Imperial College London

MSc EXAMINATION May 2017

BLACK HOLES

For MSc students, including QFFF students

Thursday, 4th May 2017: 14:30–17:30

Answer Question 1 (40%) and TWO out of Questions 2, 3 and 4 (30% each).
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the 3 answer books provided.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of the question on the front cover of its corresponding answer book.

Hand in 3 answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

Go to the next page for questions

1. (i) (*6 marks*) The Schwarzschild metric for a black hole of mass M in Schwarzschild coordinates (t, r, θ, ϕ) is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Write down two Killing vectors of the metric and describe the symmetries that these correspond to. Does the metric have any other symmetries?

(ii) (*16 marks*) Draw the Penrose diagram for the maximally extended Schwarzschild spacetime, also known as the Kruskal spacetime.

(a) For *one* of the asymptotically flat regions label $\mathcal{I}^{\pm}$ ("scri-plus "and "scri-minus"), $i_{\pm}$ and i_0. With respect to this asymptotically flat region show: the black hole region, including the horizon and singularity, as well as the white hole region, including the horizon and singularity. Mark the location of $r = 2M$ and $r = 0$.

(b) By labelling the regions of the diagram, show in which parts of the Kruskal diagram are the ingoing Eddington-Finklestein coordinates, (v, r, θ, ϕ), the outgoing Eddington-Finklestein coordinates, (u, r, θ, ϕ), and the Kruskal coordinates, (U, V, θ, ϕ), defined.

(c) Draw two lines of constant t on the Penrose diagram. Explain why the geometry of each of these surfaces is an Einstein-Rosen bridge with a minimal S^2. Also give the radius of the minimal S^2.

(d) Draw the Penrose diagram for the spacetime of a black hole formed by the spherically symmetric gravitational collapse of a star. Show which part of the Penrose diagram for the Kruskal spacetime is associated to the spacetime exterior to the collapsing star.

(iii) (*6 marks*) Calculate the entropy of the Schwarzschild black hole in terms of M. Incorporating quantum theory, the black hole will evaporate, decreasing the mass. Briefly explain why this does not violate the generalised Second Law of Thermodynamics.

(iv) (*6 marks*) Consider the Kerr metric with parameters M, a in Boyer-Lindquist coordinates (t, r, θ, ϕ). For $M > a$, draw a Penrose diagram for the non-singular totally geodesic slice $\theta = 0$ and constant ϕ. Draw a spatial surface Σ and the corresponding Cauchy horizons. Briefly comment on why this implies that most of the Penrose diagram is expected to be unphysical.

(v) (*6 marks*) If V^μ is a Killing vector, prove that $\nabla_\nu \nabla_\mu V_\rho = R_{\rho\mu\nu\sigma} V^\sigma$.

2. The five-dimensional spherically symmetric Schwarzschild black hole with parameter $\mu > 0$ is given in Schwarzschild coordinates $(t, r, \chi, \theta, \phi)$ by

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega_3^2$$

where

$$f(r) = 1 - \frac{\mu}{r^2},$$

and $d\Omega_3^2$ is the round metric on the 3-sphere given by

$$d\Omega_3^2 = d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2),$$

with $0 \leq \chi \leq \pi$, $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$.

The subquestions below refer to this black hole spacetime.

(i) (*6 marks*) Write the metric in ingoing Eddington-Finklestein coordinates $(v, r, \chi, \theta, \phi)$ where $v = t + r_*$ with $dr_* = f^{-1}dr$. Show that metric is non-singular at $r = \sqrt{\mu}$. Also show that the Killing vector $k = \partial_t$ in Schwarzschild coordinates, is given by $k = \partial_v$ in the ingoing Eddington-Finklestein coordinates.

(ii) (*6 marks*) Define a causal vector. Define a time-orientation. Given a time orientation, define a future directed causal vector.

Show that in the Eddington-Finklestein coordinates $T = -\partial_r$ defines a time orientation that agrees with that defined by the Killing vector k for $r > \sqrt{\mu}$.

(iii) (*5 marks*) Let $x^\mu(\lambda)$ be a future directed causal curve. If $r(\lambda_0) < \sqrt{\mu}$ then show $r(\lambda) < \sqrt{\mu}$ for all $\lambda \geq \lambda_0$.

(iv) (*5 marks*) For a massive particle with $r < \sqrt{\mu}$ calculate the maximum proper time that elapses before it reaches $r = 0$.

