Black hole Mechanics / Thermodynamics

1973: Bardeen, Carter, Hawking

Black hole mechanics (G.R.)

Law 0: The surface gravity κ is constant on H^+ for a stationary black hole

Law 1: $dM = \frac{\kappa}{8\pi} dA + \Omega_H dJ + \Phi_H dQ$

Law 2: $dA \geq 0$

Law 3: Can't achieve $\kappa = 0$ by physical process

Thermodynamics

Law 0: Temperature T is constant throughout a system in thermal equilibrium

Law 1: $dE = T dS + \mu dN$

Law 2: $dS \geq 0$

Law 3: Can't achieve $T = 0$ by physical process

(There is a game, you can't win, you must lose, you can't quit!)

Initial thinking: nothing can escape from a black hole, so can't have a temperature + hence no entropy.

Bekenstein: Real 2^{nd} Law would be violated if black holes had no entropy. (Because you could throw entropic objects into the black hole & hence lower total entropy). He proposed generalised 2^{nd} Law: $dS_{TOTAL} \geq 0$ $\quad S_{TOTAL} = S_{EXT} + S_{BH}$

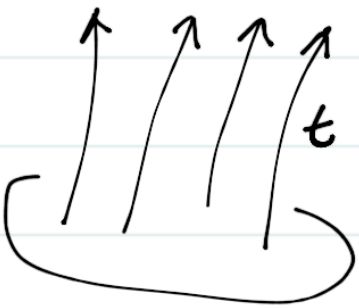
& $S_{BH} \propto A$

Hawking 1974 $\quad$ G.R. + QFT

$$\left.\begin{aligned} T_H &= \frac{\hbar}{k_B} \frac{1}{c} \frac{\kappa}{2\pi} \\ S_{BH} &= \frac{1}{4}\left(\frac{c^3}{G\hbar}\right) A = \frac{1}{4} \frac{A}{l_{Pl}^2} \end{aligned}\right\} \quad G, \hbar, c \text{ all appear}$$

We'll need to develop some (more!) tools to get at these awesome results.

Def'n: $U \subset M$. A geodesic congruence in U is a family of geodesics such that exactly one geodesic passes through U



Can introduce co-ord's $x^\mu = (y^a, \lambda)$ with $a = 1,2,3$

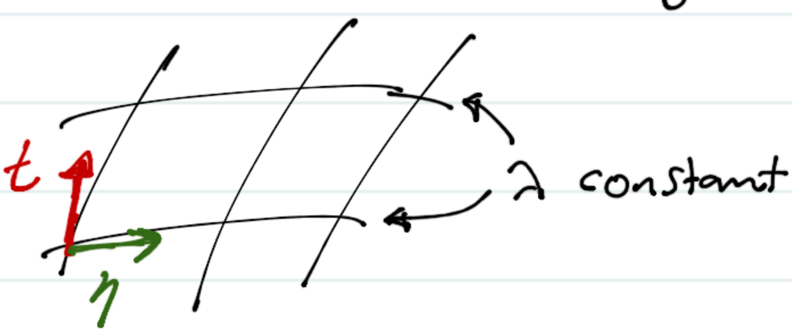
$$t = \frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x^\mu} = \frac{d}{d\lambda} \quad \text{tangent to curves}$$

ie. $t^\mu = (\underset{\sim}{0}, 1)$ & λ is an affine parameter ie. $t^\rho \nabla_\rho t^\mu = 0$

If $t^2 = -1$ have timelike geodesic congruence
$t^2 = 0$ " null " "

(will use ℓ instead of t for null case below)

The vectors $\eta_a \equiv \frac{\partial}{\partial y^a} = \frac{\partial x^\mu}{\partial y^a}\partial_\mu$ are "connecting vectors"



t & η_a commute because they form a coordinate basis.

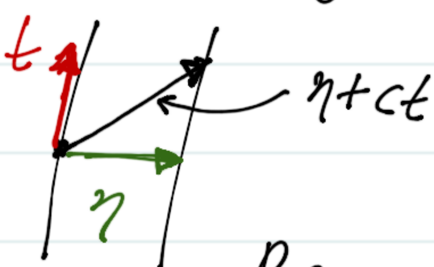
$$[t, \eta_a] = 0 \iff t^\mu \nabla_\mu \eta_a^\nu - \eta_a^\mu \nabla_\mu t^\nu = 0$$

$$\Rightarrow \frac{D}{d\lambda}\eta_a^\nu \equiv t^\mu \nabla_\mu \eta_a^\nu = B^\nu{}_\mu \, \eta_a^\mu$$

where

$B^\nu{}_\mu \equiv \nabla_\mu t^\nu$ measures geodesic deviation: the failure of the vectors η_a to be parallely transported along geodesics

<u>Note:</u> A geodesic near a fiducial geodesic is specified by η_a, but not uniquely: $\eta' = \eta + ct$ gives same geodesic.



Remove ambiguity:

<u>Timelike case:</u> Choose $\eta_a \cdot t = 0$ at a point.

Then
$$\frac{d}{d\lambda}(\eta \cdot t) = t^\nu \nabla_\nu (\eta^\rho t_\rho) = (t^\nu \nabla_\nu \eta^\rho)\, t_\rho$$
$$= B^\rho{}_\sigma \, \eta^\sigma t_\rho = \nabla_\sigma t^\rho \, \eta^\sigma t_\rho$$
$$= \frac{1}{2} \eta^\sigma \nabla_\sigma (t^2) = 0$$

Hence $\eta_a \cdot t = 0$ along the congruence

<u>Null case</u> – now denote tangent vector by l & $l^2 = 0$

$$B^\nu{}_\mu = \nabla_\mu l^\nu$$

Consider $\eta_a \cdot l = 0$. The 3 dimensional space of vectors orthogonal to l now includes l since $l^2 = 0$!

