Rotating Black Holes

In gravitational collapse, we expect isolated black holes to settle down to a "time-independent" equilibrium state (in reality this is a very fast process). We want to classify all such states. $\exists$ powerful uniqueness theorems

Defn: An asymptotically flat spacetime is stationary if $\exists$ a Killing vector k that is timelike near ∞ (i.e. near $\mathscr{I}^{+}$, i^{0}, $\mathscr{I}^{-}$)

Can normalise so that $k^2 = -1$ at ∞. Using local coord's (t, x^i) near ∞, we can write

$$ds^2 = g_{tt}(\underset{\sim}{x})\,dt^2 + 2g_{ti}(\underset{\sim}{x})\,dt\,dx^i + g_{ij}(\underset{\sim}{x})\,dx^i dx^j$$

& $g_{tt} \to -1$ as $|\underset{\sim}{x}| \to \infty$

Stationary is static if $g_{ti} = 0$

Defn: An asymptotically flat spacetime is axisymmetric if $\exists$ Killing vector m that is spacelike near ∞ & all orbits of m are closed (i.e. S^1).

Can choose local co-ord's with $m = \partial_\phi$ & $0 \le \phi < \pi$

Recall Birkhoff: Spherical symmetry $\Rightarrow$ static
Converse is NOT true (eg consider the spacetime outside a cube). BUT if the only object is a black hole we have

Theorem Israel 1967

(M,g) static, asymptotically flat, vacuum, black hole spacetime, regular on & outside an event horizon then (M,g) is (isometric to) Schwarzschild.

Notes: 1) Implies static, vacuum, multi black holes don't exist

2) $\exists$ an Einstein-Maxwell generalisation which says solution is R-N or M-P.

Theorem Hawking 1973

(M,g) stationary, non-static, asympt. flat, analytic solution of Einstein-Maxwell that is regular on & outside the event horizon then (M,g) is stationary & axisymmetric.

ie. "Stationary implies axisymmetric". Assumption of analyticity a bit unsatisfactory.

Theorem   Carter (1971)   Robinson (1975)

(M, g) stationary, axisymmetric, asymptotically flat vacuum, regular on & outside event horizon, then (M, g) is part of two parameter family of Kerr Solutions. The parameters are mass, M, & angular momentum J

This strongly implies that the final equilibrium state of gravitational collapse is Kerr. All other information about the collapse is lost - either by radiation or by falling into the black hole.

There is an Einstein-Maxwell version which states (M, g) should belong to Kerr-Newman family specified by M, J, Q & P

The above theorems roughly say "black holes have no hair".

Aside: this is NOT true for black holes in anti-de-Sitter space...

Kerr-Newman Solution 1965

Boyer - Lindquist coordinates

$$ds^2 = -\frac{(\Delta - a^2\sin^2\theta)}{\Sigma}dt^2 - 2a\sin^2\theta\frac{(r^2+a^2-\Delta)}{\Sigma}dt\,d\phi$$

$$+\left[\frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\right]\sin^2\theta\,d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma\,d\theta^2$$

$$A = -\frac{1}{\Sigma}\left[Qr(dt - a\sin^2\theta\,d\phi) + P\cos\theta\,(a\,dt - (r^2+a^2)d\phi)\right]$$

where
$$\Sigma = r^2 + a^2\cos^2\theta$$
$$\Delta = r - 2mr + a^2 + e^2$$
$$e = \sqrt{Q^2+P^2}$$

Note

1) $r \to \infty$, asymptotically flat with $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$
2) $a = 0 \to$ R-N solution
3) Killing vectors (normalised) $k = \partial_t$ & $m = \partial_\phi$
4) $t \to -t$ & $\phi \to -\phi$ discrete isometry
5) $\phi \to -\phi$ has same effect as $a \to -a$, so take $a \geq 0$
6) $a = \frac{J}{m} = \frac{\text{angular momentum}}{\text{mass}}$ (see later)

Study

Kerr Solution (1963) $P = Q = 0$

Describes all (neutral) rotating black holes once they have settled down.

Approximates spacetime around a rotating star

Co-ordinate singularities

- $\theta = 0, \pi$ usual ones on 2-sphere
- $\Delta = 0$; $\Delta = (r - r_+)(r - r_-)$ so $\Delta = 0$ at

$$r = r_\pm = m \pm \sqrt{m^2 - a^2}$$

Curvature singularity

- $\Sigma = 0$: at $r = 0$ AND $\theta = \pi/2$

Three cases (like R-N)

1) $a > M$ Super extremal

Has a naked singularity $\rightarrow$ non-physical

2) $M > a$ - of most interest

Start at $r > r_+$ & define Kerr coordinates (v, r, θ, χ) analogous to ingoing E-F

$$\begin{cases} dv = dt + \dfrac{r^2+a^2}{\Delta} dr \\ d\chi = d\phi + \dfrac{a}{\Delta} dr \end{cases}$$

Killing vectors

$$\begin{array}{cccccc} & & \text{B-L} & & \text{Kerr} & \\ k & = & \partial_t & = & \partial_v & \\ m & = & \partial_\phi & = & \partial_\chi & 0 \leq \chi < 2\pi \end{array}$$

ex:

$$ds^2 = -\frac{(\Delta - a^2\sin^2\theta)}{\Sigma} dv^2 + 2dv\,dr - 2a\sin^2\theta \frac{r^2+a^2-\Delta}{\Sigma} dv\,d\chi$$

$$-2a\sin^2\theta\, d\chi\, dr + \frac{\left((r^2+a^2)^2 - \Delta a^2\sin^2\theta\right)\sin^2\theta\, d\chi^2}{\Sigma} + \Sigma\, d\theta^2$$

