Black hole Mechanics / Thermodynamics

1973: Bardeen, Carter, Hawking

**black hole mechanics (G.R.)**

Law 0: The surface gravity $k$ is constant on $\mathcal{H}^+$ for a stationary black hole

Law 1: $dM = \frac{k}{8\pi} dA + 2\pi\Phi d\Omega + \Xi d\phi$

Law 2: $dA > 0$

Law 3: Can't achieve $k = 0$ by physical process

**Thermodynamics**

Law 0: Temperature $T$ is constant throughout a system in thermal equilibrium

Law 1: $dE = TdS + \mu dN$

Law 2: $dS > 0$

Law 3: Can't achieve $T = 0$ by physical process

(There is a game, you can't win, you must lose, you can't quit!)
Initial thinking: nothing can escape from a black hole, so can’t have a temperature & hence no entropy.

Bekenstein: Real 2nd law would be violated if black holes had no entropy. (Because you could throw entropic objects into the black hole & hence lower total entropy). He proposed generalised 2nd law: \[ d S_{\text{TOTAL}} \geq 0 \quad S_{\text{TOTAL}} = S_{\text{EXT}} + S_{\text{BH}} \]

\[ + S_{\text{BH}} \propto A \]

Hawking 1974 \quad G.R. + QFT

\[ T_{\text{H}} = \frac{\hbar}{k_B} \frac{1}{C} \frac{\kappa}{2\pi} \]

\[ S_{\text{BH}} = \frac{1}{4} \frac{\sqrt{3}C^3}{G\hbar} A = \frac{1}{4} \frac{A}{\ell_P^2} \]

\{ \text{G, } \hbar, \text{ C all appear} \}

We’ll need to develop some (more !) tools to get at these awesome results.
Defn: U ⊂ M. A geodesic congruence in U is a family of geodesics such that exactly one geodesic passes through U.

Can introduce co-ords $x^m = (y^0, 1)$ with

$$t = \frac{d}{d\lambda} \frac{dx^m}{dx^\lambda} = \frac{d}{d\lambda}$$

tangent to curves

i.e. $t^m = (0, 1)$ & $\lambda$ a an affine parameter i.e. $t \cdot \partial \phi t^m = 0$

If $t^2 = -1$ have timelike geodesic congruence

$t^2 = 0$ "null"

(Will use $\ell$ instead of $t$ for null case below)

The vectors $\eta_a = \frac{d}{dy^a} = \frac{dx^m}{dy^a} \partial_m$ are "connecting vectors".

$t$ & $\eta_a$ commute because they form a coordinate basis.
\[ \nabla_{\gamma a} t = 0 \iff t^\mu \nabla_\mu \gamma_a^\nu - \gamma_a^\nu \nabla_\mu t^\nu = 0 \]

\[ \Rightarrow \frac{D}{d\lambda} \gamma_a^\nu = t^\mu \nabla_\mu \gamma_a^\nu = B^\nu_{\mu} \gamma_a^\mu \]

where

\[ B^\nu_{\mu} = \nabla_{\mu} t^\nu \] measures geodesic deviation: the failure of the vectors \( \gamma_a \) to be parallelly transported along geodesics.

Note: A geodesic near a fiducial geodesic is specified by \( \gamma_a \), but not uniquely:

\[ \eta' = \eta + c t \] gives same geodesic.

Remove ambiguity:

Timelike case: Choose \( \eta_a \cdot t = 0 \) at a point.

Then

\[ \frac{d}{d\lambda} (\eta_a \cdot t) = t^\nu \nabla_\nu (\eta_a \cdot t) = (t^\nu \nabla_\nu \eta_a) \cdot t = B^\sigma_{\nu} \eta^\nu \cdot t = \nabla_\sigma t^\sigma \cdot t = \frac{1}{2} \eta^\sigma \nabla_\sigma (t^2) = 0 \]

Hence \( \eta_a \cdot t = 0 \) along the congruence.
Null case — now denote tangent vector by $\ell \perp \ell^2 = 0$

\[ B^\nu \nu = \ell \nu \]

Consider $\eta_i \cdot \ell = 0$. The 3-dimensional space of vectors orthogonal to $\ell$ now include $\ell$ since $\ell^2 = 0$.

