Review of Q.F.T. in Minkowski spacetime

Focus on free, real scalar field ϕ satisfying the Klein-Gordon equation

$$\nabla^2 \phi - m^2 \phi = 0$$

+ve frequency solutions

$$\psi_{\mathbf{p}} = N_{\mathbf{p}} e^{-i\omega t + i \mathbf{p}\cdot\mathbf{x}}$$

$$= N_{\mathbf{p}} e^{i p_\mu x^\mu}$$

$$p^0 \equiv \omega \equiv \sqrt{|\mathbf{p}|^2 + m^2} > 0$$

-ve frequency solutions

$$\psi_{\mathbf{p}}^*$$

$N_{\mathbf{p}} \to$ normalisation factor

These are plane waves in an inertial frame & note $k = \partial_t$ then

$$\mathcal{L}_k \psi_{\mathbf{p}} = -i\omega \psi_{\mathbf{p}}$$

$$\mathcal{L}_k \psi_{\mathbf{p}}^* = i\omega \psi_{\mathbf{p}}^*$$

$\{\psi_{\mathbf{p}}, \psi_{\mathbf{p}}^*\}$ form a complete basis, hence we can expand

$$\phi(x^\mu) = \int d^3p \left[a_{\mathbf{p}} \psi_{\mathbf{p}}(x^\mu) + a_{\mathbf{p}}^* \psi_{\mathbf{p}}^*(x^\mu)\right]$$

which is real

Introduce a bracket

$$(f, g) \equiv i \int d^3x\, f^*(t,x) \overleftrightarrow{\partial_t} g(t,x)$$

$$\equiv i \int d^3x \left(f^* \partial_t g - \partial_t f^* g \right)$$

Satisfies $\quad (f, g) = (g, f)^* = -(g^*, f^*)$

$$(\alpha f, g) = \alpha^* (f, g) \qquad \alpha, \beta \text{ constant.}$$
$$(f, \beta g) = \beta (f, g)$$

Choose normalisation $N_{\underset{\sim}{p}} = \dfrac{1}{[2 p^0 (2\pi)^3]^{1/2}}$, then

Ex:

$(\psi_{\underset{\sim}{p}}, \psi_{\underset{\sim}{q}}) = \delta^3(\underset{\sim}{p} - \underset{\sim}{q})$ ← (,) is +ve definite for +ve frequency sol's.

$(\psi_{\underset{\sim}{p}}, \psi_{\underset{\sim}{q}}^*) = 0$

$(\psi_p^*, \psi_q^*) = -\delta^3(\underset{\sim}{p} - \underset{\sim}{q})$ ← (,) is −ve definite for −ve frequency sol's.

Quantisation

Conjugate momentum to ϕ is $\pi(t, x) \equiv \dot{\phi}(t, x)$

ETCR's:

$$\begin{cases} [\phi(t,\underset{\sim}{x}),\ \pi(t,\underset{\sim}{y})] = i\,\delta^3(\underset{\sim}{x}-\underset{\sim}{y}) \\ [\phi(t,\underset{\sim}{x}),\ \phi(t,\underset{\sim}{y})] = [\pi(t,\underset{\sim}{x}),\ \pi(t,\underset{\sim}{y})] = 0 \end{cases}$$

Promote $a_{\underset{\sim}{p}}, a^*_{\underset{\sim}{p}}$ to operators $a_{\underset{\sim}{p}}, a^\dagger_{\underset{\sim}{p}}$

then ETCR's $\iff$ $\begin{cases} [a_{\underset{\sim}{p}},\ a^\dagger_{\underset{\sim}{q}}] = \delta^3(\underset{\sim}{p}-\underset{\sim}{q}) \\ [a_{\underset{\sim}{p}},\ a_{\underset{\sim}{q}}] = 0 \end{cases}$

Hilbert space $\mathcal{H}$:

- vacuum $|0\rangle$ defined via $a_{\underset{\sim}{p}}|0\rangle = 0 \quad \forall \underset{\sim}{p}$
- 1 particle state $a^\dagger_{\underset{\sim}{p}}|0\rangle$
- n particle state $a^\dagger_{\underset{\sim}{p}_1} \cdots a^\dagger_{\underset{\sim}{p}_n}|0\rangle$

$\to$ define a basis for $\mathcal{H}$

Number operator

$$N \equiv \int d^3p\ a^\dagger_{\underset{\sim}{p}} a_{\underset{\sim}{p}}$$

One can show $|0\rangle$ is Lorentz invariant. i.e. if we had used a different inertial frame $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ then can show $|0\rangle' = |0\rangle$.

QFT in curved spacetime

(no back reaction on the metric)

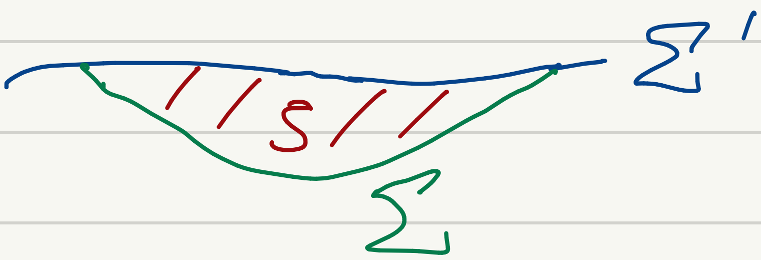
Klein Gordon equation $\quad \nabla^\mu \nabla_\mu \phi - m^2 \phi = 0$

We will assume the spacetime is globally hyperbolic i.e. $\exists$ a Cauchy surface Σ.