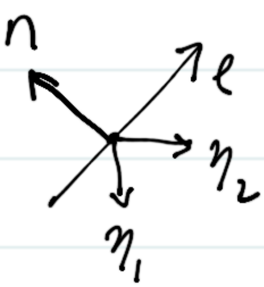
Can choose 2 <u>spacelike</u> connecting vectors η_i $i = 1, 2$

the 3rd connecting vector n chosen so that

1) $n^2 = 0$
2) $l \cdot n = -1$
3) $l^\mu \nabla_\mu n^\nu = 0$ i.e. n parallely transported

n.b. 1) & 2) consistent with 3)

Having selected n (like a gauge choice), choose η_i such that $\eta_i \cdot n = \eta_i \cdot l = 0$



The projector onto the 2-dimensional subspace spanned by η_i is

$$P^\mu{}_\nu = \delta^\mu{}_\nu + n^\mu l_\nu + l^\mu n_\nu$$

<u>check:</u> $P^2 = P$, $P \cdot n = P \cdot l = 0$, $P \cdot \eta_i = \eta_i$

We are interested in

$$\frac{D}{d\lambda} \eta_i^\mu = B^\mu{}_\nu \eta^\nu_i$$

but $B^\mu{}_\nu$ is a map on whole tangent space. We can turn it into a map on space spanned by η_i via

$$\frac{D}{d\lambda}\eta^\mu_i = \ell^\rho D_\rho \left(P^\mu{}_\nu \eta^\nu_i\right) \qquad \text{since } P\cdot\eta = \eta$$

$$= P^\mu{}_\nu\, \ell^\rho D_\rho \eta^\nu_i \qquad \text{since } \ell^\rho D_\rho \ell^\mu = 0 \text{ \& } \ell^\rho D_\rho n^\mu = 0$$

$$= P^\mu{}_\nu B^\nu{}_\rho \eta^\rho_i$$

$$= \underbrace{\left(P^\mu{}_\nu B^\nu{}_\rho P^\rho{}_\sigma\right)}_{\equiv \hat{B}^\mu{}_\sigma} \eta^\sigma_i$$

the projection of $B^\mu{}_\nu$ that we wanted.

Effectively $\hat{B}^\mu{}_\nu$ is a 2×2 matrix

Decompose:

$$\hat{B}^\mu{}_\nu = \tfrac{1}{2}\Theta P^\mu{}_\nu + \hat{\sigma}^\mu{}_\nu + \hat{\omega}^\mu{}_\nu$$

where

$$\begin{cases} \Theta = \hat{B}^\mu{}_\mu & \text{– "expansion"} \\ \hat{\sigma}_{\mu\nu} = \hat{B}_{(\mu\nu)} - \tfrac{1}{2} P_{\mu\nu} \hat{B}^\rho{}_\rho & \text{"shear"} \\ \hat{\omega}_{\mu\nu} = \hat{B}_{[\mu\nu]} & \text{"rotation} \end{cases}$$

<u>Lemma:</u> $\ell_{[\mu} \hat{B}_{\nu\rho]} = \ell_{[\mu} B_{\nu\rho]}$

Proof: Use $\ell^2 = 0$ & $\ell^\mu D_\mu \ell^\nu = 0$ to get, schematically, $\hat{B}_{\mu\nu} = P_\mu{}^\sigma B_{\sigma\rho} P^\rho{}_\nu = B_{\mu\nu} + \ell_\mu (\,)_\nu + (\,)_\mu \ell_\nu$

& the result follows.

Proposition If the congruence contains generators of null hypersurface $\mathcal{N}$ then $\hat{\omega} = 0$ on $\mathcal{N}$
Conversely, if $\hat{\omega} = 0$ everywhere then ℓ^μ is everywhere hypersurface orthogonal

Proof:

$$\ell_{[\mu} \hat{\omega}_{\nu\rho]} = \ell_{[\mu} \hat{B}_{\nu\rho]} = \ell_{[\mu} B_{\nu\rho]} = \ell_{[\mu} \nabla_\nu \ell_{\rho]} \quad \circledast$$

If ℓ^μ is normal to hypersurface $\mathcal{N}$ then Frobenius $\Rightarrow$

$\ell_{[\mu} \nabla_\nu \ell_{\rho]} = 0$ on $\mathcal{N}$. Then use $\circledast$

$\Rightarrow \ell_\mu \hat{\omega}_{\nu\rho} + \ell_\nu \hat{\omega}_{\rho\mu} + \ell_\rho \hat{\omega}_{\mu\nu} = 0$ on $\mathcal{N}$

multiply by n^μ

$\Rightarrow -\hat{\omega}_{\nu\rho} + \ell_\nu \underbrace{\hat{\omega}_{\rho\mu} n^\mu}_{0} + \ell_\rho \underbrace{n^\mu \hat{\omega}_{\mu\nu}}_{0} = 0$ on $\mathcal{N}$

since $\hat{\omega}$ has a projector P

$\Rightarrow \hat{\omega}_{\mu\nu} = 0$ on $\mathcal{N}$

Conversely if $\hat{\omega}_{\mu\nu} = 0$ everywhere then $\circledast$ implies

$\ell_{[\mu} \nabla_\nu \ell_{\rho]} = 0$ everywhere

& Frobenius $\Rightarrow$ ℓ^μ is hypersurface orthogonal

Interpretation of θ:

$\theta > 0$ neighbouring geodesics move apart

$\theta <$ " " " together

to see this: the magnitude of area element spanned by η_1 & η_2 can be written as

$$a = \varepsilon_{\mu\nu\rho\sigma} n^\mu \ell^\nu \eta_1^\rho \eta_2^\sigma$$

$$\frac{da}{d\lambda} = \ell^\alpha \nabla_\alpha a$$

$$= \varepsilon_{\mu\nu\rho\sigma} n^\mu \ell^\nu \left[\ell^\alpha \nabla_\alpha \eta_1^\rho \eta_2^\sigma + \eta_1^\rho \ell^\alpha \nabla_\alpha \eta_2^\sigma \right]$$

$$= \varepsilon_{\mu\nu\rho\sigma} n^\mu \ell^\nu \left[\hat{B}^\rho{}_\gamma \eta_1^\gamma \eta_2^\sigma + \eta_1^\rho \hat{B}^\sigma{}_\gamma \eta_2^\gamma \right]$$

$$= 2 \varepsilon_{\mu\nu\rho\sigma} n^\mu \ell^\nu \hat{B}^\rho{}_\gamma \eta_1^{[\gamma} \eta_2^{\sigma]}$$

$$= 2 \varepsilon_{\mu\nu\rho\sigma} n^\mu \ell^\nu \frac{\theta}{2} \eta_1^\rho \eta_2^\sigma$$

$$= a\,\theta$$

(to get the second last step can use explicit basis for tangent space)

Interpretation of $\hat{\sigma}$ - geodesics moving together and apart while preserving a

Interpretation of $\hat{\omega}$ - geodesics rotating.

Raychandhuri Equation

How does θ change along null congruence?

$$\frac{d\theta}{d\lambda} = \ell^\rho D_\rho (\hat{B}^\mu{}_\mu)$$

$\downarrow$ use $P^2 = P$

$$= \ell^\rho D_\rho (B^\mu{}_\nu P^\nu{}_\mu)$$

$\downarrow$ use $\ell^\rho D_\rho P^\mu{}_\nu = 0$

$$= P^\nu{}_\mu \, \ell^\rho D_\rho B^\mu{}_\nu$$

$$= P^\nu{}_\mu \, \ell^\rho D_\rho D_\nu \ell^\mu$$

$$= P^\nu{}_\mu \Big(\underbrace{\ell^\rho D_\nu D_\rho \ell^\mu}_{D_\nu(\cancel{\ell^\rho D_\rho \ell^\mu}) \to 0 \;\; - (D_\nu \ell^\rho)(D_\rho \ell^\mu)} + \underbrace{\ell^\rho [D_\rho, D_\nu] \ell^\mu}_{\ell^\rho R_{\rho\nu}{}^\mu{}_\sigma \ell^\sigma} \Big)$$

$$= \underbrace{- P^\nu{}_\mu B^\mu{}_\rho B^\rho{}_\nu}_{-P^\nu{}_\mu B^\mu{}_\rho (P^\rho{}_\sigma - n^\rho \ell_\sigma - \ell^\rho n_\sigma) B^\sigma{}_\nu} + \underbrace{P^\nu{}_\mu R_{\rho\nu}{}^\mu{}_\sigma \ell^\rho \ell^\sigma}_{-R_{\rho\sigma}\ell^\rho \ell^\sigma}$$

$$-P^\nu{}_\mu B^\mu{}_\rho (P^\rho{}_\sigma - n^\rho \ell_\sigma - \ell^\rho n_\sigma) B^\sigma{}_\nu = -P^\nu{}_\mu B^\mu{}_\rho P^\rho{}_\sigma B^\sigma{}_\nu$$

(since $\ell_\sigma B^\sigma{}_\nu = B^\mu{}_\rho \ell^\rho = 0$)

$$\Rightarrow \quad \frac{d\theta}{d\lambda} = -\hat{B}^\mu{}_\rho \hat{B}^\rho{}_\mu - R_{\rho\sigma}\ell^\rho \ell^\sigma$$

$$\Rightarrow \boxed{\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \hat{\sigma}^{\mu\nu}\hat{\sigma}_{\mu\nu} + \hat{\omega}^{\mu\nu}\hat{\omega}_{\mu\nu} - R_{\mu\nu}\ell^\mu\ell^\nu}$$

Can also get $\frac{d\hat{\sigma}}{d\lambda}$ & $\frac{d\hat{\omega}}{d\lambda}$ if desired.

The above expression has not yet used any physics: if we impose Einstein's equations

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \hat{\sigma}^{\mu\nu}\hat{\sigma}_{\mu\nu} + \hat{\omega}^{\mu\nu}\hat{\omega}_{\mu\nu} \underbrace{- 8\pi G T_{\mu\nu}\ell^\mu\ell^\nu}_{\text{constrained by the null energy condition.}}$$

<u>Proposition</u>

Einstein's eq's & the null energy condition $\Rightarrow$ the generators of a null hypersurface satisfy

$$\frac{d\theta}{d\lambda} \leq -\frac{1}{2}\theta^2$$

Proof:
- $\hat{\omega} = 0$ since ℓ is hypersurface orthogonal
- $\hat{\sigma}^{\mu\nu}\hat{\sigma}_{\mu\nu} \geq 0$ since metric on subspace of tangent space spanned by η_i is +ve definite
- $8\pi G T_{\mu\nu}\ell^\mu\ell^\nu \geq 0$ by null energy cond'n. //