Can choose 2 spacelike connecting vectors $\eta_i \perp \ell$ with $i=1,2$, the 3rd connecting vector $n$ chosen so that
1) $n^2 = 0$
2) $\ell \cdot n = -1$
3) $\ell^\nu \eta^\mu \eta_\mu = 0$ i.e. $n$ parallelly transported

n.b. 1) & 2) consistent with 3)

Having selected $n$ (like a gauge choice), choose $\eta_i$ such that $\eta_i \cdot n = \eta_i \cdot \ell = 0$

The projector onto the 2-dimensional subspace spanned by $\eta_i$ is

\[ p^\mu \nu = \delta^\mu \nu + n^\mu \ell_\nu + \ell^\mu n_\nu \]

Check: $P^2 = P$, $P \cdot n = P \cdot \ell = 0$, $P \cdot \eta_i = \eta_i$

We are interested in

\[ \frac{d}{dt} \eta_i^\mu = B^\mu \nu \eta_\nu^i \]
but $B^{\mu \nu}$ is a map on whole tangent space. We can turn it into a map on space spanned by $\eta$: via
\[
\frac{D}{\partial \eta} \eta^i = l^5 \delta_P (p^\mu \eta^\nu) \quad \text{since } p \cdot \eta = \eta
\]
\[
= p^\mu \eta \quad l^5 \delta_P \eta^\nu \quad \text{since } l^5 \delta_P \ell^m = 0
\]
\[
= p^\mu B^\nu \eta^i
\]
\[
= (p^\mu B^\nu \psi p^s \phi) \eta^s
\]
\[
\equiv \hat{B}^{\mu \sigma} \quad \text{the projection of } B^{\mu \nu} \text{ that we wanted.}
\]

Effectively $\hat{B}^{\mu \nu}$ do a $2 \times 2$ matrix

Decompose:
\[
\hat{B}^{\mu \nu} = \frac{1}{2} \sigma p^{\mu \nu} + \hat{\sigma}^{\mu \nu} + \hat{\omega}^{\mu \nu}
\]

where
\[
\begin{cases}
0 = \hat{B}^{\mu \mu} & \text{``expansion''} \\
\hat{\sigma}^{\mu \nu} = \hat{B}(\nu) - \frac{1}{2} p_{\mu \nu} \hat{B}^{\sigma \rho} & \text{``shear''} \\
\hat{\omega}^{\mu \nu} = \hat{B}[\nu] & \text{``rotation''}
\end{cases}
\]

**Lemma:** $[\mu, \nu]^{\hat{B}} = [\mu, \nu] B^{\psi \rho}$

**Proof:** Use $l^2 = 0 \Rightarrow l^m \ell^\nu = 0$ to get schematically, $\hat{B}^{\mu \nu} = p^\sigma B^{\psi \rho} p^\nu = B^{\mu \nu} + \ell^\nu \omega^{\mu}_\nu + \frac{1}{2} \omega^{\mu \nu}$
d the result follows.
Proposition: If the congruence contains generators of null hypersurface \( N \) then \( \hat{\omega} = 0 \) on \( N \).

Conversely, if \( \hat{\omega} = 0 \) everywhere then \( \mathbf{e}^m \) is everywhere hypersurface orthogonal.

Proof:

\[
\{ \Psi \hat{w}_{\nu \rho} \} = \{ \Psi \hat{b}_{\nu \rho} \} = \{ \Psi b_{\nu \rho} \} = \{ \Psi \mathbf{E}_{\nu \rho} \} \quad \ast
\]

If \( \mathbf{e}^m \) is normal to hypersurface \( N \) then Frobenius \( \Rightarrow \)

\[
\{ \Psi \mathbf{E}_{\nu \rho} \} = 0 \quad \text{on} \quad N.
\]

Then use \( \ast \)

\[
\Rightarrow \quad \mathbf{e}^m \hat{w}_{\nu \rho} + \mathbf{e}^\nu \hat{w}_{\rho m} + \mathbf{e}^\rho \hat{w}_{m \nu} = 0 \quad \text{on} \quad N
\]