Bracket: $\quad (f, g) \equiv i \int_\Sigma dS^\mu f^* \overleftrightarrow{\nabla}_\mu g$

$$= i \int_\Sigma dS^\mu n^\mu f^* \overleftrightarrow{\nabla}_\mu g$$

For solutions of the Klein Gordon equation that agree asymptotically, then this is independent of the choice of Cauchy surface.



$$(f,g)_{\Sigma'} - (f,g)_\Sigma$$

$$= i \int_{\Sigma'} dS n^\mu f^* \overleftrightarrow{\nabla}_\mu g - i \int_\Sigma dS n^\mu f^* \overleftrightarrow{\nabla}_\mu g$$

$$= i \int_{\partial S} dS n^\mu f^* \overleftrightarrow{\nabla}_\mu g$$

$$= i \int_S \nabla^\mu (f^* \overleftrightarrow{\nabla}_\mu g)$$

However, $\nabla^\mu (f^* \overleftrightarrow{\nabla}_\mu g) = \nabla^\mu (f^* \nabla_\mu g - \nabla_\mu f^* g)$

$$= \cancel{\nabla^\mu f^* \nabla_\mu g} + f^* \nabla^2 g - \nabla^2 f^* g - \cancel{\nabla_\mu f^* \nabla_\mu g}$$

$$= f^* m^2 g - m^2 f^* g$$

$$= 0.$$

Fact: there is a highly non-unique basis of solutions such that

$$\begin{cases} (\psi_i, \psi_j) = \delta_{ij} \\ (\psi_i, \psi_j^*) = 0 \\ (\psi_i^*, \psi_j^*) = -\delta_{ij} \end{cases} \qquad (*)$$

Here i is a continuous index (it was $\underline{p}$ in the flat space example).

For any such basis can expand

$$\phi = \sum_i a_i \psi_i(x) + a_i^* \psi_i^*(x)$$

and quantise

$$\phi \to \sum_i a_i \psi_i(x) + a_i^\dagger \psi_i^*(x)$$

where $a_i = (\psi_i, \phi)$

$a_i^\dagger = -(\psi_i^*, \phi)$

& $[a_i, a_j^\dagger] = \delta_{ij}$

$[a_i, a_j] = 0$

Basis for $\mathcal{H}$: $|0\rangle$ with $a_i|0\rangle = 0 \quad \forall i$

$a_i^\dagger |0\rangle$

$a_{i_1}^\dagger \cdots a_{i_n}^\dagger |0\rangle$

Note: in general $\nexists$ a preferred choice of basis $\{\psi_i, \psi_j^*\}$ satisfying (*). Hence $\nexists$ a preferred vacuum state. Hence $\nexists$ a universal notion of particle!

Consider another basis ψ_i'. Can expand

$$\left.\begin{aligned} \psi_i' &= \sum_j A_{ij}\psi_j + B_{ij}\psi_j^* \\ \psi_i'^* &= \sum_j A_{ij}^*\psi_j^* + B_{ij}^*\psi_j \end{aligned}\right\} \quad (1)$$

Then $(\psi_i', \psi_j'^*)$ satisfy (*) provided that

$$\boxed{\begin{aligned} AA^\dagger - BB^\dagger &= \mathbb{1} \\ AB^T - BA^T &= 0 \end{aligned}}$$

We can invert ① via

$$\psi_i = \sum_j A'_{ij}\psi'_j + B'_{ij}\psi'^*_j$$

with $A' = A^+$ & $B' = -B^T$

check: $$\psi'_i \overset{?}{=} \sum_j A_{ij}\left(\sum_k A'_{jk}\psi'_k + B'_{jk}\psi'^*_k\right) + B_{ij}\left(\sum_k A'^*_{jk}\psi'^*_k + B'^*_{jk}\psi'_k\right)$$

$$= \sum_k \left[(AA')_{ik} + (BB'^*)_{ik}\right]\psi'_k + \left[(AB')_{ik} + (BA'^*)_{ik}\right]\psi'^*_k$$

$$= \sum_k (AA^+ - BB^+)_{ik}\psi'_k + (-AB^T + BA^T)_{ik}\psi'^*_k$$

$$= \psi'_i \checkmark$$

Since A' & B' must satisfy same conditions as A & B we also have

$$\begin{cases} A^+A - B^TB^* = \mathbb{1} \\ A^+B - B^TA^* = 0 \end{cases}$$

note: these conditions are not directly implied by the others - they use invertibility.

Also note:

EX: $\phi = \sum_i (a_i \psi_i + a_i^\dagger \psi_i^*)$

$= \sum_i (a_i' \psi_i' + a_i'^\dagger \psi_i'^*)$

$\Rightarrow a_i' = \sum_j (A_{ij}^* a_j - B_{ij}^* a_j^\dagger)$

& $\left.\begin{array}{l}[a_i', a_j'] = 0 \\ [a_i', a_j'^\dagger] = \delta_{ij}\end{array}\right\} \Rightarrow \begin{array}{l} AA^\dagger - BB^\dagger = \mathbb{1} \\ AB^T - BA^T = 0 \end{array}$

The above change of basis involving A & B & satisfying the associated conditions is called a "Bogoliubov transformation".

Note: if $B = 0 \Rightarrow A^\dagger A = AA^\dagger = 1$

$\Rightarrow A$ is a unitary transformation

For stationary spacetimes (including Minkowski) there is a preffered choice of vacuum state that utilises +ve frequency with respect to the stationary Killing vector k.

To see this, we note.

1) $(D^2 - m^2) \mathcal{L}_k \phi = 0$ if $(D^2 - m^2)\phi = 0$

2) $\mathcal{L}_K$ is anti Hermitian: $(f, \mathcal{L}_K g) = -(\mathcal{L}_K f, g)$

$\Rightarrow \mathcal{L}_K$ has imaginary eigenvalues & we can choose a basis of +ve frequency eigenfunctions with

$$\mathcal{L}_K u_i = -i\omega_i u_i \quad \text{with } \omega_i > 0$$

& normalisation $(u_i, u_j) = \delta_{ij}$

3) The anti Hermitian action $\Rightarrow$ distinct eigenvalues are orthogonal so

$$(u_i^*, u_j) = 0$$

Particle production in non-stationary spacetimes

Consider globally hyperbolic "sandwich" spacetime

t

$t > t_2$: M_+ stationary $\longrightarrow$ preferred basis $u_i^+(x)$

t_2

M_0 - time dependent $\longrightarrow$ no preferred basis

t_1

$t < t_1$: M_- stationary $\longrightarrow$ preferred basis $u_i^-(x)$

$\phi(x)$ solves Klein-Gordon equation everywhere

Note $u_i^\pm(x)$ do <u>not</u> solve KG eq'n. everywhere.