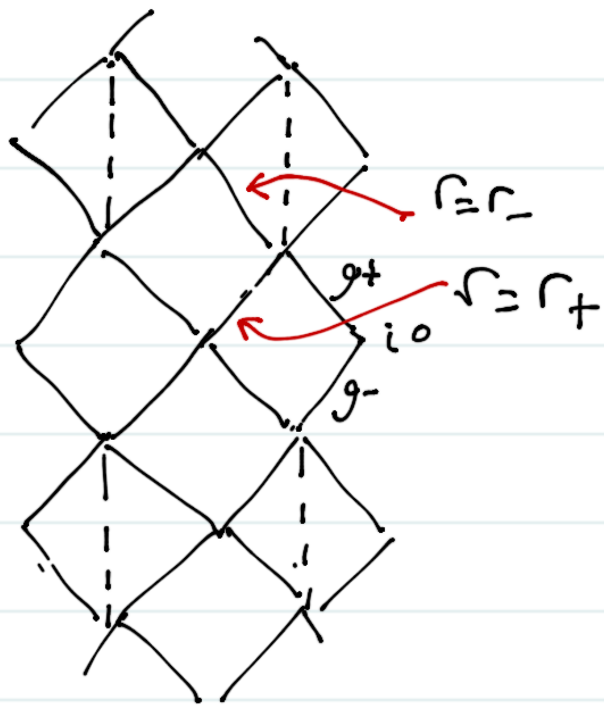
Now smooth at $r = r_+$ & also at $r = r_-$, so can continue to new regions $0 < r < r_+$. We then proceed as for R-N case.

To draw a Penrose diagram is difficult since it is not spherically symmetric. We can consider two interesting slices. × $\theta = 0$ (or π) : axis of symmetry

× $\theta = \pi/2$, $\chi = \text{const.}$: the equatorial plane (where the singularity is)

Note: these are both "totally geodesic" surfaces. i.e. a geodesic initially tangent to it, stays tangent.

By introducing Kruskal coordinates, we find in both cases:



the dashed line refers to the singularity at $r=0$ for the $\theta=\pi/2$ case.

The singularity is a ring. To see this, note for fixed v, r, θ we have

$$ds^2 = \frac{\left((r^2+a^2)^2 - \Delta a^2 \sin^2\theta\right)}{\Sigma} \sin^2\theta \, d\chi^2$$

as $r \to 0$ $ds^2 \to a^2 \sin^2\theta \, d\chi^2$

& at $\theta = \pi/2$, this circle has proper length $2\pi a$

If you travel from any other angle, $\theta \neq \pi/2$, one doesn't hit the ring singularity but emerges into a new asymptotic region.

Closed timelike curves (CTC's)

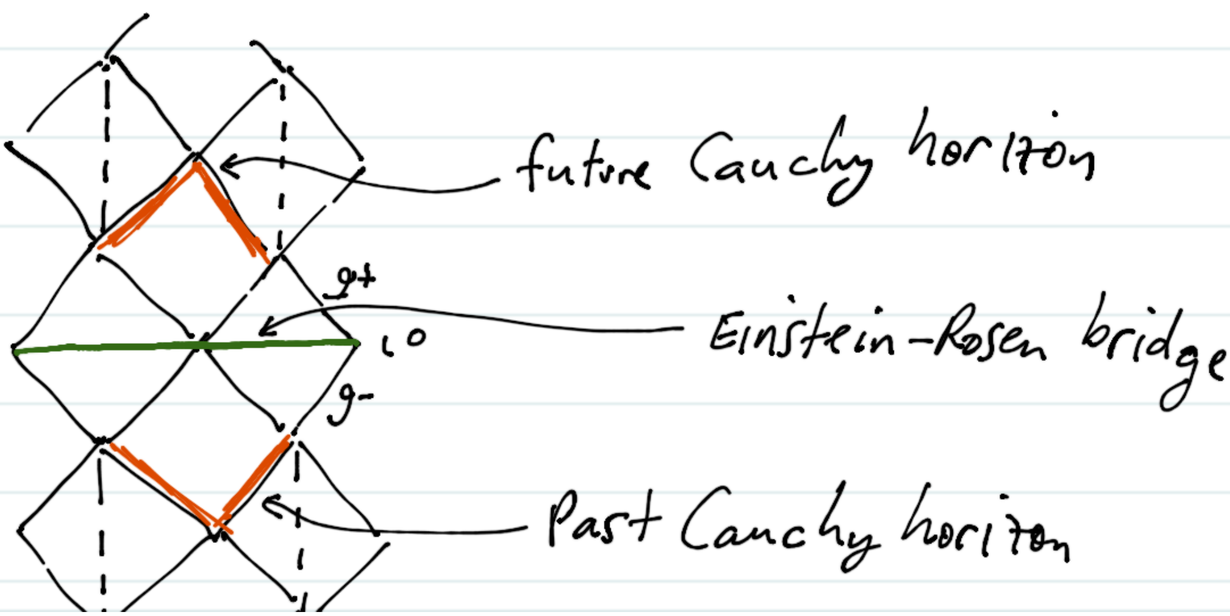
These are ubiquitous in G.R. A trivial example obtained by periodically identifying time in Minkowski spacetime.

They are almost certainly not physical spacetimes

Kerr has CTC's! Calculate the norm of $m = \partial_x$ at $\theta = \pi/2$: $\quad m^2\big|_{\theta=\pi/2} = g_{xx}\big|_{\theta=\pi/2} = r^2 + a^2 + \frac{2m}{r}a^2$

If $r<0$ & $|r| > \frac{2ma^2}{r^2+a^2}$ then m is timelike. But orbits of m are closed $\Rightarrow$ $\exists$ CTC's near the ring singularity

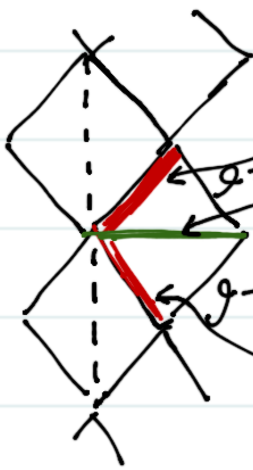
However, this is not a physical region because $\exists$ Cauchy horizons at $r = r_-$



For a collapsing rotating shell probably get a spacetime like

Spacelike? null?