Multiply by \( \mathbf{e}^m \)

\[
\Rightarrow \quad -\hat{w}_{\nu \rho} + \mathbf{e}^\nu \mathbf{E}_{\rho m} \mathbf{e}^m + \mathbf{e}^\rho \mathbf{E}_{m \nu} \mathbf{e}^\nu = 0 \quad \text{on} \quad N
\]

since \( \hat{\omega} \) has a projector \( \mathbf{P} \)

\[
\Rightarrow \quad \hat{\omega}_{\nu \rho} = 0 \quad \text{on} \quad N
\]

Conversely, if \( \hat{\omega}_{\nu \rho} = 0 \) everywhere then \( \ast \) implies

\[
\{ \Psi \mathbf{E}_{\nu \rho} \} = 0 \quad \text{everywhere}
\]

\( \Rightarrow \) Frobenius \( \Rightarrow \mathbf{e}^m \) is hypersurface orthogonal.
Interpretation of $\theta$:

- $\theta > 0$ neighboring geodesics move apart
- $\theta < 0$ together

to see that: the magnitude of area element

spanned by $\gamma_1$ and $\gamma_2$ can be written as

$$a = \sum_{\mu} \eta^\mu e^\nu \eta_1^\nu \eta_2^\nu$$

$$\frac{da}{dl} = e^\nu D_\alpha a$$

$$= \sum_{\mu} \eta^\mu e^\nu \left[ e^\nu D_\alpha \eta_1^\nu \eta_2^\nu + \eta_1^\nu e^\nu D_\alpha \eta_2^\nu \right]$$

$$= \sum_{\mu} \eta^\mu e^\nu \left[ \hat{\theta} \gamma_1^\nu \eta_2^\nu + \eta_1^\nu \hat{\theta} \gamma_2^\nu \eta_2^\nu \right]$$

$$= 2 \sum_{\mu} \eta^\mu e^\nu \hat{\theta} \gamma_1^\nu \eta_2^\nu$$

$$= 2 \sum_{\mu} \eta^\mu e^\nu \frac{\theta}{2} \eta_1^\nu \eta_2^\nu$$

$$= a \theta$$

(to get the second last step can use explicit basis for tangent space)

Interpretation of $\hat{\theta}$ - geodesics moving together and apart while preserving a

Interpretation of $\hat{\omega}$ - geodesics rotating.
Raychaudhuri Equation

How does $\theta$ change along null congruence?

$$\frac{d \theta}{d \lambda} = \ell^\mu \nabla_\mu (B^\nu_\rho R^\rho_\nu - R_{\rho \delta \epsilon \sigma} \ell^\rho \ell^\delta \ell^\epsilon \ell^\sigma)$$

Use $\ell^2 = 0$

$$= \ell^\mu \nabla_\mu (B^\nu_\rho R^\rho_\nu)$$

Use $\ell^\mu \nabla_\mu \rho^\nu = 0$

$$= \rho^\nu_\mu \ell^\mu \nabla_\nu (B^\rho_\sigma)$$

Use $\rho^\nu_\mu = 0$

$$= \rho^\nu_\mu (\ell^\mu \nabla_\nu \ell^m + \ell^\mu [\nabla_\nu \ell^m] \ell^\nu)$$

Use $\ell^\nu \ell^m \ell^\nu \ell^m = 0$

$$= -\rho^\nu_\mu B^\mu_\sigma B^\sigma_\nu + \rho^\nu_\mu R_{\rho \sigma \mu \nu} \ell^\rho \ell^\sigma \ell^\mu \ell^\nu$$

Use $\rho^\nu_\mu = 0$

$$= -\rho^\nu_\mu B^\mu_\sigma (\rho_{\sigma \nu} - \eta_{\sigma \nu} \ell^\sigma \ell^\nu) B^\sigma_\nu$$

Since $\ell^\sigma \ell^\nu B^\mu_\sigma = B^\mu_\sigma \ell^\sigma \ell^\nu = 0$

$$\Rightarrow \frac{d \theta}{d \lambda} = -\hat{B}^\mu_\sigma \hat{B}^\sigma_\mu - R_{\rho \sigma \mu \nu} \ell^\rho \ell^\sigma \ell^\mu \ell^\nu$$
\[
\frac{d\Theta}{d\lambda} = -\frac{1}{2} \Theta^2 - \delta^m_n \delta_{\mu\nu} \Theta + \Theta \hat{\Theta} - R_{\mu\nu} \Theta^\mu_n^\nu
\]

Can also get \( \frac{d\hat{\Theta}}{d\lambda} \) if desired.