In M_- : $\phi(x) = \sum_i a_i^- u_i^-(x) + a_i^{-\dagger} u_i^{-*}(x)$

In M_+: $\phi(x) = \sum_i a_i^+ u_i^+(x) + a_i^{+\dagger} u_i^{+*}(x)$

But we have seen:

$$u_i^+(x) = \sum_j A_{ij} u_j^-(x) + B_{ij} u_j^-(x)^*$$

& we showed

$$a_i^+ = \sum_j A_{ij}^* a_j^- - B_{ij}^* a_j^{-\dagger}$$

Suppose we start in the vacuum state $|0^-\rangle$ defined via $a_i^- |0^-\rangle = 0 \;\; \forall i$

What is the expected number of particles associated with late time modes defined by $N_i^+ = a_i^+ a_i^{+\dagger}$?

$$\langle 0^- | N_i^+ | 0^- \rangle = \langle 0^- | a_i^{+\dagger} a_i^+ | 0^- \rangle$$

$$= \sum_{jk} \langle 0^- | a_k^- (-B_{ik}) (-B_{ij}^* a_j^{-\dagger}) | 0^- \rangle$$

$$= \sum_{jk} \langle 0^- | a_k^- a_j^{-\dagger} | 0^- \rangle \; B_{ik} B_{ij}^*$$

$$= \sum_j B_{ij} B_{ij}^*$$

$$= (B B^\dagger)_{ii} \qquad \text{(no sum on } i)$$

The expected total number of particles is hence $\mathrm{Tr}\, BB^\dagger$. This vanishes iff $B=0$.
The time dependent gravitational field results in particle production!

Comment: One would really see $\langle N_i^+ \rangle$ particles if in a lab stationary with respect to k at late times.

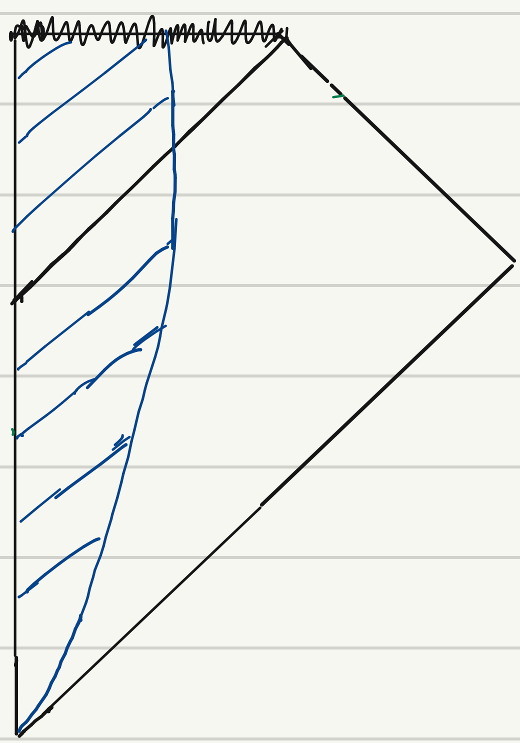
Note: the above interpretation is possible because the spacetime is stationary in the far past & the far future.

Hawking Radiation

Consider Black holes from gravitational collapse

1) It is globally hyperbolic

2) It is not stationary but it is is stationary at late times
Hence we expect particle creation at late times
One might expect this is due to the details of the collapse and be a transient phenomena.



However, the infinite time dilation at H^+ $\Rightarrow$ particles created take arbitrarily long to escape. Hence any late time particle production will be due to H^+ & be independent of the details of the collapse. Remarkably, there is such a flux of particles & moreover it turns out to be <u>thermal</u>. This is Hawking radiation.

Consider for simplicity a <u>massless</u> scalar fied

1) Spacetime is globally hyperbolic. $\mathscr{I}^- \cup i^-$ is a Cauchy surface. In fact specifying data on $\mathscr{I}^-$ is sufficient since the field is massless.

2) Can construct modes (f_i, f_i^*) on $\mathscr{I}^-$ (in the far past spacetime approximately stationary) with f_i +ve frequency.

3) $\mathscr{I}^+$ is not a Cauchy surface, but $\mathscr{I}^+ \cup H^+$ is. Define modes (g_i, g_i^*) & (h_i, h_i^*), respectively.

$\exists$ some ambiguity in defining +ve frequency on H^+. However, the result we are after doesn't depend on this choice: we will just require that the g_i & h_i form a complete basis.

Can expand: $\phi(x) = \sum_i a_i f_i + h.c.$

$$= \sum_i b_i g_i + c_i h_i + h.c.$$

Assume we are in state $|0\rangle$ corresponding to the early time vacuum:

$$a_i |0\rangle = 0 \quad \forall i$$

Notice that we have $g_i = \sum_{ij} A_{ij} f_j + B_{ij} f_j^*$ since (f_i, f_i^*) complete basis.

An observes at late times will observe the number of particles in the i^{th} mode to be

$$\langle N_i \rangle = \langle 0 | b_i^\dagger b_i | 0 \rangle = (BB^\dagger)_{ii}$$

& so we want to calculate B_{ij}.

If we could solve the KG equation in the spacetime or even Schwarzschild this would be straightforward but this is not possible.

Instead: consider a +ve frequency solution g_i to KG equation at $\mathcal{I}^+$ & ask for its form on $\mathcal{I}^-$ i.e. impose boundary condition on KG equation at $\mathcal{I}^+$ & deduce its form on $\mathcal{I}^-$.

Recall Schwarzschild metric

$$ds^2 = \left(1-\frac{2M}{r}\right)\left(-dt^2 + dr_*^2\right) + r^2 d\Omega_2$$