3) Extremal case $M=a$ not physical



future Cauchy horizon

Has an ∞ throat like R-N

past Cauchy horizon

(Note: Extreme Kerr is not supersymmetric)

Consider Killing Vector $\xi^\mu = k^\mu + \Omega_H m^\mu$

where $\Omega_H = \dfrac{a}{r_+^2+a^2}$. $\quad \xi^v = 1$, $\xi^\chi = \Omega_H$ $\quad$ & $\xi_v\big|_{r_+} = \xi_\chi\big|_{r_+} = \xi_\theta\big|_{r_+} = 0$

$\Rightarrow \xi^2 = 0$ at $r = r_+$, so horizon is null

Also, in B-L coord's $\xi = \partial_t + \Omega_H \partial_\phi$

$$\Rightarrow \xi^\mu \partial_\mu (\phi - \Omega_H t) = 0$$

$$\Rightarrow \phi = \Omega_H t + \text{const. on integral curves of } \xi$$

Recalling that $\phi = $ const. on integral curves of k, we conclude that particle on orbits of ξ rotate with angular velocity Ω_H with respect to a stationary observer (one on orbit of k) e.g. some at ∞. Ω_H is thus interpreted as the angular velocity of the black hole.

Ergosphere

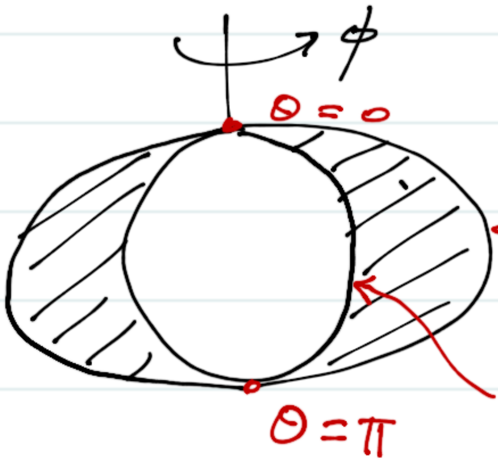
In B-L coord's: $k^2 = g_{tt} = -\left(1 - \dfrac{2mr}{r^2 + a^2\cos^2\theta}\right)$

k is timelike for $r^2 + a^2\cos^2\theta - 2mr > 0$

or $r > m + \sqrt{m^2 - a^2\cos^2\theta}$

k is spacelike outside H^+ in the ergosphere:

$$r_+ = m + \sqrt{m^2 - a^2} < r < m + \sqrt{m^2 - a^2\cos^2\theta}$$



$r = m + \sqrt{m^2 - a^2\cos^2\theta}$ the "ergosurface"

$r = r_+$ the event horizon

Stationary observers (with 4-velocity parallel to k) do NOT exist in the ergoregion. In this region they must rotate, relative to observers at ∞ in the same direction as the black hole.

To see this:

$$U^\mu = \dot{x}^\mu \quad \& \quad U^2 = -1 = \underbrace{g_{\mu\nu} U^\mu U^\nu}_{\substack{\text{all +ve in ergoregion except} \\ \text{for } g_{t\phi} U^t U^\phi}}$$

$\Rightarrow U^\phi \neq 0$. Also $U^t > 0$ for future directed

$\Rightarrow U^\phi > 0$.

This is an example of frame dragging.

Penrose Process

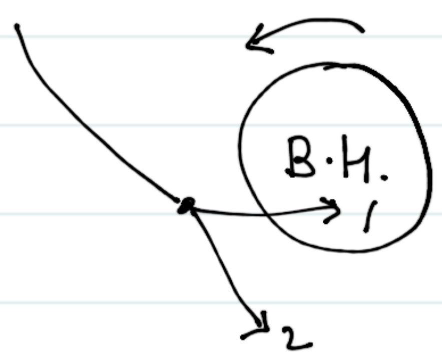
Consider a particle approaching a Kerr black hole along a geodesic with 4-momentum $p^\mu = \mu U^\mu = \mu \dot{x}^\mu$

$\uparrow$ rest mass

The conserved energy of the particle (with respect to a stationary observer at ∞) is $E = -k \cdot p$

Suppose it decays into 2 particles with momentum p_1^μ & p_2^μ. We have $p^\mu = p_1^\mu + p_2^\mu$ (Use a local inertial frame near decay)

$$\Rightarrow \quad E = E_1 + E_2$$



Normally $E_1 > 0$ ∴ $E_2 < E$

However note that $E_1 = -p_1 \cdot k$ & hence if splitting happens inside the ergoregion we can have $E_1 < 0$ & hence $E_2 > E$!

i.e. Energy is being extracted from the black hole.

To see this: choose coord's with $k = (0,1,0,0)$ & prepare decay so that $(p_1)_\mu = (1, y, 0, 0)$ with y small enough for p_1 to be timelike. Then $p_1 \cdot k = y$

Note: Can be shown that when $E_1 < 0$ the particle falls into the black hole & moreover $L < 0$ so that it reduces both mass + rotation of black hole.

How much energy can be extracted?

Particle crossing H^+ has $-P \cdot \xi \geq 0$ since both P & ξ are future directed & causal

$\Rightarrow E - \Omega_H L \geq 0$ where $L \equiv m \cdot P$ is the conserved angular momentum

$\Rightarrow L \leq E/\Omega_H$ (note if $E < 0$ then $L < 0$ as noted above)

After particle with (E, L) enters, the black hole settles down to Kerr with $\delta M = E$ & $\delta J = L$

$$\Rightarrow \delta J \leq \frac{\delta M}{\Omega_H} = \frac{2M\left(M^2 + \sqrt{M^4 - J^2}\right)}{J} \delta M \qquad \left(a = \frac{J}{M}\right)$$

& recall $a, J \geq 0$

$$\Leftrightarrow \boxed{\delta M_{IR} \geq 0} \qquad \boxed{M_{IR} \equiv \left[\frac{1}{2}\left(M^2 + \sqrt{M^4 - J^2}\right)\right]^{1/2}}$$

The "irreducible mass"

Special case of the 2^{nd} law – see later

Note $M^2 = M_{IR}^2 + J^2/4M_{IR}^2 \geq M_{IR}^2$

Thus, we cannot reduce the mass below the initial value of M_{IR}.

Suppose we have $M_I, J_I \to M_{IR}(M_I, J_I)$ to start & M_F, J_F to end.