The above expression has not yet used any physics: if we impose Einstein's equations

\[
\frac{d\Theta}{d\lambda} = -\frac{1}{2} \Theta^2 - \delta^m_n \delta_{\mu\nu} \Theta + \Theta \hat{\Theta} - 8\pi G T_{\mu\nu} \Theta^\mu_n^\nu
\]

constrained by the null-energy condition.

**Proposition**

Einstein's eq's & the null-energy condition \( \rightarrow \) the generators of a null hypersurface satisfy

\[
\frac{d\Theta}{d\lambda} \leq -\frac{1}{2} \Theta^2
\]

**Proof:**

1. \( \hat{\Theta} = 0 \) since \( \lambda \) is hypersurface orthogonal.
2. \( \delta^m_n \delta_{\mu\nu} \geq 0 \) since metric on subspace of tangent space spanned by \( \eta_i \) is positive definite.
3. \( 8\pi G T_{\mu\nu} \lambda^\mu_n^\nu \geq 0 \) by null energy cond'n.
Corollary

\( \theta = \theta_0 < 0 \) at some point on a generator \( \gamma \) of a null hypersurface, then \( \theta \to -\infty \) along \( \gamma \) within an affine distance \( \frac{2}{1001} \) (provided \( \theta \) extends this far)

Proof: Take affine parameter \( \lambda \) so that \( \lambda = 0 \) at pf \( P \)

\[
\frac{d(\theta^{-1})}{d\lambda} = -\theta^{-2} \frac{d\theta}{d\lambda} > \frac{1}{2} \Rightarrow \theta^{-1} > \frac{3}{2} + \theta_0^{-1}
\]

\[
\Rightarrow \theta < \frac{\theta_0}{(1 + \frac{3\theta_0}{2})}
\]

If \( \theta_0 < 0 \) then RHS \( \to -\infty \) as \( \lambda \to \frac{2}{1001} \)

Thus, if null geodesics are converging at a point, they continue to converge as \( \theta \to -\infty \). This signals a breakdown of the congruence of geodesics, which will meet at some point called a caustic.
Proposition

Consider a null geodesic congruence that contains the generators of a Killing horizon \( N \). Then \( \overset{\sim}{\gamma}_{\mu\nu} = 0 \) on \( N \).

Proof: First, \( \overset{\sim}{\gamma}_{\mu\nu} = 0 \) on \( N \) because the generators are hypersurface orthogonal.

Next, let \( \xi \) be the Killing vector.

On \( N \) we have \( \xi^m = f l^m \) for some \( f \) with \( l^2 = 0 \) and \( l \xi = 0 \).

Suppose \( N \) is specified by an equation \( S = 0 \). Then off \( N \) \( l^m = f^{-1} \xi^m + S V^m \) some smooth vector field

\[ \beta_{\mu\nu} = \nabla_\nu l_\mu = \left( \nabla_\nu f^{-1} \right) \xi^m + f^{-1} \nabla_\nu \xi_m + \left( \nabla_\nu S \right) V_m + S \nabla_\nu V_m \]

\[ \Rightarrow \beta_{(\mu\nu)} = \left( \nabla_\nu f^{-1} \right) \xi_m + \left( \nabla_\nu S \right) V_m + S \nabla_\nu V_m \]

\[ \Rightarrow \beta_{(\mu\nu)} \big|_N = \left[ \nabla_\nu f^{-1} \xi_m + \nabla_\nu S V_m \right] \big|_N \]

Now \( \xi_m = f l^m \) and \( \nabla_\nu S \) are both proportional to \( l^m \) on \( N \)

\[ \Rightarrow \beta_{(\mu\nu)} \big|_N = P^\rho l_\mu \beta_{(\rho\nu)} \big|_N = 0 \]

Since \( p.l = l.p = 0 \). Thus, \( \beta_{\mu\nu} = 0 \) and \( \overset{\sim}{\gamma}_{\mu\nu} = 0 \) on \( N \).

\[ \therefore \beta_{\mu\nu} = 0 \]
Notice, in particular, if \( \Theta = 0 \) everywhere on \( \mathcal{N} \) then \( \zeta \) is an affine parameter for generator of \( \mathcal{N} \), \( \frac{d\Theta}{d\zeta} = 0 \) on \( \mathcal{N} \) too.

