Imperial College London

MSc EXAMINATION May 2014

BLACK HOLES (ADVANCED GENERAL RELATIVITY)

For MSc students, including QFFF students

Wednesday, 14th May 2014: 14:00–17:00

Answer Question 1 (40%) and TWO out of Questions 2, 3 and 4 (30% each).
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the 3 answer books provided.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of the question on the front cover of its corresponding answer book.

Hand in 3 answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

Go to the next page for questions

1.

(a) (*6 marks*) Write down the Schwarzschild metric for a black hole of mass M in Schwarzschild coordinates (t, r, θ, ϕ) in units in which the speed of light $c = 1$ and Newton's constant $G = 1$. Write down two Killing vectors of the metric and describe the physical symmetries that these correspond to. Does the metric have any other symmetries?

(b) (*10 marks*) Define the *ingoing* radial null Eddington-Finkelstein (EF) coordinate, v and calculate the Schwarzschild metric in terms of (v, r, θ, ϕ). Find the inverse metric. Hence show that $r = 2M$ is only a coordinate singularity. Explain how we know that $r = 0$ is not a coordinate singularity. Does the metric have any other singularities?

Define the *outgoing* radial null Eddington-Finkelstein coordinate u and calculate the Schwarzschild metric in terms of (u, r, θ, ϕ). Show that $r = 2M$ is a coordinate singularity of this metric.

(c) (*12 marks*)

Draw the Penrose diagram for the maximally extended Schwarzschild spacetime, in other words the Kruskal spacetime. Show on the diagram: the black hole (interior, horizon and singularity) the white hole (interior, horizon and singularity), the exterior regions, $\mathcal{I}^{\pm}$ ("scri-plus and scri-minus"), $i_{\pm}$ and i_0, $r = 2M$ and $r = 0$. Explain the *physical meaning* of all these features. Also draw an ingoing and an outgoing radial null geodesic.

(d) (*4 marks*)

Consider a model for a collapsing star which is a spherically symmetric ball of dust. Explain how the Schwarzschild metric is relevant to this model. Sketch the Penrose diagram for the spacetime of the star collapsing to form a black hole.

(e) (*8 marks*)

Astronaut Bob falls radially, feet first into a large Schwarzschild black hole. Does he lose sight of his feet as his feet cross the event horizon? Justify your answer with reference to a Penrose diagram showing relevant worldlines.

A space-station is stationary at a fixed large radius, far away from a very big Schwarzschild black hole. A shuttlecraft falls from the space-station radially towards and into the black hole and sends regular pulses of electromagnetic radiation to the space-station as it does so. The spacestation also sends regular electromagnetic communication to the shuttlecraft.

Draw the worldlines of the space-station and the shuttlecraft on a new Penrose diagram for the black hole. Can the shuttlecraft receive the signals from the space-station once it is inside the black hole? Can the shuttlecraft avoid the singularity? Use the diagram to explain your answers.

What does an observer on the space-station see as the shuttlecraft falls into the black hole? Use the diagram to explain your answer.

2.

(a) (*10 marks*)

Astronaut Alice falls from rest radially towards and into a Schwarzschild black hole of mass M. Derive the geodesic equation for Alice's trajectory:

$$\frac{dR}{d\tau} = -(1-\epsilon^2)^{\frac{1}{2}} \left(\frac{2M}{R(1-\epsilon^2)} - 1 \right)^{\frac{1}{2}}$$

where R is a coordinate which is Alice's Schwarzschild radial coordinate outside the black hole, τ is proper time along Alice's worldline and ϵ is a constant of the motion. You may use the particle action,

$$S = \frac{m}{2} \int d\tau\, g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \,.$$

What is the physical meaning of ϵ? Find the initial value of the Schwarzschild radial coordinate that Alice starts at in terms of M and ϵ. What is the range of possible values for ϵ?

(b) (*10 marks*)

Calculate the proper time along Alice's worldline from the moment she starts falling (from rest outside the black hole) till she reaches the singularity.

Calculate the proper time along Alice's worldline from the moment she crosses the horizon to the moment she reaches the singularity.

Explain briefly why, if Alice does anything other than falls radially while inside the black hole, the proper time between crossing the horizon and reaching the singularity will be less than if she falls radially.

(c) (*10 marks*)

Alice can choose to fall from rest starting just outside the horizon of the black hole or starting very far away from the black hole. Which should she choose if she wants to maximise her proper time spent *inside* the black hole before she reaches the singularity? What is the ratio of the proper times in the two cases?