What's the smallest M_F can be?

$$M_F^2 \geq M_{IR}^2(F) \geq M_{IR}^2(I)$$

Hence smallest is

$$\underset{\substack{\uparrow \\ J_F = 0}}{M_F^2 = M_{IR}^2(F)} \underset{\substack{\uparrow \\ L = E/\Omega_H}}{=} M_{IR}^2(0)$$

$L = E/\Omega_H$ for all decayed particles entering black hole.

i.e. $\begin{cases} M_F = M_{IR}(M_I, J_I) \\ J_F = 0 \end{cases}$

$$\eta = \text{Efficiency} = \frac{M_I - M_F}{M_I} = 1 - \frac{M_{IR}(M_I, J_I)}{M_I}$$

$$= 1 - \left[\frac{1}{2} \left(1 + \sqrt{1 - \frac{J_I^2}{M_I^4}} \right) \right]^{1/2}$$

This is maximised if $J_I = M_I$ i.e. initial black hole is extremal

& then $\eta = 1 - \frac{1}{\sqrt{2}} \simeq 29\%$ (huge!)

Energy & Angular Momentum: Komar Integrals

Maxwell's Eq.'s

$$\begin{cases} \nabla_\mu F^{\mu\nu} = -4\pi j^\nu & \Rightarrow \nabla_\mu j^\mu = 0 \\ \nabla_{[\mu} F_{\nu\rho]} = 0 & \end{cases}$$

Continuity equation

Rewrite using differential forms $F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$

$j = j_\mu dx^\mu$

$$\begin{cases} d * F = -4\pi * j & \Rightarrow d * j = 0 \\ dF = 0 & \end{cases}$$

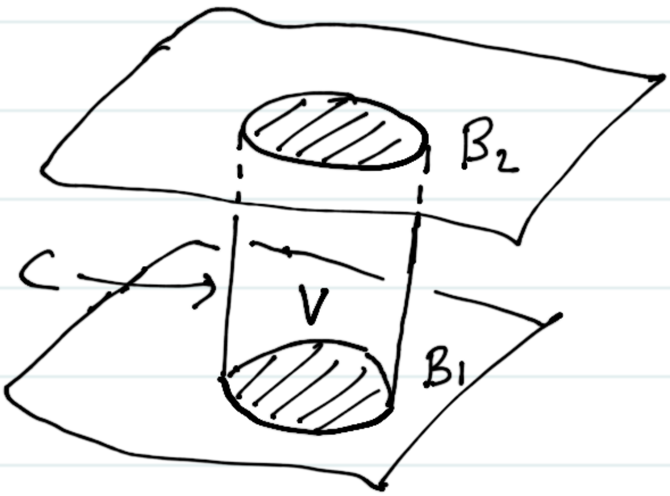
Also, note $F = dA \Rightarrow dF = 0$.

Take spacelike surface Σ and define Q_B to be the electric charge in some spacelike region B

$$Q_B \equiv -\int_B * j$$

As usual this charge is time independent if j vanishes on the boundary. To see this:

2 spacelike surfaces



$$0 = \int_V d*j$$
$$= \int_{B_2} *j - \int_{B_1} *j + \underbrace{\cancel{\int_C *j}}_{\text{integral on surface of cylinder}} \to 0$$

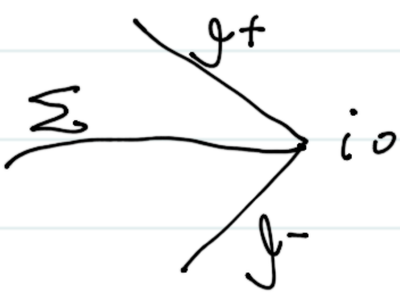
sign arises because normal to B_1 points inwards

Hence: $\displaystyle\int_{B_2} *j = \int_{B_1} *j$

Invoking Maxwell's equations we can also write

$$Q_B = \frac{1}{4\pi}\int_B d*F = \frac{1}{4\pi}\int_{\partial B} *F$$

Now consider an asymptotically flat spacetime



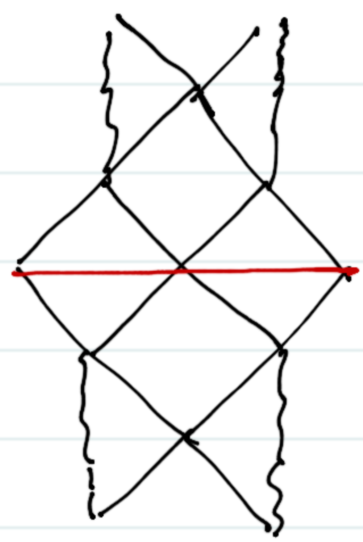
Surface Σ has an induced metric which becomes flat at ∞. Consider the two-sphere boundary at ∞, S^2_∞ & define

Electric Charge $Q \equiv \frac{1}{4\pi}\int_{S^2_\infty} * F$

Magnetic Charge $P = \frac{1}{4\pi}\int_{S^2_\infty} F$

Ex: show this agrees with notation we used for Reissner-Nordstrom

Comment: R-N has $Q \neq 0$ despite the fact that $j = 0$. E.g. consider a t=constant slice



Before discussing Komar integrals, let us first consider conserved stress tensor

$\nabla_\mu T^{\mu\nu} = 0$

where $T^{\mu\nu}$ could be test matter or the RHS of Einstein's equations

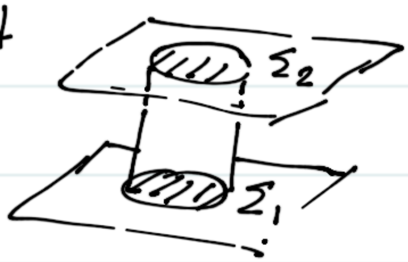
If spacetime is stationary, $\exists$ Killing vector k^μ (timelike at ∞). Then

$$J^\mu \equiv -T^\mu{}_\nu k^\nu$$

$$\& \quad \nabla_\mu J^\mu = -\nabla_\mu T^\mu{}_\nu k^\nu - T^\mu{}_\nu \nabla_\mu k^\nu = 0$$

Can then define $E(\Sigma) = -\int_\Sigma * J$ for a spacelike surface Σ;

&



$E(\Sigma_2) = E(\Sigma_1)$ if J^μ vanishes on the boundary of the cylinder as above.