(v) (*8 marks*) Working in Schwarzschild coordinates calculate the Komar integral

$$-\int_{S^3_\infty} *dk$$

where in this expression k is the one-form associated with the Killing vector $k = \partial_t$. You should take the orientation $\epsilon_{tr\chi\theta\phi} = \sqrt{-g}$.

3. (i) (*10 marks*) Consider a family of hypersurfaces defined by the condition $S = constant$ for some smooth function $S(x^\mu)$. Explain why the vector $n^\mu = f g^{\mu\nu}\partial_\nu S$, for some function f, is normal to the hypersurfaces. Assume now that we have a specific *null* hypersurface $\mathcal{N}$ with normal vector which we write as $l^\mu = f g^{\mu\nu}\partial_\nu S$ with $l^2 = 0$. Explain why l^μ is tangent to null curves in $\mathcal{N}$ and show that these curves are geodesics.

(ii) (*4 marks*) Assume now that $\mathcal{N}$ is a Killing horizon for a Killing vector field ξ. Explain why we can write

$$\nabla_\mu(\xi^2)\big|_{\mathcal{N}} = -2\kappa\xi_\mu|_{\mathcal{N}}$$

where κ is the surface gravity.

(iii) (*16 marks*) Consider the non-extremal Reissner-Nordström black hole solution with parameters M, Q with $M > Q$. The metric is given by

$$ds^2 = -\frac{\Delta}{r^2}dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

where $\Delta = r^2 - 2Mr + Q^2 \equiv (r - r_+)(r - r_-)$ and vector potential $A_v = -Q/r$.

(a) Show that $r = r_+$ is a null hypersurface and a Killing horizon. Show that the surface gravity of this Killing horizon is $\kappa = (r_+ - r_-)/(2r_+^2)$ and express this in terms of M, Q.

(b) Define the area A of the event horizon at $r = r_+$ and calculate it in terms of M, Q.

(c) Calculate Φ_H, the potential difference between infinity and the horizon, in terms of M, Q.

(d) Use these results to prove the first Law of black hole mechanics for a charged, non-rotating black hole

$$dM = \frac{1}{8\pi}\kappa dA + \Phi_H dQ$$

carefully stating what result you are assuming.

4. Consider a real, free scalar field ϕ in a globally hyperbolic spacetime that satisfies the Klein-Gordon equation

$$(\nabla^2 - m^2)\phi = 0$$

(i) (*5 marks*) Let Σ be a Cauchy surface with future pointing normal n^μ. For two solutions f, g of the Klein-Gordon equation define the Klein-Gordon product by

$$(f, g) = i \int_\Sigma dS n^\mu f^* \overleftrightarrow{\nabla}_\mu g$$

Show that this definition does not depend on the choice of Σ.

(ii) (*5 marks*) Now consider the quantum theory and write

$$\begin{aligned}\phi &= \sum_i a_i \psi_i + a_i^\dagger \psi_i^* \\ &= \sum_i a_i' \psi_i' + a_i'^\dagger \psi_i'^*\end{aligned}$$

where ψ_i, ψ_i' are an appropriately normalised complex basis of solutions to the Klein Gordon equation and

$$\begin{aligned}[a_i, a_j^\dagger] &= [a_i', a_j'^\dagger] = \delta_{ij} \\ [a_i, a_j] &= [a_i', a_j'] = 0\,.\end{aligned}$$

Assuming

$$a_i' = \sum_j \alpha_{ij} a_j + \beta_{ij} a_j^\dagger$$

give conditions on α_{ij}, β_{ij} to ensure that a_i', $a_i'^\dagger$ satisfy the above commutation relations given that a_i, $a_i^\dagger$ do.

(iii) (*6 marks*) Writing

$$a_i = \sum_j \gamma_{ij} a_j' + \rho_{ij} a_j'^\dagger$$

obtain expressions for γ_{ij}, ρ_{ij} in terms of α_{ij}, β_{ij}.

(iv) (*5 marks*) Explain how, using the Klein Gordon product, one can obtain α_{ij}, β_{ij} from the relation between the solutions ψ_i and ψ_i'.

(v) (*4 marks*) Given the system is in the state $|0\rangle$ with $a_i|0\rangle = 0$, for all i, calculate $\langle 0|a_i'^\dagger a_i'|0\rangle$ and interpret the result.

(vi) (*5 marks*) Illustrate your results by briefly describing particle production in a sandwich spacetime; namely one that is stationary in the past and the future.

 End of examination paper