$$= \left(1-\frac{2M}{r}\right)\left(-du\,dv\right) + r^2 d\Omega_2$$

where $u = t - r_*$ & $v = t + r_*$

Ex. Solutions to KG equation have the form

$$\phi = e^{-i\omega t} R_{\omega\ell}(r_*)\, Y_{\ell m}(\theta, \phi)$$

spherical harmonic

with

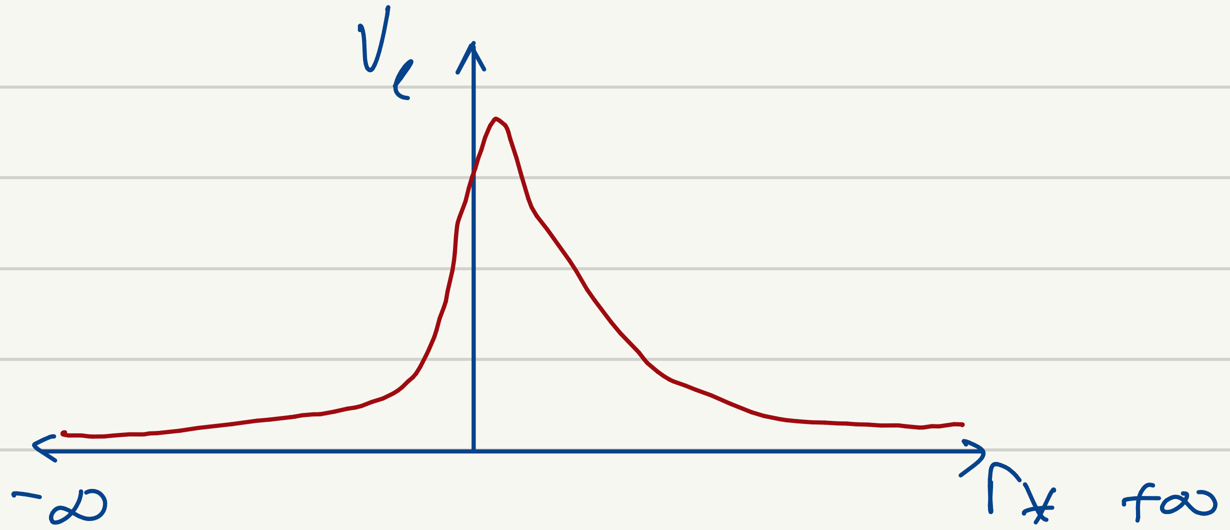
$$\left(\partial_{r_*}^2 + \omega^2\right) R_{\omega\ell}(r_*) = V_\ell(r_*)\, R_{\omega\ell}(r_*)$$

where $$V_\ell(r_*) = \left(1-\frac{2M}{r}\right)\left(\frac{\ell(\ell+1)}{r^2} + \frac{2M}{r^3}\right)$$

Notice:

At H^+ $r_* \to -\infty$, $V_\ell \to 0$

At $\mathscr{I}^\pm$ $r_* \to +\infty$, $V_\ell \to 0$



Near $\mathscr{I}^{\pm}$ solutions are plane waves.
On $\mathscr{I}^{-}$ define

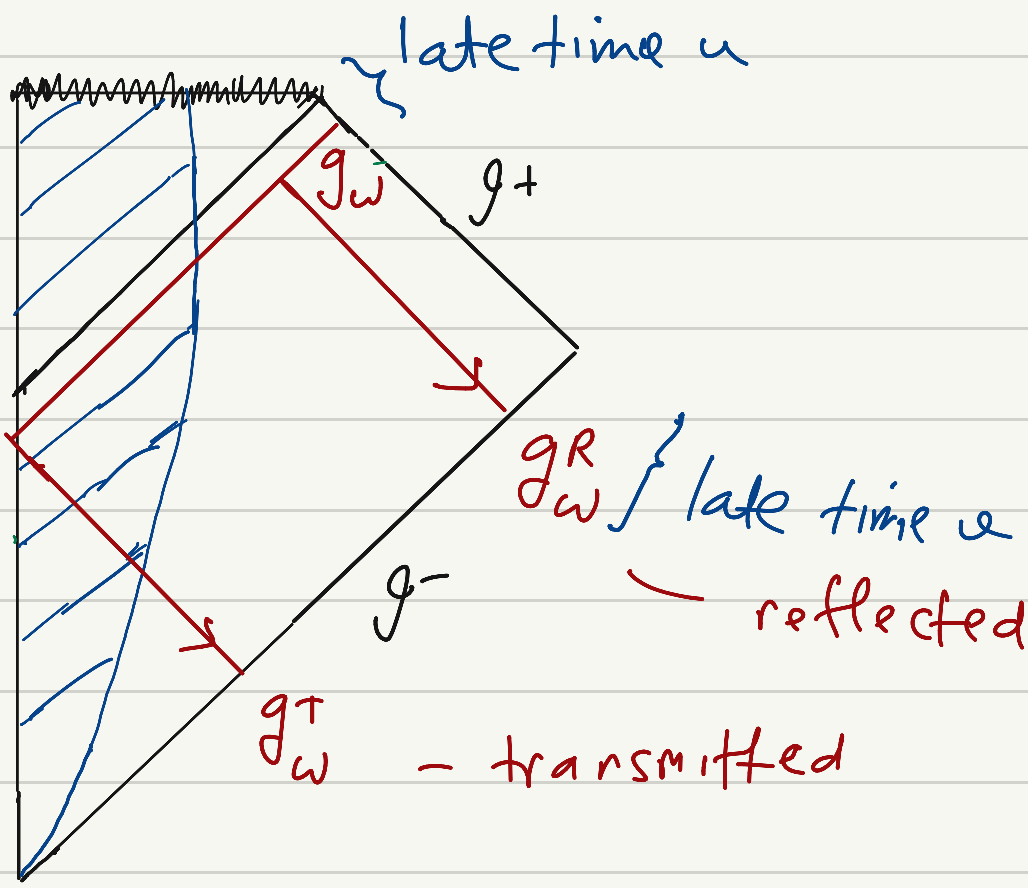
$$\begin{cases} f_{\ell m\omega}^{(O)} = \dfrac{1}{\sqrt{2\pi\omega}} e^{-i\omega u} \dfrac{Y_{\ell m}}{r} & \rightarrow \text{outgoing} \\ f_{\ell m\omega}^{(I)} = \dfrac{1}{\sqrt{2\pi\omega}} e^{-i\omega v} \dfrac{Y_{\ell m}}{r} & \longrightarrow \text{Ingoing} \end{cases}$$

On $\mathscr{I}^{+}$ define

$$\begin{cases} g_{\ell m\omega}^{(O)} = \dfrac{1}{\sqrt{2\pi\omega}} e^{-i\omega u} \dfrac{Y_{\ell m}}{r} & \rightarrow \text{outgoing} \\ g_{\ell m\omega}^{(I)} = \dfrac{1}{\sqrt{2\pi\omega}} e^{-i\omega v} \dfrac{Y_{\ell m}}{r} & \longrightarrow \text{Ingoing} \end{cases}$$

