Imperial College London

MSc EXAMINATION May 2018

This paper is also taken for the relevant Examination for the Associateship

BLACK HOLES

For Students in Quantum Fields and Fundamental Forces

Friday, 4th May 2018: 14:00–17:00

Answer all three questions. Question 1 is worth 40% and Questions 2 and 3 are worth 30% each.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the 3 answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in 3 answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

Go to the next page for questions

1. The Schwarzschild metric for a black hole of mass M in Schwarzschild coordinates (t, r, θ, ϕ) is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \qquad (1.1)$$

(i) Explicitly show that the metric can be written in ingoing Eddington-Finklestein coordinates in the form below (you do not need to explicitly determine any r_* coordinate in an intermediate stage). Also explain why $r = 2m$ is not singular.

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2) \qquad (1.2)$$

[8 marks]

(ii) Draw the Penrose diagram for the maximally extended Schwarzschild spacetime, also known as the Kruskal spacetime.

(a) For *one* of the asymptotically flat regions label $\mathcal{I}^\pm$ ("scri-plus" and "scri-minus") and i_0. With respect to this asymptotically flat region show the black hole horizon and singularity.

(b) By labelling the regions of the diagram, show in which parts of the Kruskal diagram are the ingoing Eddington-Finklestein coordinates, (v, r, θ, ϕ) are defined.

[12 marks]

(iii) Consider three observers outside a large Schwarzschild black hole of mass M, labelled A,B and C. A remains at a fixed large radius, far away from the black hole. B and C both fall freely from rest from the same radial coordinate $r = R > 2M$, B first and then C, and both fall into the black hole. All three spaceships are at the same fixed angular coordinates, θ, ϕ. B sends signals by sending photons at regular intervals of proper time along its worldline. Sketch their three worldlines on a Penrose diagram. Also sketch some representative signal photon worldlines between them. Briefly describe what A sees of B, as B falls into the black hole. Briefly describe what C sees of B, as B falls into the black hole. [8 marks]

(iv) Define a causal vector. Define a time orientation. Define a future directed causal vector field. [4 marks]

(v) Define the causal past of a set. Define the future event horizon for an asymptotically flat spacetime. [4 marks]

(vi) State the Dominant energy condition. [4 marks]

[Total 40 marks]

2. The Kerr metric in Boyer Lindquist coordinates for a rotating black hole of mass M and angular momentum $J = Ma$ is given by

$$ds^2 = -\frac{(\Delta - a^2\sin^2\theta)}{\Sigma}dt^2 - 2a\sin^2\theta\frac{(r^2+a^2-\Delta)}{\Sigma}dtd\phi + \frac{\left[(r^2+a^2)^2 - \Delta a^2\sin^2\theta\right]}{\Sigma}\sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2$$

where

$$\Delta = r^2 - 2Mr + a^2 \equiv (r - r_+)(r - r_-)$$
$$\Sigma = r^2 + a^2\cos^2\theta$$

(i) In these coordinates identify the event horizon and also where the spacetime has a singularity. [3 marks]

(ii) State the reason why it is possible to draw a Penrose diagram for the slice $\theta = 0$. Draw the extended Penrose diagram for this slice when $M > a$. On this diagram, for one asymptotically flat region, label past and future null infinity as well as spacelike infinity. [4 marks]

(iii) On a new Penrose diagram draw an Einstein-Rosen bridge B and indicate where the past and future Cauchy horizons for B are. Briefly explain the physical significance of these Cauchy horizons. [4 marks]

(iv) Show that the Killing vector $k = \partial_t$ is timelike for large r and that it becomes spacelike outside the horizon in the ergoregion. Derive the equation for the location of the ergoregion. [4 marks]

(v) The Kerr metric has an additional Killing vector, $m = \partial_\phi$. Write down the equation for an affinely parametrised geodesic, which describes the motion of a freely falling massive particle, and show that there are two conserved quantities. Show that inside the ergoregion the energy of the massive particle can be negative with respect to infinity. [4 marks]

(vi) Define the area A of the event horizon. In the above coordinates, using the fact that we can obtain the area by considering surfaces of constant t, r, show that for the Kerr metric $A = 8\pi(M^2 + \sqrt{M^4 - J^2})$. State the second law of black hole mechanics. [6 marks]

(vii) Describe the Penrose process using the above results. Using the second law show that the maximum efficiency of the Penrose process, achieved when the area stays the same and the final state of the black hole is not rotating, is $1 - 1/\sqrt{2}$. What is the requirement on the initial state of the black hole in order to achieve this maximum efficiency? [5 marks]

[Total 30 marks]

3. (i) The Lie derivative of the metric with respect to a vector k can be written

$$\mathcal{L}_k g_{\mu\nu} = k^\rho \partial_\rho g_{\mu\nu} + g_{\rho\nu}\partial_\mu k^\rho + g_{\mu\rho}\partial_\nu k^\rho \,.$$

If k is a Killing vector, satisfying $\mathcal{L}_k g_{\mu\nu} = 0$, show that $\nabla_{(\mu}k_{\nu)} = 0$. [4 marks]

(ii) If k is a Killing vector prove that

$$\nabla_\nu \nabla_\mu k_\rho = R_{\rho\mu\nu\sigma} k^\sigma$$

and hence $\nabla_\nu \nabla_\mu k^\nu = R_{\mu\nu} k^\nu$. You may use $(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)V^\rho = R_{\mu\nu}{}^\rho{}_\sigma V^\sigma$, valid for an arbitrary vector V^μ. [5 marks]

(iii) Using Einstein's equations, $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu}$, show that the condition $\nabla_\nu\nabla_\mu k^\nu = R_{\mu\nu}k^\nu$ can be rewritten in the notation of differential forms as $*d*dk = J$, where the one-form J should be identified. Also show that $d*J = 0$. [5 marks]

(iv) For an asymptotically flat stationary spacetime with Killing vector k, the Komar mass is defined to be

$$M_{Komar} = -\frac{1}{8\pi}\int_{S^2_\infty} *dk$$

Show that the mass parameter entering the Schwarzschild metric is the same as the Komar mass. You should use Schwarzschild coordinates, as in (**??**), and you may assume $\epsilon_{tr\theta\phi} = +\sqrt{-g}$. [4 marks]

(v) Define a Killing horizon, $\mathcal{N}$, associated with a Killing vector ξ. Explain why we can write

$$\nabla_\mu(\xi^2)\big|_{\mathcal{N}} = -2\kappa\xi_\mu|_{\mathcal{N}}$$

where κ is the surface gravity, and hence

$$\xi^\mu\nabla_\mu\xi_\rho|_{\mathcal{N}} = \kappa\xi_\rho|_{\mathcal{N}}$$

For the Schwarzschild metric show that the event horizon is a Killing horizon for the vector ∂_v in ingoing Eddington-Finklestein coordinates and calculate κ. [6 marks]

(vi) Show, in general, that κ is constant on the orbits of ξ on $\mathcal{N}$. [6 marks]

[Total 30 marks]