Corollary

$\theta = \theta_0 < 0$ at some point on a generator γ of a null hypersurface, then $\theta \to -\infty$ along γ within an affine distance $\frac{2}{|\theta_0|}$ (provided γ extends this far)

Proof: Take affine parameter λ so that $\lambda = 0$ at pt p

$$\frac{d(\theta^{-1})}{d\lambda} = -\theta^{-2}\frac{d\theta}{d\lambda} \geq \frac{1}{2} \Rightarrow \theta^{-1} \geq \frac{\lambda}{2} + \theta_0^{-1}$$

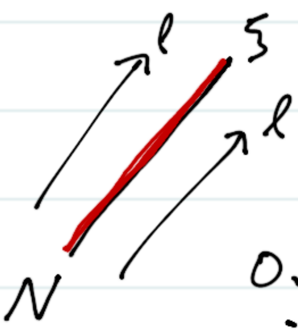
$$\Rightarrow \theta \leq \frac{\theta_0}{\left(1 + \frac{\lambda\theta_0}{2}\right)}$$

If $\theta_0 < 0$ then RHS $\to -\infty$ as $\lambda \to \frac{2}{|\theta_0|}$ //

Thus, if null geodesics are converging at a point, they continue to converge & $\theta \to -\infty$. This signals a breakdown of the congruence of geodesics, which will meet at some point called a caustic.

Proposition

Consider a null geodesic congruence that contains the generators of a Killing horizon $\mathcal{N}$. Then $\hat{B}_{\mu\nu}=0$ on $\mathcal{N}$.



Proof First, $\hat{\omega}_{\mu\nu}=0$ on $\mathcal{N}$ because the generators are hypersurface orthogonal.

Next. Let ξ be the Killing vector.

On $\mathcal{N}$ we have $\xi^\mu = f\ell^\mu$ for some f with $\ell^2=0$ & $\ell^\mu\nabla_\mu \ell^\rho=0$.

Suppose $\mathcal{N}$ is specified by an equation $S=0$.

Then off $\mathcal{N}$ $\ell^\mu = f^{-1}\xi^\mu + S V^\mu$ ← some smooth vector field

& we have

$$B_{\mu\nu} = \nabla_\nu \ell_\mu$$

$$= (\nabla_\nu f^{-1})\xi_\mu + f^{-1}\nabla_\nu\xi_\mu + (\nabla_\nu S)V_\mu + S\nabla_\nu V_\mu$$

$$\Rightarrow B_{(\mu\nu)} = (\nabla_{(\nu} f^{-1})\xi_{\mu)} + \nabla_{(\nu} S V_{\mu)} + S\nabla_{(\nu}V_{\mu)}$$

$$\Rightarrow B_{(\mu\nu)}\Big|_{\mathcal{N}} = \Big[\nabla_{(\nu} f^{-1}\xi_{\mu)} + \nabla_{(\nu} S V_{\mu)}\Big]_{\mathcal{N}}$$

Now $\xi_\mu = f\ell^\mu$ & $\nabla_\mu S$ are both proportional to ℓ_μ on $\mathcal{N}$

$$\Rightarrow \hat{B}_{(\mu\nu)}\Big|_{\mathcal{N}} = P_\mu{}^\rho B_{(\rho\sigma)} P^\sigma{}_\nu\Big|_{\mathcal{N}} = 0$$

since $P\cdot\ell = \ell\cdot P = 0$. Thus, $\hat{B}_{[\mu\nu]}=0$ & $\hat{B}_{(\mu\nu)}=0 \Rightarrow \hat{B}_{\mu\nu}=0$ //

Notice, in particular, if $\theta = 0$ everywhere on $\mathcal{N}$ then if λ is an affine parameter for generator of $\mathcal{N}$ $\frac{d\theta}{d\lambda} = 0$ on $\mathcal{N}$ too.

Corrolary: For a Killing horizon $\mathcal{N}$ of ξ, we have

$$R_{\mu\nu}\xi^\mu \xi^\nu \big|_{\mathcal{N}} = 0$$

Proof: Use $\frac{d\theta}{d\lambda} = 0$ & $\hat{B}_{\mu\nu} = 0$ in Raychaudhuri eq'n. //

We will need the following

Theorem: Hawking 1972

In a stationary asymptotically flat, analytic black hole spacetime, H^+ is a Killing horizon.

Note: H^+ is not necessarily Killing horizon with respect to stationary Killing vector. e.g. for Kerr we have already seen $\xi^\mu = k^\mu + \Omega_H m^\mu$.

$\boxed{\text{Zeroth Law}}$ κ is constant on H^+ for a stationary black hole provided $T_{\mu\nu}$ satisfies the dominant energy condition

Note: We might appeal to the uniqueness theorems to get stationary $\Rightarrow$ Kerr-Newman family of solutions & then verify directly κ is constant on H^+. The following proof is much simpler as it does not assume assumptions entering uniqueness theorems.

Proof Hawking's theorem $\Rightarrow \exists$ Killing vector ξ normal to H^+. We have shown $R_{\mu\nu}\xi^\mu\xi^\nu|_{H^+} = 0$
If we now use Einstein's equations as well as $\xi^2 = 0$ on H^+ we deduce $T_{\mu\nu}\xi^\mu\xi^\nu|_{H^+} = 0$

So if $J_\mu = -T_{\mu\nu}\xi^\nu$ we have $J_\mu \xi^\mu|_{H^+} = 0$
Since ξ^μ is future directed & causal, the dominant energy condition $\Rightarrow J^\mu$ is also (unless $J^\mu = 0$).