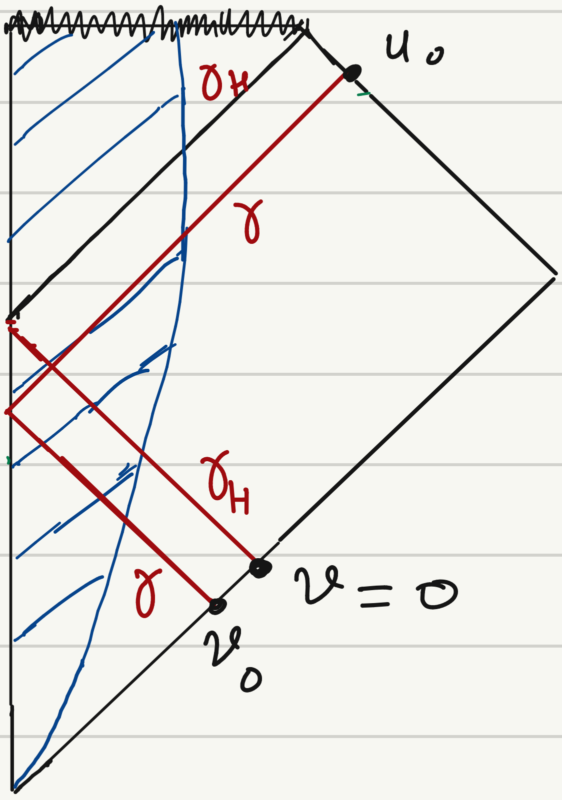
We are interested in $g_\omega \equiv g_{\ell m\omega}^{(O)}$ at late times on $\mathscr{I}^{+}$ in order to get late time radiation. We will determine how it is related to $f_\omega \equiv f_{\ell m\omega}^{(I)}$ on $\mathscr{I}^{-}$ – this is the key idea.

Subtlety: Plane waves such as g_ω are delocalised – they have support everywhere on $\mathscr{I}^{+}$. However, we can superpose & construct wave packets on $\mathscr{I}^{+}$ that are localised around some ω_0 & u_0. Keeping this in mind, we phrase the arguments below in terms of g_ω to simplify the presentation.



As the wave g_ω travels inwards from $\mathcal{I}^+$ i.e. decreasing values of r_*, it encounters the potential barrier in V_ℓ.
One part of the wave will be reflected & end up on $\mathcal{I}^-$. this will not expesience any of the time dependent geometry of the collapsing matter and hence gives +ve frequency mode (in fact this is the same calculation as in the schwarzschild metric). This gives rise to Bogoliubov coefficients A_{ij} which are not our focus.

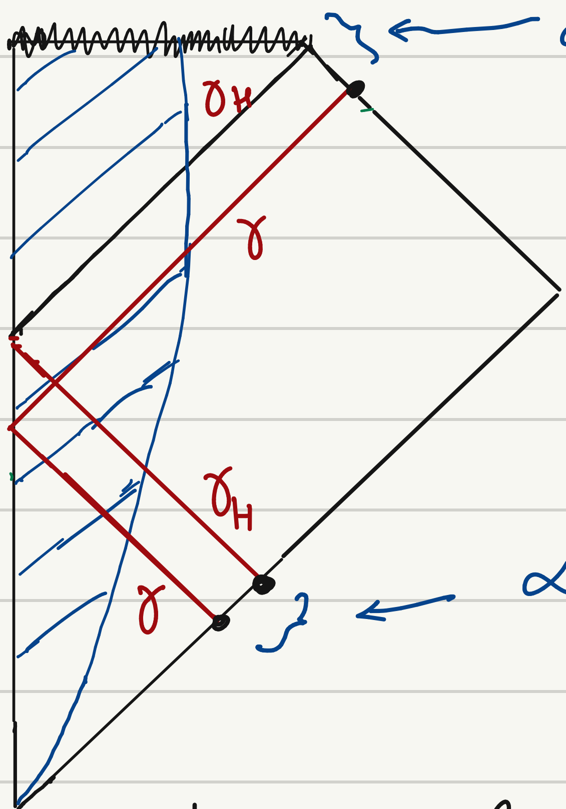
However, the transmitted wave enters the time dependent geometry & hence will be a mixture of +ve & -ve frequency modes on $\mathcal{I}^-$.



Let γ_H be a generator of $\mathcal{H}^+$ - continue it back until it hits $\mathcal{I}^-$. Choose this point to be at $v=0$ (can always shift v by a constant).

Now for fixed large u_0 the wave packet is localised around $v_0 < 0$ on $\mathcal{I}^-$

Also note



∞ many oscillations of field for $u \in (u_0, \infty)$

∞ Many oscillations of field for <u>finite</u> range of v: $v \in (v_0, 0)$

Hence the field is oscillating very rapidly near γ all the way back to $\mathcal{I}^-$. Thus we can use the geometric optics approximation write $\phi(x) = A(x)\, e^{i\lambda S(x)}$ & assume $\lambda \gg 1$

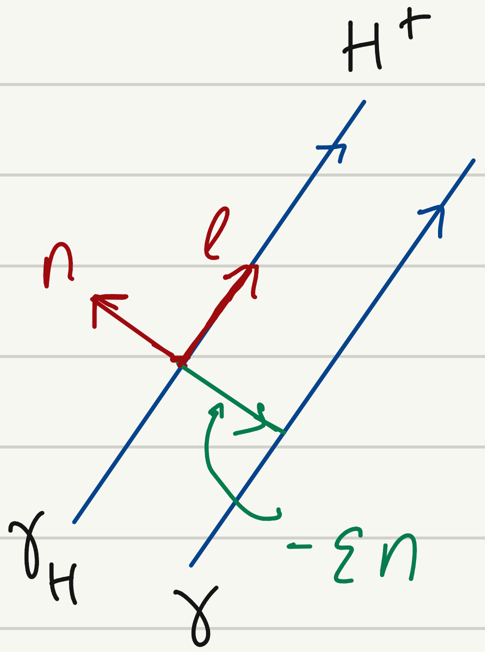
At leading order in λ, $\nabla^2 \phi = 0 \Rightarrow (\nabla S)^2 = 0$

$\Rightarrow$ surfaces of constant S are null hypersurfaces & the generators are null geodesics.