We cannot, however, express this J (as a one-form) as $d * \omega_2$ for some 2-form ω_2 & write $E(\Sigma)$ as an integral on $\partial \Sigma$.

Physical reason: gravitational field has energy.

Komar Integrals

Killing vector k^μ: $\nabla_{(\mu} k_{\nu)} = 0$

$$\Rightarrow \nabla_\mu k_\nu = \nabla_{[\mu} k_{\nu]} = \frac{1}{2}(dk)_{\mu\nu}$$

Also,

$$\nabla_\nu \nabla_\mu k^\nu = R_{\mu\nu} k^\nu \qquad \text{(prob. sheet 1)}$$

$$\Rightarrow \nabla_\nu \left(\tfrac{1}{2} dk\right)^{\mu\nu} = R^\mu{}_\nu k^\nu$$

$$\Rightarrow (*d*dk)_\mu = -2 R_{\mu\nu} k^\nu$$

$$= -2 \cdot 8\pi \cdot \left(T_{\mu\nu} - \tfrac{1}{2} T g_{\mu\nu}\right) k^\nu$$

$$= 8\pi \hat{J}_\mu$$

where

$$\hat{J}_\mu \equiv -2\left(T_{\mu\nu} - \tfrac{1}{2} T g_{\mu\nu}\right) k^\nu$$

& here $T_{\mu\nu}$ is the RHS of Einstein's eq's.

$$\boxed{*d*dk = 8\pi J}$$

note: $*J$ is exact & hence $d*J = 0$

Hence: can define a Komar integral

$$Q_k(\Sigma) = -\frac{c}{8\pi} \int_{\partial\Sigma} *dk$$

convenient constant

for spacelike region Σ

Definition Asymptotically flat spacetime, stationary define the Komar mass - the total energy of the spacetime - to be

$$M_{Komar} = -\frac{1}{8\pi}\int_{S^2_\infty} *dk$$

Definition Asymptotically flat spacetime, axisymmetric define Komar angular momentum

$$J_{Komar} = \frac{1}{16\pi}\int_{S^2_\infty} *dm$$

where m^μ is the K. vector

Exercise: For Kerr, $k = \partial_t$ & $m = \partial_\phi$, then

$$\begin{cases} M_{Komar} = M \\ J_{Komar} = J \end{cases}$$

Let's show for Schwartschild

$$k = \partial_t \Rightarrow k^t = 1$$

$$k_\mu = g_{\mu\nu} k^\nu = g_{\mu t}$$

$$\Rightarrow k = -\left(1 - \frac{2M}{r}\right) dt$$

$$\Rightarrow dk = -\frac{2M}{r^2}\, dr \wedge dt \qquad \& \quad (dk)_{rt} = -\frac{2M}{r^2}$$

$$(*dk)_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\, g^{\rho\rho'} g^{\sigma\sigma'} (dk)_{\rho'\sigma'}$$

$$= \varepsilon_{\mu\nu\rho\sigma}\, g^{\rho r} g^{\sigma t} (dk)_{rt}$$

$$= \varepsilon_{\mu\nu r t}\, g^{rr} g^{tt} (dk)_{rt}$$

$$\Rightarrow (*dk)_{\theta\phi} = \underbrace{\underbrace{\varepsilon_{\theta\phi r t}}_{-\varepsilon_{tr\theta\phi}}}_{-\sqrt{-g}} \cdot (-1) \cdot (-2m/r^2) \qquad \sqrt{-g} = r^2 \sin\theta$$

& all other components are zero

$$\Rightarrow *dk = -2m \sin\theta \, d\theta \wedge d\phi$$

$$\Rightarrow M_{\text{KOMAR}} = -\frac{1}{8\pi}\int_{S^2_\infty} *dk$$

$$= \frac{m}{4\pi}\int_{S^2_\infty} \sin\theta\, d\theta \wedge d\phi$$

$$= \frac{m}{4\pi}\int_{S^2_\theta} \sin\theta\, d\theta d\phi$$

$$= m$$

Comment: For asymptotically flat spacetimes without Killing vectors, can still define the total energy using the fact $\exists$ an asymptotic Killing vector ∂_t as we approach $r \to \infty$ (in A.F. coordinates) & this is equivalent to "ADM Energy".

Energy Conditions

In G.R. we only want to consider "Physical matter" which means $T^{\mu\nu}$ should satisfy some conditions. E.g. Consider an observer with 4-velocity V^μ who would measure energy momentum current $J^\mu = -T^\mu{}_\nu V^\nu$. We would expect physical matter not to move faster than c, so we might demand $J^2 \leq 0$

Dominant Energy Condition $-T^\mu{}_\nu V^\nu$ is a future directed causal vector (or zero) for all future directed timelike vectors V^μ

e.g. Massless scalar field $T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} (\partial \phi)^2$

$$J^\mu = -T^\mu{}_\nu V^\nu = -(V\cdot\partial\phi)\,\nabla^\mu\phi + \tfrac{1}{2}V^\mu(\partial\phi)^2$$

$$\Rightarrow J^2 = \tfrac{1}{4}\,V^2\underbrace{\left[(\partial\phi)^2\right]^2}_{\geqslant 0} \leqslant 0 \quad \text{if } V^\mu \text{ timelike}$$

$\Rightarrow J$ causal or zero

Also,

$$V\cdot J = -(V\cdot\partial\phi)^2 + \tfrac{1}{2}V^2(\partial\phi)^2$$

$$= \underbrace{-\tfrac{1}{2}(V\cdot\partial\phi)^2}_{\leqslant 0} + \underbrace{\tfrac{1}{2}V^2}_{\leqslant 0}\underbrace{\left(\partial\phi - \frac{V\cdot\partial\phi}{V^2}V\right)^2}$$

This vector is orthogonal to V^μ
$\Rightarrow$ it is spacelike or zero
$\Rightarrow$ norm squared of vector is $\geqslant 0$

$\Rightarrow V\cdot J \leqslant 0$

$\Rightarrow J^\mu$ is future directed (or zero) //

We can impose less restrictive condition – that only the energy density measured by observers is positive – and we have...