$J_\mu \xi^\mu |_{H^+} = 0 \Rightarrow J^\mu$ can be expanded on basis of tangent vectors for H^+:

$$J = a\xi + b_1 \eta_1 + b_2 \eta_2 \qquad \text{with } \xi \cdot \eta_i = 0$$

This is spacelike or null (when $b_i = 0$).

But we need J to be causal $\Rightarrow$ timelike or null

$\Rightarrow b_i = 0$ & $J^\mu \propto \xi^\mu$ on H^+

Thus,

$$0 = \xi_{[\mu} J_{\nu]} |_{H^+}$$

$$= - \xi_{[\mu} T_{\nu]\rho} \xi^\rho \Big|_{H^+}$$

Einstein's eq's

$$= -\frac{1}{8\pi} \xi_{[\mu} R_{\nu]\rho} \xi^\rho \Big|_{H^+}$$

See problem sheet 5

$$\Rightarrow \quad 0 = \xi_{[\mu} \nabla_{\nu]} K \Big|_{H^+}$$

Now if $A \wedge B = 0$ for two one-forms then $A \propto B$ (use local coord's $A = f dx^1$ & $B = B_\mu dx^\mu$)

$\Rightarrow \nabla_\nu K \propto \xi_\nu$ on H^+

$\Rightarrow t^\nu \nabla_\nu K \propto t^\nu \xi_\nu = 0$ for any tangent vector to H^+

$\Rightarrow K$ is constant on H^+ //

First Law

Take a stationary black hole of mass M, charge Q and angular momentum J. Perturb it and it will settle down to a stationary black hole with parameters $M+dM$, $Q+dQ$, $J+dJ$ with

$$dM = \frac{\kappa}{8\pi} dA + \Omega_H dJ + \Phi_H dQ$$

κ - surface gravity
Ω_H - angular velocity
Φ_H - electric potential } define below
A - area of black hole

Proof: Uniqueness theorems $\Rightarrow$ the black hole before & after is Kerr-Newman (with $P=0$). Using Kerr coord's

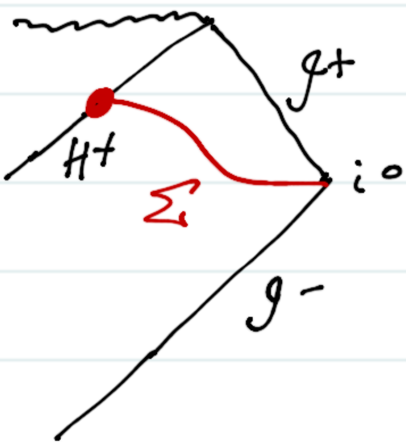
$$ds^2 = -\frac{\Delta - a^2\sin^2\theta}{\Sigma} dv^2 + 2dv\,dr - 2a\sin^2\theta\,\frac{r^2+a^2-\Delta}{\Sigma} dv\,d\chi$$

$$-2a\sin^2\theta\, d\chi\, dr + \left(\frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\right)\sin^2\theta\, d\chi^2 + \Sigma d\theta^2$$

$$A = -\frac{Qr}{\Sigma}\left(dv - a\sin^2\theta\, d\chi\right) + \cancel{\frac{Qr}{\Delta}dr}$$ delete by gauge trans'n.

$$\Delta = (r-r_+)(r-r_-)\,, \quad r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2}\,, \quad a = \frac{J}{M}$$

Area of a black hole horizon is the area of the intersection of H^+ with partial Cauchy surface



e.g. for Kerr set $v =$ constant & $r = r_+$

$$ds^2 \to \frac{(r_+^2+a^2)^2}{\Sigma}\sin^2\theta\, d\chi^2 + \Sigma\, d\theta^2$$

$$\equiv h_{ij}\, dx^i dx^j$$

$$\Rightarrow \sqrt{h} = (r_+^2 + a^2)\sin\theta$$

$$\text{Area} = \int d\theta\, d\chi \sqrt{h} = 4\pi\,(r_+^2 + a^2)$$

$$= 4\pi\left(2M^2 - Q^2 + 2\sqrt{M^4 - J^2 - M^2Q^2}\right)$$

(note: would get the same answer in B-L coord's if we set $t =$ const & $r = r_+$, but this is dodgy).

Have already:

$$\Omega_H = \frac{a}{r_+^2 + a^2} = \frac{J}{M}\,\frac{1}{2M^2 - Q^2 + 2\sqrt{M^4 - J^2 - M^2Q^2}}$$

d $$K = \frac{r_+ - r_-}{2(r_+^2 + a^2)} = \frac{1}{M} \frac{\sqrt{M^4 - J^2 - M^2 Q^2}}{2M^2 - Q^2 + 2\sqrt{M^4 - J^2 - M^2 Q^2}}$$

Φ_H is defined as the co-rotating potential difference between infinity & the horizon: the work required to move a unit charge from $r = \infty$ to $r = r_+$.