Consider the null congruence containing these hypersurfaces & also H^+ (which is located at $S=\infty$)

Let l be a tangent vector to the congruence of null geodesics & introduce a connecting null vector n with $n \cdot l = -1$

& $l^\mu \nabla_\mu n^\nu = 0$



Spherical symmetry $\Rightarrow$ can choose $n^\theta = n^\phi = 0$

Outside matter we know that we can choose $l = \frac{\partial}{\partial V}$ as affinely parametrised generator of H^+ & hence can choose $n = c \frac{\partial}{\partial U}$ for some constant $c > 0$.

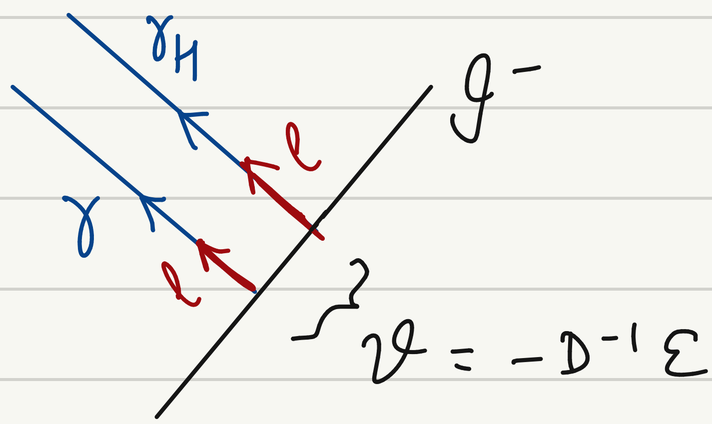
(Here U, V are Kruskal coordinates)

Hence outside matter, $-\varepsilon n^\mu$ connects γ_H to a null geodesic γ located at $U = -c\varepsilon$.

Now we know from the definition of U that $u = -\frac{1}{\kappa}\log(-U)$. $\Rightarrow$ at late times γ is an outgoing null geodesic with $u = -\frac{1}{\kappa}\log(c\varepsilon)$

Moreover, the phase of the mode $g_\omega \sim e^{-i\omega u}$ is thus $-i\omega u = \frac{i\omega}{\kappa}\log(c\varepsilon)$ & this is the phase everywhere on γ.

Next, at $\mathcal{I}^-$, in u, v coordinates we have $\ell \propto \partial_u$ & $n = D^{-1}\partial_v$ for constant $D > 0$



Thus the phase at $\mathcal{I}^-$ is given by $\frac{i\omega}{\kappa}\log(-cDv)$

$$\Rightarrow \text{ on } \mathcal{I}^-: \quad g_\omega^T \sim \begin{cases} 0 & v > 0 \\ e^{i\frac{\omega}{\kappa}\log(-v)} & v < 0 \end{cases}$$

where we have ignored unimportant constant phase & also the normalisation (for now)

Now we want to decompose g^T_ω in terms of f & f^*

First take the Fourier transform

$$\tilde{g}^T_\omega(\omega') = \int_{-\infty}^{\infty} e^{i\omega' v}\, g^T_\omega(v)\, dv$$

$$= \int_{-\infty}^{0} \exp\left(i\omega' v + \frac{i\omega}{\kappa}\log(-v)\right) dv$$

<u>Lemma</u> (problem sheet 6)

$$\tilde{g}^T_\omega(-\omega') = -\exp(-\pi\omega/\kappa)\, \tilde{g}^T_\omega(\omega') \qquad \omega' > 0$$

Take the inverse Fourier transform:

$$g^T_\omega(v) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi}\, e^{-i\omega' v}\, \tilde{g}^T_\omega(\omega')$$

$$= \int_0^\infty \frac{d\omega'}{2\pi}\, e^{-i\omega' v}\, \tilde{g}^T_\omega(\omega')$$

$$+ \int_0^\infty \frac{d\omega'}{2\pi}\, e^{+i\omega' v}\, \tilde{g}^T_\omega(-\omega')$$

$$= \int_0^\infty d\omega'\, N_{\omega'} f_{\omega'}(v)\, \tilde{g}^T_\omega(\omega')$$

N_ω normalisation factor

$$+ \int_0^\infty d\omega'\, N^*_{\omega'} f^*_{\omega'}(v)\, \tilde{g}^T_\omega(-\omega')$$

$$\Rightarrow \begin{cases} A_{\omega\omega'} = N_{\omega'}\, \tilde{g}^T_\omega(\omega') \\ B_{\omega\omega'} = N^*_{\omega'}\, \tilde{g}^T_\omega(-\omega') \end{cases} \qquad \omega > 0,\ \omega' > 0$$

$$\Rightarrow \; |B_{\omega\omega'}| = \exp\left(-\frac{\pi\omega}{k}\right) |A_{\omega\omega'}|$$

using the lemma.

Return to normalisation of g^T_ω:

$$\Gamma_\omega \equiv T^2_\omega \equiv (g^T_\omega, g^T_\omega)$$

$$= \sum_{\omega', \omega''} \left(A_{\omega\omega'} f_{\omega'} + B_{\omega\omega'} f^*_{\omega'}, \, A_{\omega\omega''} f_{\omega''} + B_{\omega\omega''} f^*_{\omega''}\right)$$

$$= \sum_{\omega'} |A_{\omega\omega'}|^2 - |B_{\omega\omega'}|^2$$

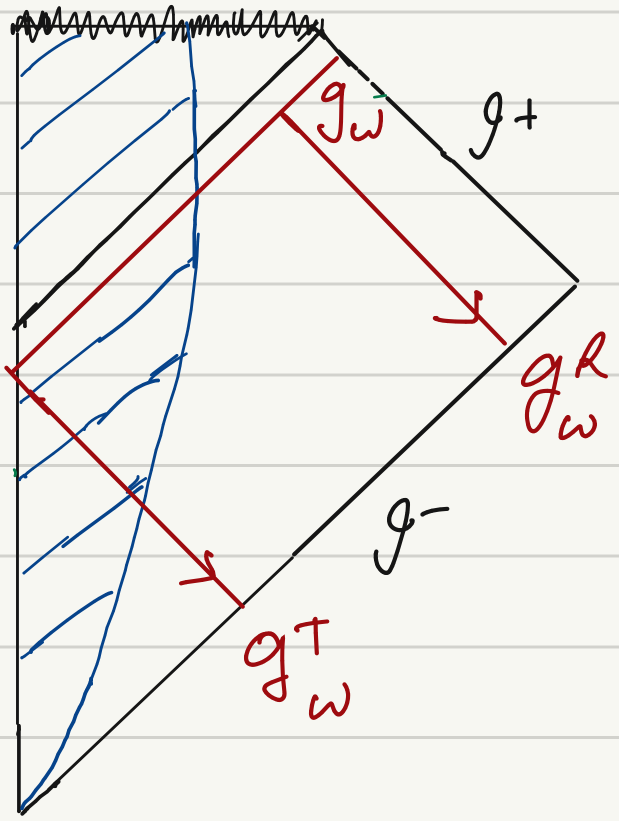
$$= \left(e^{2\pi\omega/k} - 1\right) \sum_{\omega'} |B_{\omega\omega'}|^2$$