<u>Weak Energy Condition</u>: $T_{\mu\nu} V^\mu V^\nu \geq 0$ for any causal vector V^μ

A special case is

<u>Null Energy Condition</u>: $T_{\mu\nu} V^\mu V^\nu \geq 0$ for any null vector V^μ

Note: Dominant $\Rightarrow$ weak $\Rightarrow$ null

Another condition

<u>Strong Energy Condition</u>: $\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right) V^\mu V^\nu \geq 0$ for all causal vectors V^μ

Note: Einstein's equations $\Rightarrow R_{\mu\nu} V^\mu V^\nu \geq 0$ or "gravity is attractive"

<u>Notes</u>: * S.E.C. $\Rightarrow$ N.E.C. but S.E.C. $\not\Rightarrow$ W.E.C.

* S.E.C. needed for <u>some</u> singularity theorems
* S.E.C. violated by a +ve cosmological constant.
* D.E.C. used in +ve mass theorem...

Positive Mass Theorem : Schoen & Yau 1979
Witten 1981

The ADM mass M of an asymptotically flat spacetime satisfying Einstein's equations has $M \geq 0$ & $M = 0$ only for Minkowski spacetime with $T_{\mu\nu} = 0$, provided

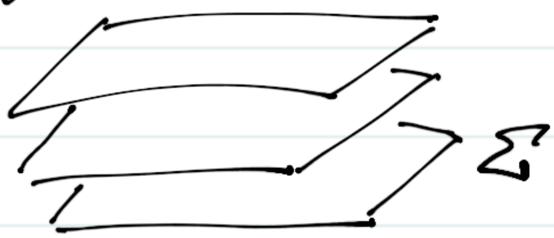
i) $T_{\mu\nu}$ satisfies dominant energy condition

ii) $\exists$ non-singular Cauchy surface (otherwise -ve mass Schwarzschild would be a counterexample

iii) some other technical assumptions.

Hypersurfaces

Let $S(x)$ be a smooth function on M & consider family of hypersurfaces $\{\Sigma_S\}$ defined by $S=$ constant



The one-form $dS = \partial_\mu S\, dx^\mu$ is normal to Σ_S

Proof: If T is a tangent vector to Σ_S then can write $T^\mu = \dot{x}^\mu(\lambda)$ for some curve $x^\mu(\lambda)$ in Σ_S

Then: $T\cdot dS = T^\mu (dS)_\mu = T^\mu \partial_\mu S$

$$= \dot{x}^\mu \partial_\mu S = \frac{d}{d\lambda} S = 0$$

since S is constant on Σ_S //

Write $n^\mu = f g^{\mu\nu} \partial_\nu S$ for some function f

Hypersurface $\begin{cases} \underline{\text{spacelike}}: & \text{choose } f \text{ so that } n^2 = -1 \\ \underline{\text{timelike}}: & \text{" \quad " \quad " \quad "} \quad n^2 = +1 \\ \underline{\text{null}}: & n^2 = 0 \end{cases}$

For timelike & spacelike cases, $n^2 = \pm 1$, we can define induced metric $h_{\mu\nu} \equiv g_{\mu\nu} \mp n_\mu n_\nu$ on Σ_S

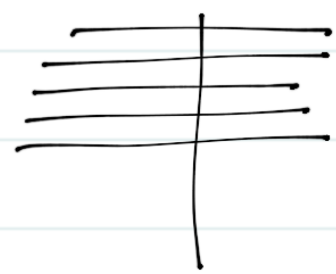
Note: $h_{\mu\nu}$ is degenerate on full tangent space at a point on Σ, but non-degenerate on subspace of vectors tangent to curves in Σ (i.e. satisfying $V \cdot n = 0$)

Note: $h^{\mu}{}_{\nu} h^{\nu}{}_{\rho} = h^{\mu}{}_{\rho} \Rightarrow h^{\mu}{}_{\nu}$ is a projection

e.g. Minkowski spacetime: coord's (t, x), $S(x) = t$

Have $n^{\mu} = (1, 0, 0, 0)$

$$h_{\mu\nu} = \begin{pmatrix} 0 & \underset{\sim}{0} \\ \underset{\sim}{0} & \mathbb{1} \end{pmatrix}$$



Aside: In the Hamiltonian formulation of G.R., one foliates spacetime with spacelike surfaces & uses $h_{\mu\nu}$ as configuration space variable. One also defines the extrinsic curvature $K_{\mu\nu} = h_{\mu}{}^{\sigma} h_{\nu}{}^{\rho} \nabla_{\sigma} n_{\rho}$ & this can be used to obtain the canonical momenta. The diffeomorphism invariance of GR implies that the Hamiltonian $= N H + N_{\mu} H^{\mu}$ with Lagrange multipliers N & N_{μ}. Thus it is a sum of constraints $H = 0$ "hamiltonian constraint"

$H_{\mu} = 0$ "Momentum constraint"

Null Hypersurfaces: We are particularly interested in the case where one of the Σ_i is null

Write null hypersurface as $\mathcal{N}$ & normal as ℓ^μ

e.g. Schwarzschild in ingoing E-F coord's

$$S = S(r) \equiv r - 2M$$

then $\ell^\mu = f g^{\mu\nu}\partial_\nu S = f g^{\mu r}\partial_r S$

$\Rightarrow \ell^r = f(1-2M/r)$, $\ell^v = f$, $\ell^\theta = \ell^\phi = 0$

$\Rightarrow$ normal is

$$\ell = f\left[\left(1-\frac{2M}{r}\right)\partial_r + \partial_v\right]$$

$\ell^2 = f^2\left(1-\frac{2M}{r}\right) \Rightarrow$ $r = 2M$ is null H.S. & $\ell|_{r=2M} = f\partial_v$

$r < 2M$ spacelike H.S.