**Corollary:** For a Killing horizon \( \mathcal{N} \) of \( \mathcal{I} \), we have
\[
R_{\mu
u} \tilde{\xi}^\mu \tilde{\xi}^\nu / N = 0
\]

**Proof:** Use \( \frac{d\Theta}{d\lambda} = 0 \) and \( \tilde{\Theta}_{\mu\nu} = 0 \) in Raychaudhuri eqn.

We will need the following

**Theorem:** Hawking 1972

In a stationary asymptotically flat, analytic black hole spacetime, \( H^+ \) is a Killing horizon.

**Note:** \( H^+ \) is not necessarily Killing horizon with respect to stationary Killing vector. E.g. for Kerr we have already seen \( \xi^m = k^m + \Omega H M^m \).
Zeroth Law

K is constant on $H^+$ for a stationary black hole provided $T^{\mu \nu}$ satisfies the dominant energy condition.

Note: We might appeal to the uniqueness theorems to get stationary $\Rightarrow$ Kerr-Newman family of solutions $\Rightarrow$ then verify directly $K$ is constant on $H^+$.

The following proof is much simpler as it does not assume assumptions entering uniqueness theorems.

Proof: Hawking's theorem $\Rightarrow$ Each Killing vector $\xi$ normal to $H^+$. We have shown $R_{\mu \nu} \xi^\mu \xi^\nu |_{H^+} = 0$

If we now use Einstein's equations as well as $\xi^2 = 0$ on $H^+$ we deduce $T_{\mu \nu} \xi^\mu \xi^\nu |_{H^+} = 0$

So if $J^m = -T_{\mu \nu} \xi^\nu$ we have $J_m \xi^m |_{H^+} = 0$

Since $\xi^m$ is future directed and causal, the dominant energy condition $\Rightarrow J^m$ is also (unless $J^m = 0$).
\( J_\Sigma^m \mid_{H^+} = 0 \Rightarrow J^m \) can be expanded on basis of tangent vectors for \( H^+ \):

\[
J = a \xi + b_1 \eta_1 + b_2 \eta_2 \quad \text{with} \quad \xi \cdot \eta_i = 0
\]

This is space like or null (when \( b_i = 0 \)).
But we need \( J \) to be causal \( \Rightarrow \) timelike or null
\( \Rightarrow b_i = 0 \) \( \Rightarrow J^m \propto \xi^m \) on \( H^+ \)

Thus,

\[
0 = \frac{\partial}{\partial t} J^m \bigg|_{H^+}
\]

\[
= -\frac{\partial}{\partial t} \xi^m \bigg|_{H^+}
\]

\[
= -\frac{1}{8\pi} \frac{\partial}{\partial t} R_{\xi\xi} \xi^m \bigg|_{H^+} \quad \text{Einstein's eq's}
\]

\[
\Rightarrow 0 = \frac{\partial}{\partial t} K^m \bigg|_{H^+} \quad \text{See problem sheet 5}
\]

