Imperial College London

MSc EXAMINATION May 2013

BLACK HOLES (ADVANCED GENERAL RELATIVITY)

For MSc students, including QFFF students

Wednesday, 15th May 2013: 14:00–17:00

Answer Question 1 (40%) and TWO out of Questions 2, 3 and 4 (30% each).
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the 3 answer books provided.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of the question on the front cover of its corresponding answer book.

Hand in 3 answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

© Imperial College London 2013

Go to the next page for questions

1.

(a) (*4 marks*) Write down the Schwarzschild metric for a black hole of mass M in Schwarzschild coordinates (t, r, θ, ϕ) in units in which the speed of light $c = 1$ and Newton's constant $G = 1$.

What is the metric with the G and c dependence put in explicitly?

(c) (*14 marks*) Define the ingoing null Eddington-Finkelstein (EF) coordinate, v and calculate the Schwarzschild metric in terms of (v, r, θ, ϕ). Prove that $r = 2M$ is a coordinate singularity of this metric.

Prove that in the spacetime described by the metric in ingoing EF coordinates, no future directed timelike worldline that starts in region $r < 2M$ can leave that region. (Future directed means $dv > 0$).

Now define the *o*utgoing Eddington-Finkelstein coordinate u and calculate the Schwarzschild metric in terms of (u, r, θ, ϕ). Show that $r = 2M$ is a coordinate singularity of this metric and prove that in the spacetime described by the metric in outgoing EF coordinates, every future directed timelike or worldline that starts in region $r < 2M$ must eventually leave that region. (Future directed means $du > 0$.)

(d) (*14 marks*) Draw the Penrose diagram for the maximally extended Schwarzschild spacetime, in other words the Kruskal spacetime. Show on the diagram: the black hole (interior, horizon and singularity) the white hole (interior, horizon and singularity), the exterior regions, $\mathcal{I}^{\pm}$ ("scri-plus and scri-minus"), $i_{\pm}$ and i_0, $r = 2M$ and $r = 0$. Explain the physical meaning of all these features. Also draw an ingoing and an outgoing null geodesic.

Indicate on the diagram where the wormhole (or Einstein-Rosen bridge) is and describe its geometry, illustrating your answer with a sketch. Would such a wormhole form when a black hole forms from the collapse of a star? Explain where in the Kruskal spacetime ingoing EF coordinates are defined and where outgoing EF coordinates are defined. Hence explain why the results of part (c) are not contradictory.

(f) (*8 marks*) There are three small spaceships, A, B and C. A remains at a fixed large radius, far away from a very big Schwarzschild black hole. B and C both fall freely from rest from the same radial coordinate $r = R > 2M$, B first and then C, and both fall into the black hole. All three spaceships are at the same fixed angular coordinates, θ and ϕ. B sends signals by sending photons at regular intervals of proper time along its worldline.

Sketch their three worldlines on a Penrose diagram. And sketch some representative signal photon worldlines between them.

Describe what A sees of B, as B falls into the black hole. Describe what C sees of B, as B falls into the black hole.

2. Raychaudhuri's equation for null geodesic congruences is

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \hat{\sigma}^{\mu\nu}\hat{\sigma}_{\mu\nu} + \hat{\omega}^{\mu\nu}\hat{\omega}_{\mu\nu} - R_{\mu\nu}l^\mu l^\nu \, ,$$

where l^μ is the tangent field along the congruence, θ is the expansion, $\hat{\sigma}$ is the shear, $\hat{\omega}$ is the twist and $R_{\mu\nu}$ is the Ricci tensor,.

(a) (*10 marks*) State the Weak Energy Condition.

Assuming that the tangent vector to the null geodesic congruence is everywhere normal to a family of null hypersurfaces and that the spacetime is a solution of Einstein's equations in which the energy momentum tensor satisfies the Weak Energy Condition, show that

$$\frac{d\theta}{d\lambda} \leq -\frac{1}{2}\theta^2 \, .$$

Hence prove that if $\theta < 0$ at any point on a null generator of a null hypersurface θ diverges to infinity within a finite affine parameter along that null generator.

(You may assume that $\hat{\omega}_{\mu\nu}$ vanishes for null hypersurfaces.)

(b) (*10 marks*) State the black hole Area Theorem, also known as the "Second Law of Black Hole Mechanics". Sketch a proof of the Theorem.

You can use results you do not prove if you state them clearly.

(c) (*10 marks*)

Consider two widely separated nonrotating black holes of masses M_1 and M_2. They merge to form a single black hole. What is the maximum energy that could be radiated during this process?