$r > 2M$ timelike H.S. //

A vector T is tangent to a hypersurface Σ with normal $\ell \Leftrightarrow T\cdot\ell = 0$

For null hypersurface $\mathcal{N}$ we have ℓ is also tangent to $\mathcal{N}$! (since $\ell^2 = 0$). i.e. we have $\ell^\mu = \frac{dx^\mu}{d\lambda}$ for some $x^\mu(\lambda) \subset \mathcal{N}$

Claim: These curves $x^\mu(\lambda)$ are null geodesics

Proof:

$$\ell^\mu \nabla_\mu \ell^\nu = \ell^\mu \nabla_\mu \left(f g^{\nu\rho} \partial_\rho S\right) = \ell^\mu \nabla_\mu \left(f \nabla^\nu S\right)$$

$$= \left(\ell^\mu \nabla_\mu f\right) \nabla^\nu S + f \ell^\mu \nabla_\mu \nabla^\nu S$$

$$= \left[\left(\ell^\mu \nabla_\mu f\right) f^{-1}\right] \ell^\nu + \underbrace{f \ell^\mu \nabla^\nu \nabla_\mu S}_{\underbrace{f \ell^\mu \nabla^\nu \left(f^{-1} \ell_\mu\right)}_{\left(f \nabla^\nu f^{-1}\right) \ell^2 + \frac{1}{2} \nabla^\nu \left(\ell^2\right)}}$$

Now $\ell^2 = 0$ on $\mathcal{N}$

But $\nabla^\nu(\ell^2) \neq 0$ on $\mathcal{N}$ (ℓ^2 can be non-zero off $\mathcal{N}$)

However, $\ell^2 =$ constant on $\mathcal{N}$

$$\Rightarrow \nabla^\rho (\ell^2) \propto \ell^\rho$$

Hence

$$\ell^\mu \nabla_\mu \ell^\nu \propto \ell^\nu$$

$\Rightarrow$ $x^\mu(\lambda)$ are geodesics with $\ell^\mu = \dot{x}^\mu$ //

Can choose the function f, so that $\ell^\mu \nabla_\mu \ell^\nu \big|_{\mathcal{N}} = 0$

Definition: Null geodesics $x^\mu(\lambda)$ with affine parameter λ for which $\ell^\mu = \dot{x}^\mu$ are normal to a null hypersurface $\mathcal{N}$ are the "generators of $\mathcal{N}$"

eg Schwartzschild in Kruskal co-ords

$U = \text{const} \to$ family of null hypersurfaces

Let $\mathcal{N}$ be $U=0$ (the event horizon)

Normal $\quad \ell = -f \dfrac{r}{32m^3} e^{r/2m} \partial_V$

$$\ell\big|_{\mathcal{N}} = -f\left(\frac{e}{16m^2}\right)\partial_V$$

In this example $\ell^2 \equiv 0$ (not just on $\mathcal{N}$)

$\Rightarrow D^\rho(\ell^2) = 0$ on $\mathcal{N}$

Hence, $\ell^\mu D_\mu \ell^\nu = 0$ if $f = $ constant. choose $f = -\dfrac{16m^2}{e}$

Then $\ell\big|_{\mathcal{N}} = \partial_V$ is normal to $\mathcal{N}$

& V is the affine parameter for generator of $\mathcal{N}$

Killing Horizon

Null hypersurface $\mathcal{N}$ is a Killing Horizon of a Killing vector field ξ, if on $\mathcal{N}$, ξ is normal to $\mathcal{N}$

note: not necessarily the same as event horizon!

Since $\xi^2 = 0$ on $N \Rightarrow \partial_\rho(\xi^2)$ is normal to N & hence proportional to ξ_ρ

Thus, $\nabla_\rho(\xi^2)\Big|_N = -2K\,\xi_\rho\Big|_N$

where K is the "surface gravity"

EX: Also have $\xi^\mu \nabla_\mu \xi_\rho = K\,\xi_\rho$ on N

K is the force that an observer at ∞ would need to exert to keep a unit mass at the horizon.

Example: Schwarzschild in ingoing E-F. coord's (v, r, θ, ϕ)

$\xi = \partial_v$ Killing vector & $r = 2M$ is N

$\xi^2 = g_{vv} = -(1 - 2M/r)$ $\Rightarrow$ N is a Killing horizon

$\nabla_\rho(\xi^2)$: $\nabla_r(\xi^2) = -\frac{2M}{r^2}$ all other components 0

Next: $\xi_\rho = g_{\rho\mu}\xi^\mu = g_{\rho v} \Rightarrow \begin{cases} \xi_v = -(1-2M/r) \\ \xi_r = 1 \\ \xi_\theta = \xi_\phi = 0 \end{cases}$

$$\xi_\rho\Big|_N = \begin{matrix} 1 & \text{for } \rho = r \\ 0 & \text{otherwise} \end{matrix}$$

$$\nabla_\rho(\xi^2)\Big|_{r=2M} = -2K\,\xi_\rho\Big|_{r=M} \Rightarrow \boxed{K = \frac{1}{4M}}$$

Note: normalisation of ξ is important: if $\tilde{\xi} = c\xi$ then $\tilde{K} = cK$. In the example ξ was normalised so that $\xi^2 = -1$ as $r \to \infty$

Fact K is constant on the orbits of ξ on N

Proof: ξ is tangent to N (& normal!)

Therefore $\underline{\underline{\text{on } N}}$ $\quad \xi^\sigma \nabla_\sigma \left[\xi^\mu \nabla_\mu \xi_\rho \right] = \xi^\sigma \nabla_\sigma (K \xi_\rho)$

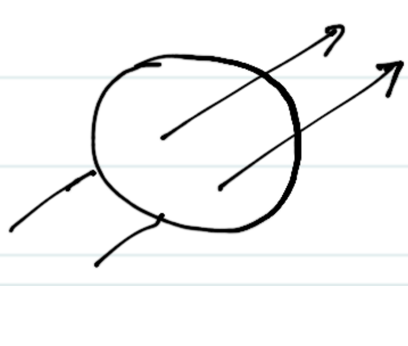
$$\Rightarrow \underbrace{\underbrace{(\xi^\sigma \nabla_\sigma \xi^\mu)}_{K\xi^\mu} (\nabla_\mu \xi_\rho)}_{K^2 \xi_\rho} + \xi^\sigma \xi^\mu \nabla_\sigma \nabla_\mu \xi_\rho = (\xi^\sigma \nabla_\sigma K)\xi_\rho + \underbrace{K \xi^\sigma (\nabla_\sigma \xi_\rho)}_{K^2 \xi_\rho}$$