Now if \( A \wedge B = 0 \) for two one-forms then \( A \propto B \)
(Use local coords \( A = \xi^m dx^m \quad B = B_n dx^n \))

\[
\Rightarrow \partial_\nu K \propto \xi_\nu \text{ on } H^+
\]

\[
\Rightarrow t^\nu \partial_\nu K \propto t^\nu \xi_\nu = 0 \text{ for any tangent vector to } H^+
\]

\[
\Rightarrow K \text{ is constant on } H^+
\]
First Law

Take a stationary black hole of mass $M$, charge $Q$ and angular momentum $J$. Perturb it and it will settle down to a stationary black hole with parameters $M+dM$, $Q+dQ$, $J+dJ$ with

$$dM = \frac{K}{8\pi} dA + \Omega_H dJ + \Phi_H dQ$$

$K$ - surface gravity

$\Omega_H$ - angular velocity

$\Phi_H$ - electric potential

$A$ - area of black hole

Proof: Uniqueness theorem $\Rightarrow$ the black hole before and after $\Rightarrow$ Kerr-Newman (with $P=0$). Using Kerr coord's

$$ds^2 = \frac{a^2 \sin^2 \varphi}{\Sigma} dv^2 + 2dv dr - 2a\sin^2 \varphi \frac{r^2+a^2}{\Sigma} dv dx$$

$$-2a\sin^2 \varphi dx dr + \left(\frac{r^2+a^2}{\Sigma} - a^2 \sin^2 \varphi \right) \sin^2 \varphi dx^2 + \Sigma d\sigma^2$$

$$A = \frac{\phi r}{\Sigma} (dv - a\sin^2 \varphi dx) + \frac{\phi r}{\Delta}$$

Delete by gauge trans'n.
\[ \Delta = (r-r_+)(r-r_-), \quad r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2}, \quad a = \frac{J}{M} \]

Area of a black hole horizon is the area of the intersection of \( H^+ \) with partial Cauchy surface.

\[ \begin{align*}
\Sigma \quad \Sigma^+ \\
\rightarrow \quad \text{e.g. for Kerr set } v = \text{constant} \\
\quad + r = r_+ \\
\end{align*} \]

\[ dS^2 = \left( \frac{r_+^2 + a^2}{r_+^2 - a^2} \right) \sin^2 \theta \, dx^i \, dx^j + \Sigma \, d\theta^2 \]

\[ \Sigma = h_{ij} \, dx^i \, dx^j \]

\[ \Rightarrow \sqrt{h} = (r_+^2 + a^2) \sin \theta \]

Area = \( \int d\sigma \, \sqrt{h} = 4\pi (r_+^2 + a^2) \)

\[ = 4\pi \left( 2M^2 - Q^2 + 2\sqrt{M^4 - J^2 - M^2 Q^2} \right) \]

(Note: Would get the same answer in B-L coords if we set \( t = \text{const} \times r = r_+ \), but this is dodgy.)

Have already:

\[ Q^2 H = \frac{a}{r_+^2 + a^2} = \frac{J}{M} \, \frac{1}{2M^2 - Q^2 + 2\sqrt{M^4 - J^2 - M^2 Q^2}} \]
\[ K = \frac{r_+ - r_-}{2 (r_+^2 + a^2)} = \frac{1}{M} \frac{\sqrt{M^4 - J^2 - M^2 Q^2}}{2M^2 - Q^2 + 2\sqrt{M^4 - J^2 - M^2 Q^2}} \]

\( \Phi_H \) is defined as the co-rotating potential difference between infinity and the horizon: the work required to move a unit charge from \( r = \infty \) to \( r = r_+ \).

\[ \Phi_H = \left. 5^m A_m \right|_\infty - \left. 5^m A_m \right|_{r = r_+} \]

with \( 5 = k + \omega H m = \partial \nu + \omega H \partial \chi \)

Calculate

\[ 5^m A_m = -\frac{\Phi r}{\Sigma} \left( 1 - a \sin^2 \theta \omega H \right) \]

\[ = -\frac{\Phi r}{\Sigma} \frac{1}{r_+^2 + a^2} \left( r_+^2 + a^2 \cos^2 \theta \right) \]

\[ 5^m A_m \bigg|_\infty \rightarrow 0 \]

\[ 5^m A_m \bigg|_{r_+} \rightarrow -\frac{\Phi \sqrt{r_+}}{r_+^2 + a^2} \]

\[ \Rightarrow \Phi_H = \frac{\Phi \sqrt{r_+}}{r_+^2 + a^2} = \frac{\Phi}{M} \frac{M^2 + \sqrt{M^4 - J^2 - M^2 Q^2}}{2M^2 - Q^2 + 2\sqrt{M^4 - J^2 - M^2 Q^2}} \]
Now: \( A = A(M, J, \varphi) \)

\[ \Rightarrow dA = \frac{\partial A}{\partial M} dM + \frac{\partial A}{\partial J} dJ + \frac{\partial A}{\partial \varphi} d\varphi \]

\[ \Rightarrow dM = \frac{1}{8\pi} k dA + q_H dJ + \Phi_H d\varphi \]

Comments

1) The reason this argument is so simple is that we use all the hard work that has gone into proving the uniqueness theorems.