$$\Phi_H \equiv \xi^\mu A_\mu \big|_\infty - \xi^\mu A_\mu \big|_{r = r_+}$$

with
$$\xi = k + \Omega_H m = \partial_v + \Omega_H \partial_x$$

Calculate
$$\xi^\mu A_\mu = -\frac{Qr}{\Sigma} \left(1 - a \sin^2\theta\, \Omega_H\right)$$
$$= -\frac{Qr}{\Sigma} \frac{1}{r_+^2 + a^2} \left(r_+^2 + a^2 \cos^2\theta\right)$$

$$\xi^\mu A_\mu \big|_\infty \to 0$$
$$\xi^\mu A_\mu \big|_{r_+} \longrightarrow \frac{-Q r_+}{r_+^2 + a^2}$$

$$\Rightarrow \Phi_H = \frac{Q r_+}{r_+^2 + a^2} = \frac{Q}{M} \frac{M^2 + \sqrt{M^4 - J^2 - Q^2 M^2}}{2M^2 - Q^2 + 2\sqrt{M^4 - J^2 M^2 Q^2}}$$

Now: $A = A(M, J, Q)$

$$\Rightarrow dA = \frac{\partial A}{\partial M} dM + \frac{dA}{dJ} dJ + \frac{dA}{\partial Q} dQ$$

$\downarrow$ ex.

$$\Rightarrow dM = \frac{1}{8\pi} \kappa \, dA + \Omega_H dJ + \Phi_H dQ$$

//

Comments

1) The reason this argument is so simple is that we use all the hard work that has gone into proving the uniqueness theorems.

2) This version of the first Law compares 2 different spacetimes. $\exists$ other versions of the first Law - "physical process" versions - in which one relates change in horizon area to mass, angular momentum & charge of matter that is thrown into a black hole.

$\boxed{\text{Second Law}}$ Hawking's Area theorem

will need some topological concepts

- A set S is <u>open</u> if $\forall p \in S$ $\exists$ neighbourhood U of p with $U \subset S$ (e.g. (a,b) open interval $[a,b)$ & $[a,b]$ not open)

- Define $\bar{S}$ as the <u>topological closure</u> of S: the union of S & it's limit points

- Define $\mathrm{Int}(S)$ as the <u>interior</u> of S: the largest open set contained in S

- Define $\dot{S}$ as the <u>topological boundary</u> of S:
$$\dot{S} \equiv \bar{S} / \mathrm{Int}(S)$$

Recall

chronological $\binom{\text{future}}{\text{past}}$ of set U is $I^{\pm}(U)$ $\binom{\text{does not}}{\text{include } U}$

Causal $\binom{\text{future}}{\text{past}}$ of set U is $J^{\pm}(U)$ $\binom{\text{does}}{\text{include } U}$

$I^{\pm}(U)$ are open sets : small deformations of timelike curves remain timelike

$J^{\pm}(U)$ may or may not be closed

e.g. point p in Minkowski spacetime, $J^-(p)$ is closed

eg remove a point r in Minkowski spacetime

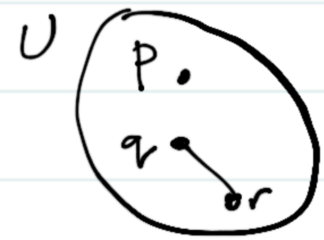
$J^-(p)$ is not closed as limit points here are missing

In general we have

$$I^-(U) = \mathrm{Int}\left(J^-(U)\right)$$

$$\overline{I^-(U)} = \overline{J^-(U)}$$

Equivalence principle $\Rightarrow$ every spacetime is locally like Minkowski $\Rightarrow$ for any point $p \in M$ $\exists$ a convex normal neighbourhood U which has causal structure of M

$\Rightarrow$ 

for any $q, r \subset U$ $\exists$ a unique geodesic connecting q & r.

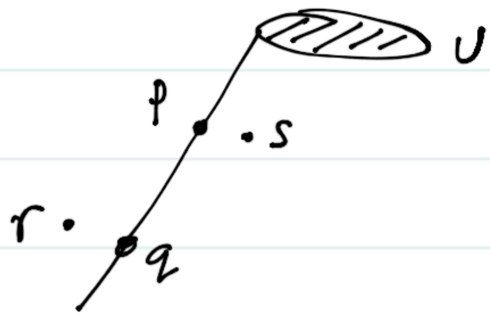
$\exists$ a stronger result: $q \in J^-(p) / I^-(p)$ then $\exists$ a null geodesic from q to p.

Idea of proof: use a finite cover of convex normal neighbourhoods & use result above.

Also can show $\dot{I}^\pm(U) = \dot{J}^\pm(U)$

<u>Theorem</u> $\dot{J}^-(U)$ is an achronal submanifold i.e. no 2 points are timelike separated.

Proof:



Let $p, q \in \dot{J}^-(U)$ & $q \in I^-(p)$

Want a contradiction.

$I^-(p)$ is open $\Rightarrow$ $\exists$ r near q, $r \in I^-(p)$ & $r \notin J^-(U)$

$I^+(r)$ is open $\Rightarrow$ $\exists$ s near p, $s \in I^+(p)$ & $s \in J^-(U)$

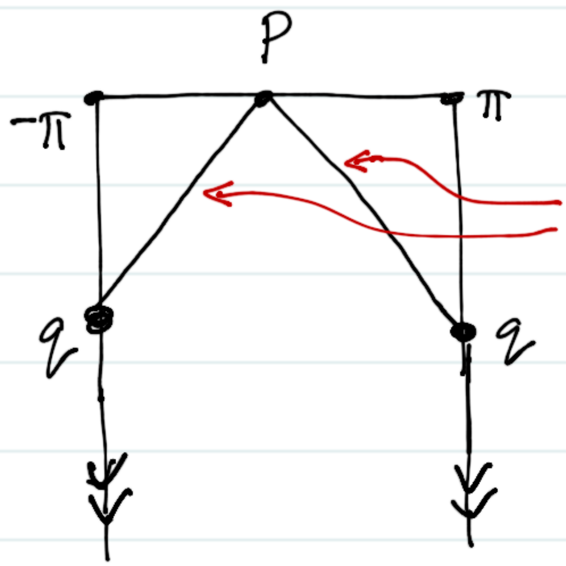
Hence, $\exists$ causal curve from r to s to U

$\Rightarrow r \in J^-(U)$, a contradiction.