$$= \left(e^{2\pi\omega/k} - 1\right) (BB^+)_{\omega\omega}$$

$$\Rightarrow \; (BB^+)_{\omega\omega} = \frac{\Gamma_\omega}{\left(e^{2\pi\omega/k} - 1\right)}$$

Can argue that Γ_ω is the absorption cross section for the mode f_ω. (see over). Then

this is exactly the black body spectrum with temperature $T = \frac{k}{2\pi}$!



The modes g^R_ω & g^T_ω are orthogonal.
Define

$$T_\omega^2 = (g^T_\omega, g^T_\omega)$$

$$R_\omega^2 = (g^R_\omega, g^R_\omega)$$

Then

$$T_\omega^2 + R_\omega^2 = 1$$

Now start with a mode f_ω. Some of it gets absorbed & some of it gets reflected. The reflected part is the same as R_ω. Hence, $\Gamma_\omega = T_\omega^2$ is the absorption cross section for the mode f_ω – ie. the fraction absorbed by the black hole.

Comments

1. Derivation can be generalised to
   i) other fields: fermions & bosons
   ii) $m \neq 0$
   iii) non-spherical collapse to Schwartschild
   iv) Collapse to rotating & or charged black holes. In this case the black

hole preferentially emits particles with the same J & Q as the black hole.

2. $T_H \sim 6 \times 10^{-8} \left(\frac{M_\odot}{M}\right) K$

- Astrophysical black holes will be cooler than the CMB & hence absorb CMB.

- Do micro black holes exist? perhaps formed after the big bang?

3. $\frac{\partial T_H}{\partial M} < 0 \Rightarrow$ black holes have negative specific heat ie. they heat up as they evaporate!

4. Thermodynamics: the 1st Law of black hole mechanics is promoted to a thermodynamical law:

$$dE = T_H \, dS_{BH} + \Omega_H \, dJ + \Phi \, dQ$$

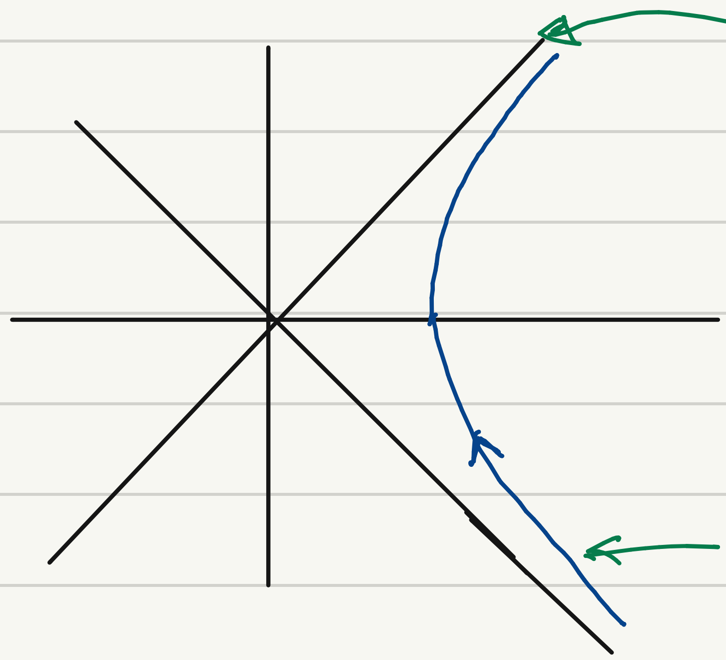
with $S_{BH} = \frac{1}{4} A$.

Also have generalised 2nd Law

$$S_{BH} + S_{\substack{\text{matter outside} \\ \text{black hole}}} \geq 0$$

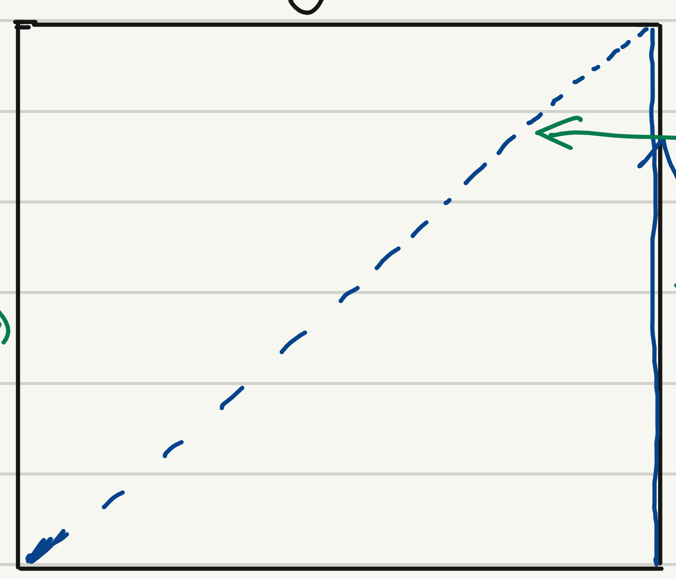
5. The derivation emphasises the key role of the event horizon.
There are closely related phenomena associated with observer dependent horizons

e.g. Rindler observer



eg. de Sitter spacetime

$$ds^2 = \frac{1}{\cos^2 t}\left(-dt^2 + d\Omega_3\right)$$



6. Can obtain T_H by "euclideanisation" $t \to i\tau$ & make sure geometry is regular. Works for both kinds of horizons.

7. In anti-de Sitter spacetime, black hole thermodynamics is thermodynamics of the boundary CFT

8. $S_{BH} = k_B 10^{77} \left(\frac{M}{M_\odot}\right)^2$

By comparison $S_{sun} \sim k_B 10^{58}$

Entropy would be much greater if all matter was in black holes. i.e. the universe is in a low entropy state.