Using Killing vector lemma

$$\Rightarrow \underbrace{\xi^\sigma \xi^\mu R_{\rho\mu\sigma\nu} \xi^\nu}_{0} = (\xi^\sigma \nabla_\sigma K) \xi_\rho$$

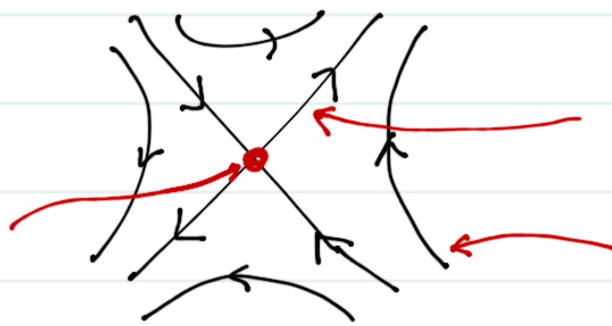
$\Rightarrow$ if $\xi_\rho \neq 0$ on N then $\xi^\sigma \nabla_\sigma K = 0$

In fact points with $\xi = 0$ arise as limit points of orbits with $\xi \neq 0$, so result is true even when $\xi = 0$ on N



Non-degenerate bifurcate Killing horizons $\kappa \neq 0$

Suppose $\kappa \neq 0$ on one orbit of ξ in $\mathcal{N}$, then can show that this only coincides with part of a null generator of $\mathcal{N}$. We find that we must have what we saw for Kruskal:



Furthermore: for bifurcate Killing horizons κ is constant on $\mathcal{N}$ (i.e. value of κ on all orbits are the same)

To prove this we need some preliminary results

<u>Frobenius theorem:</u> Vector field χ^μ is hypersurface orthogonal iff $\chi_{[\mu} \nabla_\nu \chi_{\rho]} = 0$ ($\Leftrightarrow \chi \wedge d\chi = 0$). See e.g. Wald.

Now

ξ is normal to $\mathcal{N}$ $\Rightarrow$ $\xi_{[\mu} \nabla_\nu \xi_{\rho]} = 0$ on $\mathcal{N}$

$$\Rightarrow \xi_\rho \nabla_\mu \xi_\nu \big|_{\mathcal{N}} = \left[-\xi_\mu \nabla_\nu \xi_\rho + \xi_\nu \nabla_\mu \xi_\rho \right]_{\mathcal{N}}$$

using $\nabla_\mu \xi_\nu = \nabla_{[\mu} \xi_{\nu]}$ for K. vectors multiply by $\nabla^\mu \xi^\nu$

$$\Rightarrow \underbrace{-2\nabla^\mu\xi^\nu(\xi_\mu \nabla_\nu \xi_\rho)\Big|_N}_{\underbrace{-2K\xi^\nu\nabla_\nu\xi_\rho|_N}_{-2K^2\xi_\rho|_N}} = \underbrace{\nabla^\mu\xi^\nu(\xi_\rho\nabla_\mu\xi_\nu)\Big|_N}_{(\nabla^\mu\xi^\nu\nabla_\mu\xi_\nu)\xi_\rho|_N}$$

$\Rightarrow$ for points in N with $\xi^\rho \neq 0$

$$\boxed{K^2 = -\frac{1}{2}(\nabla^\mu\xi^\nu)(\nabla_\mu\xi_\nu)\Big|_N}$$

Also true for points with $\xi^\rho = 0$ by continuity

Let t be arbitrary tangent vector to N

$$(t\cdot\nabla)K^2\Big|_N = -t^\rho(\nabla^\mu\xi^\nu)\nabla_\rho\nabla_\mu\xi_\nu|_N$$

$$= -t^\rho\nabla^\mu\xi^\nu R_{\nu\mu\rho\sigma}\xi^\sigma|_N \quad (*)$$

Now assume we have bifurcate Killing Horizon. K^2 is constant on each orbit of ξ. This constant is the same value of K^2 at the limit point of ξ where $\xi^\rho = 0$, the bifurcate S^2

But on this S^2 we have from $(*)$ that RHS $= 0$

Hence $(t\cdot\nabla)K^2 = 0$ on the S^2 for any tangent vector t $\Rightarrow$ K^2 is constant on S^2

& hence on the whole of N //

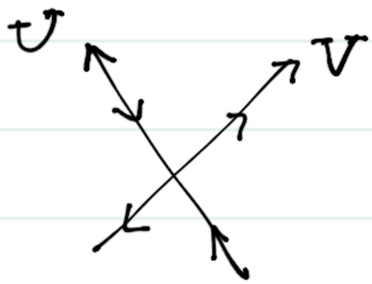
Schwartschild in Kruskal coord's

$$k = \partial_t = \frac{1}{4m}\left(V\partial_V - U\partial_U\right)$$

write $k = f\ell$ with $\ell^\mu \nabla_\mu \ell^\nu = 0$

Then $k^\mu \nabla_\mu k_\nu = (k^\mu \ln f)\, k_\nu$ (ex)

$\Rightarrow K = k^\mu \partial_\mu \ln f$



on $U = 0$ $\quad k = \frac{V}{4m}\partial_V$ & $\ell = \partial_V$

ie. $f = V/4m$ & $K = \frac{V}{4m}\partial_V \ln \frac{V}{4M} = \frac{1}{4M}$

on $V = 0$ $\quad k = -\frac{U}{4m}\partial_U$ & $\ell = \partial_U$

ie $f = -U/4M$ & $K = -1/4M$

Note that on each null hypersurface K is indeed constant.

Degenerate Horizons

$K = 0$ as in extreme R-N & Kerr

No bifurcate S^2

Note for these case $\xi^\mu \nabla_\mu \xi_\nu = 0$ on N

ie. the Killing vector on N is affinely parametrised