(That $\dot{J}^-(U)$ is a submanifold is shown in Wald

Example: take flat cylinder $ds^2 = -dt^2 + d\phi^2$

$-\pi \leq \phi < \pi$



$\dot{J}^-(p)$ is 2 null geodesic segments (have past endpoints – if we extend them back further, they leave $\dot{J}^-(p)$)

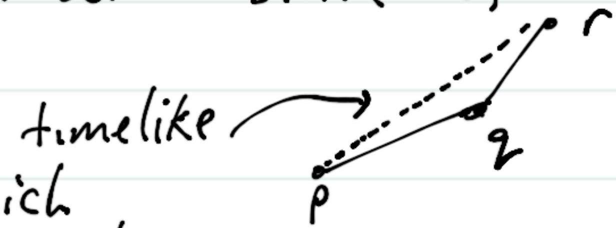
Theorem: Let $\mathcal{U}$ be closed. Every $p \in \dot{J}^-(\mathcal{U})$ lies on a null geodesic in $\dot{J}^-(\mathcal{U})$ that is future inextendible or has a future endpoint on $\mathcal{U}$.

Idea of proof:

sequence of timelike curves. By considering the limit, construct a q with γ null.

One got from p to q, keep going, from q to r.

Note that p to q to r must all lie on same null generator since if



which can't happen.

From the above discussion:

<u>Properties</u> of the future event horizon

$H^+ = \dot{J}^-(\mathscr{I}^+)$ – topological boundary of the causal past of $\mathscr{I}^+$

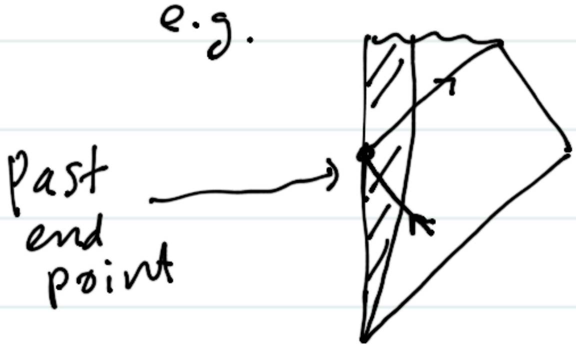
i) H^+ is a null hypersurface

ii) H^+ is achronal (note this only follows locally from (i))

iii) Null geodesic generators of H^+ have no future end points.

(They can have past end points

e.g.



2nd Law: Hawking's Area Theorem

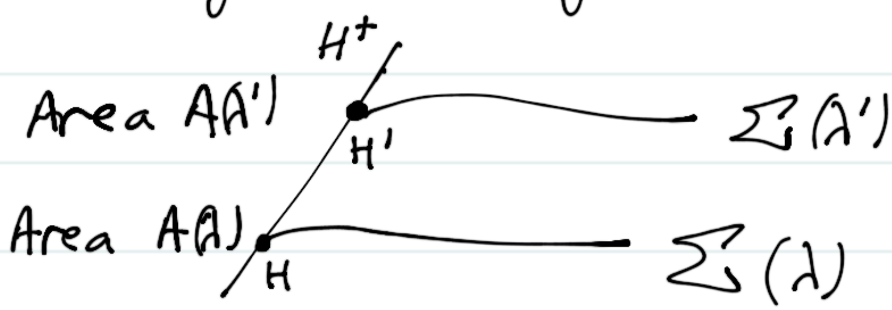
Assume
- $T_{\mu\nu}$ satisfies null energy condition
- Cosmic censorship

then, the area of H^+ is non-decreasing in time

Sketch of proof

Cosmic censorship $\Rightarrow \exists$ a family of Cauchy surfaces $\Sigma_3(\lambda)$ with $\Sigma_3(\lambda') \subset D^+(\Sigma_3(\lambda))$ if $\lambda' > \lambda$.

We can choose λ to be an affine parameter on a generator of H^+:



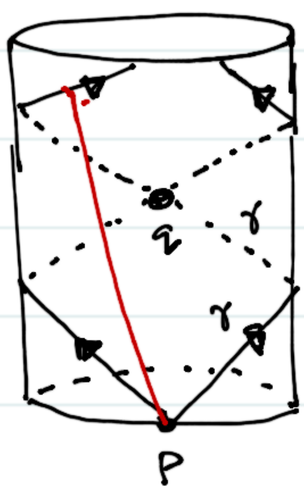
Want to show $A(\lambda') \geq A(\lambda)$ when $\lambda' \geq \lambda$.

It is sufficient to show that each area element a of H has this property.

Since $\dfrac{da}{d\lambda} = \theta a$, want to show $\theta \geq 0$ on H^+

Suppose this was not true: if $\theta < 0$ at a point p, then the null energy condition $\Rightarrow$ geodesics must form a caustic. This implies geodesics that are nearby to a given one passing through p will intersect at a finite distance along it. The first point q for which this happens is called the point "conjugate to p on γ". Can show that all points beyond q are no longer null separated.