2) This version of the first law compares 2 different spacetimes. Other versions of the first law — "physical process" versions — in which one relates change in horizon area to mass, angular momentum, change of matter that is thrown into a black hole.
**Second Law** Hawking's Area theorem

will need some topological concepts

- A set $S$ is open if $\forall p \in S$ there exists $\forall U$ open neighborhood of $p$ with $U \subseteq S$ (e.g. $(a, b)$ open interval $[a, b)$ & $[a, b]$ not open)

- Define $\overline{S}$ as the topological closure of $S$: the union of $S$ & its limit points

- Define $\text{Int}(S)$ as the interior of $S$: the largest open set contained in $S$

- Define $\partial S$ as the topological boundary of $S$: $\partial S = \overline{S} / \text{Int}(S)$

Recall

chronological (future) of set $U$ is $\text{I}^+(U)$ (does not include $U$)

Causal (future past) of set $U$ is $\text{J}^+(U)$ (does include $U$)
$I^\pm (U)$ are open sets: small deformations of timelike curves remain timelike

$J^\pm (U)$ may or may not be closed
e.g. point $p$ in Minkowski spacetime, $J^-(p)$ is closed

E.g. remove a point $r$ in Minkowski spacetime
$J^-(p)$ is not closed as limit points here are missing

In general we have

$I^-(U) = \text{Int} (J^-(U))$
$I^-(U) = \overline{J^-(U)}$

Equivalence principle $\Rightarrow$ every spacetime is locally like Minkowski $\Rightarrow$ for any point $p \in M$ $\exists$ a convex normal neighbourhood $U$ which has causal structure of $M$
for any \( q, r \in U \) there is a unique geodesic connecting \( q \) to \( r \).

There is a stronger result: \( q \in J^{-}(p) \cap I^{-}(p) \) then \( q \) is a null geodesic from \( q \) to \( p \).

Idea of proof: use a finite cover of convex normal neighbourhoods and use result above.

Also can show \( i^{\pm}(U) = j^{\pm}(U) \).

Theorem \( J^{-}(U) \) is an achronal submanifold i.e. no points are timelike separated.

Proof:

\[ i^{\pm}(U) \]

Let \( p, q \in J^{-}(U) \) \& \( q \in I^{-}(p) \)

Want a contradiction.

\[ i^{-}(p) \text{ is open} \Rightarrow J_{r \text{ near } q, r \in I^{-}(p)} \cap r \notin J^{-}(U) \]

\[ i^{+}(r) \text{ is open} \Rightarrow J_{s \text{ near } p, s \in I^{+}(p)} \cap s \notin J^{-}(U) \]

Hence, \( J \) causal curve from \( r \to s \) to \( U \)

\[ r \in J^{-}(U) \text{, a contradiction.} \]

(That \( J^{-}(U) \) is a submanifold is shown in Wald)
Example: take flat cylinder $ds^2 = -dt^2 + d\phi^2$
$-\pi < \phi < \pi$

$\mathcal{J}^-(P)$ is 2 null geodesic segments
I have past endpoints -
if we extend them back further, they leave $\mathcal{J}^-(P)$

Theorem: Let $U$ be closed. Every $p \in \mathcal{J}^-(U)$ lies on a null geodesic in $\mathcal{J}^-(U)$ that is future inextendible or has a future endpoint on $U$.

Idea of proof:

One got from $P$ to $q$, keep going, from $q$ to $r$.
Note that $p$ to $q$ to $r$ must all lie on same
null generator since if

which can't happen.

