

where

$$f(r) = 1 - \frac{\mu}{r^2},$$

and $d\Omega_3^2$ is the round metric on the 3-sphere

$$d\Omega_3^2 = d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2).$$

The subquestions below refer to this black hole spacetime.

(i) (*15 marks*) State the definitions of a null hypersurface, a Killing vector field and a Killing horizon.

Prove that $r = \sqrt{\mu}$ is a null hypersurface and a Killing horizon. Calculate the surface gravity, κ, of this horizon.

(ii) (*9 marks*)

(a) The Stefan-Boltzmann law for the power, P, radiated by a hot body at temperature T in 4 spacetime dimensions is

$$P = \sigma_4 A T^4,$$

where A is the surface area of the body and σ_4 is a constant.
Given that $\sigma_4 = \zeta \hbar^p k^q c^r$ where ζ is a dimensionless constant, k is Boltzmann's constant and c is the speed of light, find the integers p, q and r. (Hint: the average kinetic energy of a particle in an ideal gas of temperature T is equal to a dimensionless constant times kT.) Why does this tell us that this is a relativistic, quantum law?

(b) The surface gravity of a 4-d Schwarzschild black hole of mass M is $\kappa = \frac{1}{4M}$. Show that, if a black hole loses energy as it radiates according to Stefan's law and if it can be approximated by a Schwarzschild black hole at all times, the time it takes to radiate away all its mass is proportional to M_0^3, where M_0 is the original mass.

(c) Assuming that the exact same relationship holds in 5 dimensions between the temperature of a black hole and its surface gravity as in 4 dimensions, what is the temperature of the 5 dimensional black hole?
Suppose that in 5 dimensions the power radiated by a hot body of temperature T is proportional to T^5. Given that the mass parameter μ is proportional to the mass M of a 5-d black hole – $\mu = \xi M$ where ξ is a constant, show that the time a 5-d black hole of original mass M_0 takes to radiate away all its mass is proportional to M_0^α. What is the value of α?

[This question continues on the next page . . .]

(iii) (*6 marks*) Let $\tau = it$ be Euclidean (or imaginary) time in which the metric becomes

$$ds^2 = +f(r)d\tau^2 + f(r)^{-1}dr^2 + r^2 d\Omega_3^2 \,,$$

with signature $(+,+,+,+)$.

Consider coordinate r' where

$$r' = r - \sqrt{\mu} \,, \qquad r' > 0$$

and expand the metric in small r' to find the leading order behaviour of the metric close to the horizon. Find a new coordinate ρ as a function of r' in which the leading order behaviour of the metric is

$$ds^2 = 2\sqrt{\mu}\left[d\rho^2 + \rho^2 \frac{d\tau^2}{\mu}\right] + \mu d\Omega_3^2 + \ldots \,.$$

Explain why considering τ as a periodic coordinate with a particular period makes this geometry nonsingular at $\rho = 0$.

Why is this result physically consistent with the black hole radiating with the temperature calculated previously?

3. The Painlevé-Gullstrand (PG) metric in Painlevé-Gullstrand (PG) coordinates $\{T, R, \theta, \phi\}$ is

$$ds^2 = -dT^2 + (dR + \sqrt{\frac{2Z}{R}}dT)^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2)\,,$$

where $Z > 0$ is a constant. The spacetime is spherically symmetric and a solution of the vacuum Einstein equations. It is not flat.

(i) (5 marks) Calculate the inverse metric and show the metric is regular for *all* values of $R > 0$. What is the metric of a T = constant hypersurface? Show that this hypersurface is spacelike.

(ii) (2 marks) Explain why Birkoff's theorem means that the PG metric must be a Schwarzschild black hole in alternative coordinates.

(iii) (8 marks) The event horizon is located at $R = R_0$. Find the value of R_0 and prove, using only the metric given above in PG coordinates, that the surface $R = R_0$ is an event horizon. In particular, prove that if a massive or massless particle is in region $R < R_0$ then it can never reach region $R \geq R_0$. Hint: T = constant surfaces are spacelike everywhere and so T must increase along any future pointing worldline.

Given that the angular coordinates θ and ϕ of the PG coordinates are equal to the angular Schwarzschild coordinates θ and ϕ, what is the PG parameter Z in terms of the mass of the black hole M?

(iv) (*15 marks*)

Consider a radial (*i.e.* constant θ and ϕ) worldline, γ, in the PG coordinates along which $dR = -\sqrt{\frac{2Z}{R}}dT$.

(a) Show that γ is a timelike trajectory.

(b) Show that T is proper time along γ.

(c) Show that γ is a geodesic (you can assume the θ and ϕ Euler Lagrange equations hold for radial motion).

(d) Show that the tangent vector of γ is normal to the T = constant hypersurfaces.

(e) Show that the total proper time taken for a massive particle to fall freely along one of these geodesics radially in from initial radial coordinate $R = R_{\max}$ to $R = 0$ is finite. Calculate the value of this proper time. [Note: the particle does not fall from rest at $R = R_{\max}$.]

4. Consider the Kerr metric for a rotating black hole of mass M and angular momentum $J = Ma$,

$$ds^2 = -\frac{(\Delta - a^2 \sin^2\theta)}{\Sigma} dt^2 - 2a\sin^2\theta \frac{(r^2 + a^2 - \Delta)}{\Sigma} dt\, d\phi$$
$$+ \frac{((r^2 + a^2)^2 - \Delta a^2 \sin^2\theta)}{\Sigma} \sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2$$

where

$$\Sigma = r^2 + a^2 \cos^2\theta$$
$$\Delta = r^2 - 2Mr + a^2$$

and $M > a$.

(i) (*5 marks*) Show that the Killing vector $k = \frac{\partial}{\partial t}$ is timelike for large enough r and any value of the angular coordinates. Show also that k is spacelike in a region, called the ergoregion, outside the horizon. Derive the equation for the boundary of the ergoregion and sketch the ergoregion and show its relation to the event horizon.

(ii) (*15 marks*)

(a) Consider a process in which a black hole, well approximated by a Kerr metric with mass M_1 and angular momentum J_1 changes to another black hole well approximated by a Kerr metric with mass M_2 and angular momentum J_2. Assuming that the Second Law of Black Hole Mechanics holds, prove that if $J_1 = J_2 = 0$ then $M_2 \geq M_1$, but if $J_1 > 0$ the black hole's mass can decrease.

(b) Describe the Penrose process for extraction of energy from a Kerr black hole with initial mass M and angular momentum J.
What is the limiting minimum mass that the resulting black hole can have if the initial mass and angular momentum are M and J? Deduce the limiting maximum amount of energy that can be extracted from the black hole.

(iii) (*10 marks*)

(a) In the event observed by the LIGO experiment on Sept 14th 2015, two black holes merged to form a single black hole. The data is consistent with predictions from General Relativity if the final black hole is a Kerr black hole of mass $M = 62M_s$, where M_s is the mass of the sun, and the angular momentum per unit mass is $a = 0.67M$. The best fit values of the masses of the two initial black holes are $M_1 = 29M_s$ and $M_2 = 36M_s$.
Assume that initially the two black holes are far enough apart that they can each be well-approximated by a Kerr black hole. Show that no matter what initial angular momenta they each had, this event does not violate the Second Law of black hole mechanics.

(b) Will we ever observe an event in which a single uncharged, non-rotating black hole splits into two uncharged black holes? Justify your answer.