(To get a flavour consider again the two-d cylindrical spacetime



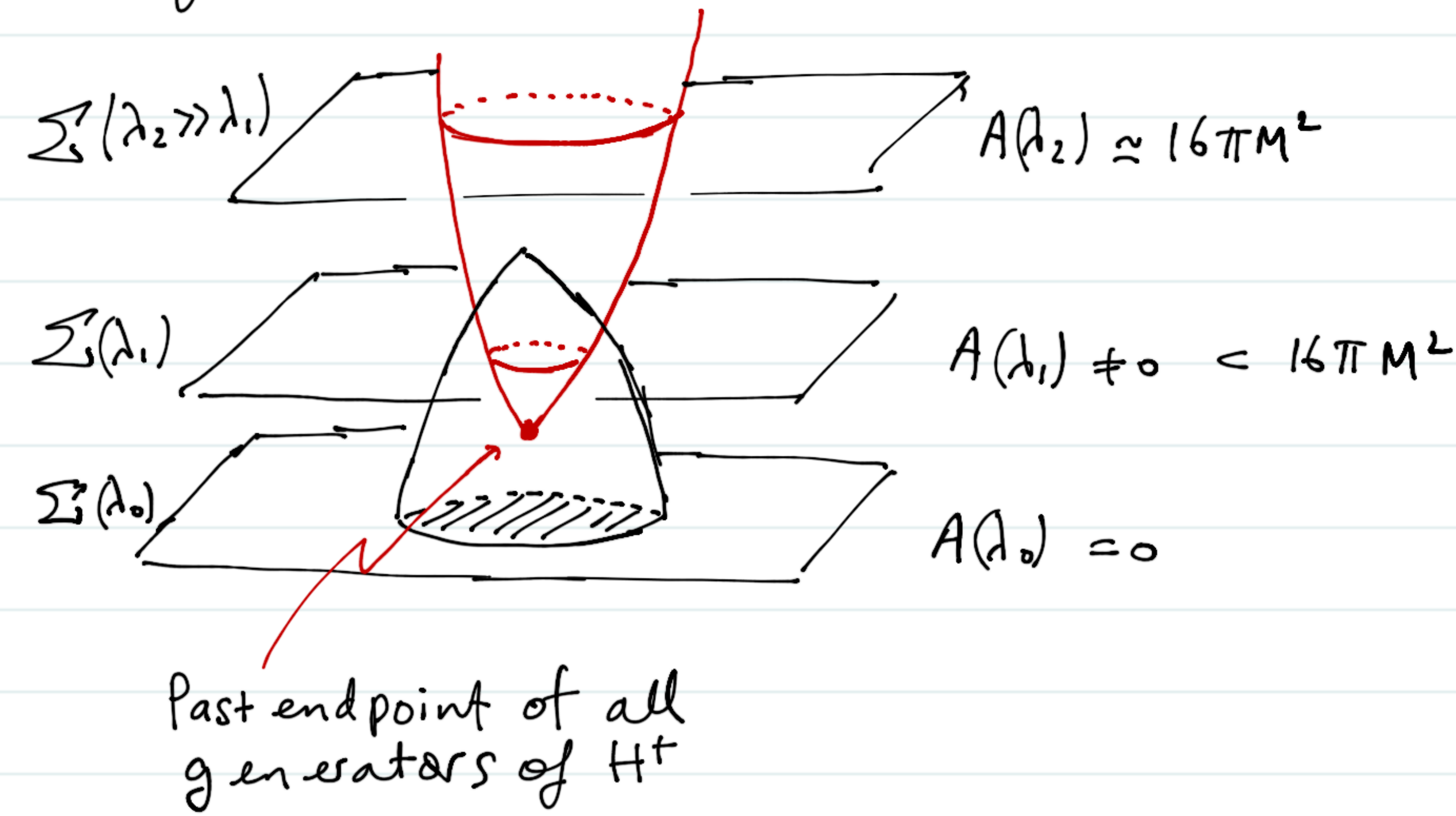
Red curve is timelike

)

$\Rightarrow$ γ must have left H^+ (otherwise H^+ not achronal). But this can't happen. //

Example. Spherically symmetric collapse

Eddington - Finklestein diagram



Consequences of 2^{nd} Law

1) For Schwarzschild $A = 16\pi M^2$

Thus if we start with schwarzschild & end with schwarzschild, M must increase.

(clearly shows the need for an energy cond'n. in the 2^{nd} Law, otherwise could throw in -ve energy).

2) Consider 2 non-rotating black holes, each approximating schwarzschild with masses M_1 & M_2. Suppose the coalesce to form third schwartschild black hole with mass M_3.

$$A_3 \geq A_1 + A_2$$
$$\Rightarrow M_3^2 \geq M_1^2 + M_2^2$$
$$\Rightarrow M_3 \geq \sqrt{M_1^2 + M_2^2}$$

Energy radiated is $M_1 + M_2 - M_3$ which could be used to do work. The efficiency of this is

$$\eta = \frac{M_1 + M_2 - M_3}{M_1 + M_2} \leq 1 - \frac{\sqrt{M_1^2 + M_2^2}}{M_1 + M_2} = \frac{\sqrt{1 + (M_2/M_1)^2}}{1 + (M_2/M_1)}$$
$$\leq 1 - \frac{1}{\sqrt{2}}$$

& saturated when $M_1 = M_2$.

3) Black holes can't bifurcate.

eg Sch. $M_3 \rightarrow M_1 + M_2$

2nd Law: $M_3 \leq \sqrt{M_1^2 + M_2^2} \leq M_1 + M_2$

& equality for $M_1 = M_2 = 0$

Energy conservation: $M_3 \geq M_1 + M_2$

Hence, can't happen.