9. In statistical mechanics

$$S = k_B \ln W \leftarrow \text{\# of microstates}$$

What are the microstates? To answer this we need a theory of quantum gravity.

A state counting interpretation has been achieved for a special class of supersymmetric & near supersymmetric black holes in string theory. The key idea is that the black holes can be built from configurations of intersecting branes wrapped on cycles of the internal compactified space.

10. Hawking's calculation neglected back reaction on the spacetime geometry. This should be ok when $\frac{dM}{dt} << M$ but will not be ok in the final stages of evaporation. We again need a theory of quantum gravity to address this issue.

Can estimate the time taken to evaporate using the Stefan - Boltzmann law for the rate of energy loss for a black body:

$$\frac{dE}{dt} = -\sigma A T^4$$

T_H

$E = Mc^2$

Area of event horizon $\propto \left(\frac{MG}{c^2}\right)^2$

$$\Rightarrow \frac{dM}{dt} \sim -\frac{\hbar c^4}{G^2}\frac{1}{M^2}$$

$$\Rightarrow \tau_{evap} \sim \frac{G^2}{\hbar c^4}\left(\frac{M}{M_\odot}\right)^3 M_\odot^3$$

$$\sim 10^{70}\left(\frac{M}{M_\odot}\right)^3 s$$

$$\sim 10^{-44}\left(\frac{M}{M_{pl}}\right) s$$

(n.b. lifetime of universe $\sim 10^{17}s$)

11. Information loss paradox.

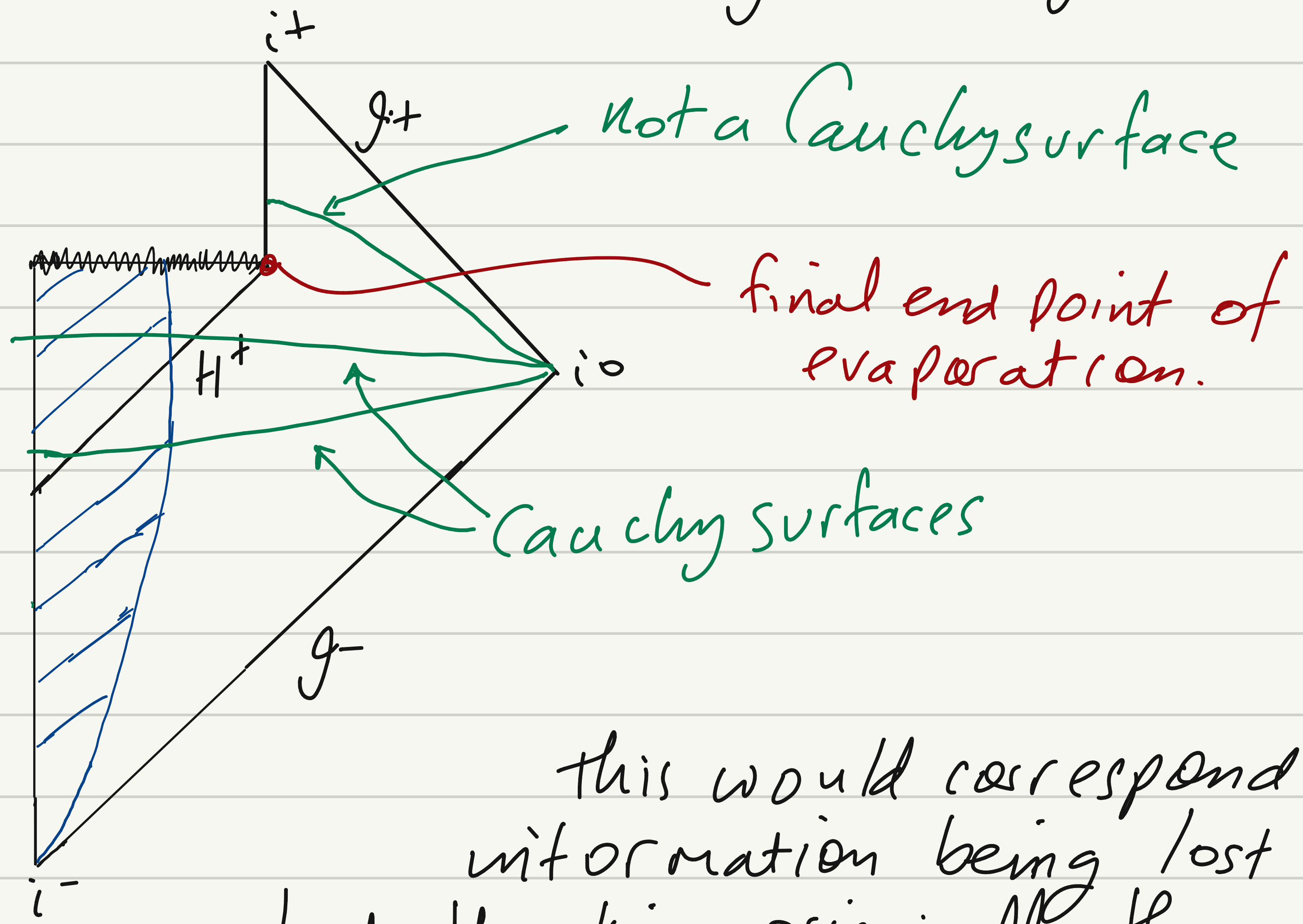
Consider an initial configuration with matter in a pure state. Let it collapse, form a black hole & then radiate. The radiation is thermal so it appears that

$$\text{pure state} \longrightarrow \text{Mixed state} \quad !$$

This is impossible with unitary evolution in quantum mechanics.

Contrast this with burning a lump of coal. If one carefully studied the final radiation one would find that we have a pure state evolving to a pure state. However this point of view is hard to implement for black holes as the information going into making the black holes seems to be lost at least in GR (uniqueness theorems) & for very long times the radiation is exactly thermal.

A possible Penrose diagram might be



this would correspond to information being lost. This is what Hawking originally thought & that quantum mechanics would need modifying in quantum gravity.

However, most physicists think there are somehow (!) subtle correlations in the Hawking radiation & that information is not lost.

To be continued....