The area of the horizon of a black hole with electric charge Q and mass M is given by

$$A = 4\pi r_+^2$$

where $r_+ = M + \sqrt{M^2 - Q^2}$. Can a single charged black hole split into two charged black holes? Explain your answer.

3.

(a) (*7 marks*) What does it mean for a spacetime to be stationary?

A free massive real scalar field, ϕ satisfies the Klein Gordon equation

$$\nabla^2\phi - m^2\phi = 0\,.$$

Describe how this field theory is quantised in a stationary spacetime with a Cauchy surface. Define the vacuum state. Why does it make sense to call it *the* vacuum state in a stationary spacetime?

(b) (*8 marks*) Suppose a spacetime (M, g) has the form of a sandwich: there are two non-intersecting Cauchy surfaces, Σ_1 and Σ_2 such that the spacetime is stationary to the past of Σ_1 and to the future of Σ_2, and in between it is time dependent.

Describe how this may lead to particle production. Derive the formula, in terms of Bogoliubov coefficients, for the expectation value of the number of particles in a certain mode as measured by an observer in the far future, in the vacuum state as defined by an observer in the far past?

(c) (*6 marks*) Explain why the sandwich spacetime example above is relevant to the derivation of late time Hawking radiation in the spacetime of gravitational collapse to a black hole? Illustrate your answer with a Penrose diagram.

Briefly explain what the discovery of black hole radiation means for the Laws of Black Hole Mechanics.

(c) (*9 marks*)

Consider an uncharged spherically symmetric black hole which starts with initial mass M and radiates with Hawking radiation. Assume that it evaporates completely in a finite time τ. Assuming that τ is proportional to a power of M, $\tau = \alpha M^b$, is the exponent b positive or negative? Find the dependence of the constant α on b, Newton's constant G, the speed of light c and Planck's constant $\hbar$ by dimensional arguments. What else can you deduce about b given that this a quantum process?

Assume that Stefan's Law for the power radiated by a hot body holds for the black hole, in other words the power is proportional to the fourth power of the temperature, T, and to the surface area, A, of the body:

$$Power = \sigma A T^4$$

where σ is a constant of order 1 in units where $c = \hbar = k_B = 1$ and k_B is Boltzmann's constant. Ignoring the potential barrier around the black hole, and assuming it evaporates completely, estimate the lifetime of the black hole, τ, and determine the exponent b.

Why does this process not violate the Second Law of Black Hole Mechanics (the Area Theorem)? Why can we not be sure what happens to the black hole in the final stages of the evaporation?

4. The metric for a Schwarzschild-anti-de Sitter black hole in 5 dimensions is:

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega_3^2$$

where L is the radius of curvature of the anti-de Sitter spacetime to which this black hole metric asymptotes at large r and

$$f(r) = \frac{r^2}{L^2} + 1 - \frac{r_0^2}{r^2},$$

and r_0 is a constant.

$d\Omega_3^2$ is the round metric on the 3-sphere

$$d\Omega_3^2 = d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2).$$

(a) (*4 marks*) For this spacetime define the analogues of ingoing and outgoing Eddington-Finkelstein coordinates, v and u, respectively. (Do not solve explicitly for the intermediate coordinate r_*.)

(b) (*12 marks*) Define a Killing horizon.

The black hole horizon is at $r = r_+$ where r_+ is the positive real root of the polynomial $r^2 f(r)$. Show that the black hole horizon is a Killing horizon of the Killing vector $k = \frac{\partial}{\partial v} = \frac{\partial}{\partial t}$.

Using the formula for the surface gravity

$$\kappa^2 = \lim_{r \to r_+} g^{\mu\nu}\nabla_\mu V \nabla_\nu V$$

where $V^2 = -g_{\mu\nu}k^\mu k^\nu$, prove that the surface gravity is

$$\kappa = \frac{2r_+}{L^2} + \frac{1}{r_+}.$$

(c) (*3 marks*) Assuming the same relation between the surface gravity and the temperature of the black hole holds as for asymptotically flat 4 dimensional black holes, what is the temperature? What calculation might one do to check that this is the physical temperature?

(d) (*11 marks*) What does it mean for a body to have positive or negative specific heat? For a black hole in anti-de Sitter space with fixed L and variable mass, there is a critical mass M_c, where the specific heat changes from positive to negative. Conjecture whether the anti-de Sitter black hole has negative specific heat for M smaller or larger than M_c and explain your reasoning briefly given what you know about the Schwarzschild black hole.

Assuming that the mass of the anti-de Sitter black hole is $M = ar_0^2$ where a is a constant, prove your conjecture, and calculate the critical mass M_c in terms of L. What is the temperature of a black hole with this critical mass?

 End of examination paper