Estimate an order of magnitude value of the mass of the black hole in kilograms such that the proper time Alice will spend inside the black hole is 100 years. You may use values of Newton's constant of gravitation, G, and the speed of light, c:

$$G = 6.67 \times 10^{-11} N(m/kg)^2$$
$$c = 3 \times 10^8 m/s$$

3. Consider the Kerr metric for a rotating black hole of mass M and angular momentum $J = Ma$,

$$ds^2 = -\frac{(\Delta - a^2 \sin^2\theta)}{\Sigma} dt^2 - 2a\sin^2\theta \frac{(r^2 + a^2 - \Delta)}{\Sigma} dt d\phi + \frac{((r^2+a^2)^2 - \Delta a^2 \sin^2\theta)}{\Sigma} \sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2$$

where

$$\Sigma = r^2 + a^2\cos^2\theta$$
$$\Delta = r^2 - 2Mr + a^2$$

and $M > a$.

(a) (*6 marks*) Where is the event horizon? Show that the Killing vector $k = \frac{\partial}{\partial t}$ is timelike at large r. Show also that k becomes spacelike in a region, called the ergoregion, outside the horizon. Derive the equation for the boundary of the ergoregion and sketch the ergoregion and show its relation to the event horizon.

(b) (*7 marks*)

The Kerr metric is stationary and axisymmetric. Show that this implies that a freely falling massive particle has two conserved quantities and calculate them. Show that a particle inside the ergoregion can have negative energy with respect to infinity.

You may use the particle action

$$S = \frac{m}{2}\int d\tau \, g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \, .$$

(c) (*8 marks*) State the Second Law of Black Hole Mechanics.

Calculate the area of the event horizon of the Kerr black hole. The First Law of Black Hole Mechanics (for uncharged black holes) is

$$dM = \frac{\kappa}{8\pi} dA + \Omega_H dJ$$

where κ is surface gravity and Ω_H is angular velocity of the black hole horizon. Express the area of the horizon as a function of M and J and hence find κ and Ω_H in terms of M and a. Explain why this derivation depends on the Black Hole Uniqueness Theorem.

Show that, in a process in which the mass and angular momentum of the black hole both change, the mass of the black hole can decrease if $J \neq 0$. What is the limiting minimum mass that the resulting black hole can have? Deduce the limiting maximum amount of energy that can be extracted from the black hole.

[This question continues on the next page . . .]

(d) (*9 marks*) Describe the Penrose process for extraction of energy from a rotating black hole. Assume that the black hole is very big and the change in mass, ΔM, and angular momentum, ΔJ, due to the process are very small compared to the mass and angular momentum of the black hole, respectively. Consider the Killing vector $\zeta = \frac{\partial}{\partial t} + \Omega_H \frac{\partial}{\partial \phi}$. Given that ξ is future pointing and timelike in the ergoregion, without using the Second Law of Black Hole Mechanics, prove that the Penrose process cannot decrease the area of the black hole.

4.

(a) (*6 marks*) Define (i) a Killing vector (ii) a null hypersurface and (iii) a Killing horizon.

(b) (*14 marks*) The Reissner-Nordstrom metric for a spherically symmetric electrically charged black hole with mass M and charge Q is

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega_2^2 \tag{1}$$

where

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} = r^{-2}(r - r_-)(r - r_+)\,, \tag{2}$$

and $d\Omega_2^2$ is the round metric on the 2-sphere.

Prove that the event horizon, $r = r_+$, is a Killing horizon and that the surface gravity κ of this horizon is

$$\kappa = \frac{\sqrt{M^2 - Q^2}}{(M + \sqrt{M^2 - Q^2})^2}\,. \tag{3}$$

What is the physical interpretation of κ for a classical Reissner-Nordstrom black hole? Sketch a graph of κ against M for fixed Q.

(c) (*10 marks*) Taking the quantum nature of all matter fields in the theory into account implies that a charged black hole radiates as a *perfect black body* at Hawking temperature $T_H = 2\pi\hbar\kappa$. That means that it will radiate all existing species of particles since it absorbs all particles. However, the probability that a particle of mass m is radiated is negligible unless $T_H > m$.

Assume that as well as electromagnetism there are electrically charged particles of mass m. Show that one possible fate of an electrically charged black hole is that it tends to an equilibrium state of non-zero mass and zero temperature. What conditions on the black hole's initial mass and charge will guarantee this happens?

Describe another possible fate for an electrically charged black hole.

A magnetically charged black hole, with magnetic charge Q, has the same metric as the Reissner-Nordstrom solution for an electrically charged black hole. Comment on the possible fate or fates of a radiating magnetically charged black hole.

 End of examination paper