From the above discussion:

Properties of the future event horizon

\( H^+ = J^- (\partial^+) \) - topological boundary of the causal past of \( \partial^+ \)

i) \( H^+ \) is a null hypersurface

ii) \( H^+ \) is achronal (note this only follows locally from (i))

iii) null geodesic generators of \( H^+ \) have no future endpoints.

(They can have past end points
e.g.

```
\[ \text{Past endpoint} \]
```
2nd Law: Hawking's Area Theorem

Assume \( T_{\mu\nu} \) satisfies null energy condition

- Cosmic censorship

then, the area of \( H^+ \) is non-decreasing in time

Sketch of proof

Cosmic censorship \( \Rightarrow \) \( \exists \) a family of Cauchy surfaces \( \Sigma (\lambda) \) with \( \Sigma (\lambda') \subset D^+ (\Sigma (\lambda)) \) if \( \lambda' > \lambda \).

We can choose \( \lambda \) to be an affine parameter on a generator of \( H^+ \):

\[
\text{Area } A(\lambda') \quad \Sigma (\lambda')
\]
\[
\text{Area } A(\lambda) \quad \Sigma (\lambda)
\]

Want to show \( A(\lambda') \geq A(\lambda) \) when \( \lambda' \geq \lambda \).

It is sufficient to show that each area element \( a \) of \( H \) has this property.

Since \( \frac{da}{d\lambda} = \Theta a \), want to show \( \Theta \geq 0 \) on \( H^+ \)
Suppose this was not true: if $\theta < 0$ at a point $p$, then the null energy condition $\Rightarrow$ geodesics must form a caustic. This implies geodesics that are nearby to a given one passing through $p$ will intersect at a finite distance along it. The first point $q$ for which this happens is called the point "conjugate to $p$ on $\gamma$". Can show that all points beyond $q$ are no longer null separated.

(To get a flavour consider again the two-d cylindrical spacetime

\[ \text{Red curve is timelike} \]

$\Rightarrow \gamma$ must have left $H^+$ (otherwise $H^+$ not achronal). But this can't happen. \( \Box \)
Example. Spherically symmetric collapse

Eddington–Finkelstein diagram

\[ \Sigma (\lambda_2 \gg \lambda_1) \]

\[ A(\lambda_2) \approx 16\pi M^2 \]

\[ \Sigma (\lambda_1) \]

\[ A(\lambda_1) \neq 0 < 16\pi M^2 \]

\[ \Sigma (\lambda_0) \]

\[ A(\lambda_0) = 0 \]

Past endpoint of all generators of \( H^+ \)

Consequences of 2\(^{nd}\) Law

i) For Schwarzschild \( A = 16\pi M^2 \)

Thus if we start with Schwarzschild \& end with Schwarzschild, \( M \) must increase.

(Clearly shows the need for an energy cond'n.

in the 2\(^{nd}\) Law, otherwise could throw in \(-ve\) energy).
2) Consider 2 non-rotating black holes, each approximating Schwartzschild with masses $M_1$ and $M_2$. Suppose they coalesce to form a third Schwartzschild black hole with mass $M_3$. 

$$A_3 \geq A_1 + A_2$$

$$\Rightarrow M_3^2 \geq M_1^2 + M_2^2$$

$$\Rightarrow M_3 \geq \sqrt{M_1^2 + M_2^2}$$

Energy radiated is $M_1 + M_2 - M_3$ which could be used to do work. The efficiency of this is 

$$\eta = \frac{M_1 + M_2 - M_3}{M_1 + M_2} \leq 1 - \frac{\sqrt{M_1^2 + M_2^2}}{M_1 + M_2} = \frac{\sqrt{1 + \left(\frac{M_2}{M_1}\right)^2}}{1 + \left(\frac{M_2}{M_1}\right)^2}$$

$$\leq 1 - \frac{1}{2}$$

and saturated when $M_1 = M_2$.

3) Black holes can’t bifurcate.

eg. Sch. $M_3 \rightarrow M_1 + M_2$

2nd law: $M_3 \leq \sqrt{M_1^2 + M_2^2} \leq M_1 + M_2$

+ equality for $M_1 = M_2 = 0$

Energy conservation: $M_3 \geq M_1 + M_2$

Hence, can’t happen.