Spherically symmetric Collapse & the Schwarzschild black hole

Consider a ball of pressure free dust that undergoes gravitational collapse: an idealised model of a star.

Recall Birkhoff's theorem: spherical symmetry combined with Einstein's equations $\Rightarrow$ outside the star we have Schwarzschild geometry

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega$$

$$d\Omega = d\theta^2 + \sin^2\theta\, d\phi^2$$

round metric on S^2

Killing vectors?

- S^2: ∂_ϕ is obvious. $\exists$ 2 more & the algebra under $[\,,]$ is $SO(3)$ (EX.)
- ∂_t is another K.V. It is timelike (i.e. $\|\partial_t\|^2 < 0$) for $r > 2M$

$\Rightarrow$ stationary spacetime

Also $t \to -t$ is a <u>discrete</u> symmetry

$\Rightarrow$ static spacetime

Interesting fact that spherical symmetry $\Rightarrow$ static.

Outside star we have schwarzschild.
Surface of star at $r = R(t)$ is also Schwarzschild by continuity.
A point on the surface follows a radial time like geodesic

$$\dot{\theta} = \dot{\phi} = 0$$

$$X^\mu(\tau) = [t(\tau), r(\tau), \theta_0, \phi_0], \quad \tau \text{ proper time}$$

We want $R(t) \equiv r(\tau(t))$

Now,

$$d\tau^2 = -ds^2 \Leftrightarrow 1 = \left[\left(1-\frac{2M}{R}\right) - \left(1-\frac{2M}{R}\right)^{-1}\dot{R}^2\right]\left(\frac{dt}{d\tau}\right)^2 \quad (*)$$

Also, ∂_t is a Killing vector & have

$$Q \equiv P_\mu k^\mu = P_t = e^{-1} g_{tt} \frac{dt}{d\tau}$$ is constant of motion

with e constant (since using τ) we have

$$\mathcal{E} \equiv -g_{tt}\frac{dt}{d\tau}$$ is constant on the geodesic

It is the energy/unit mass.

We can substitute into $(*)$ & rearrange to get

$$\boxed{\dot{R}^2 = \frac{1}{\mathcal{E}^2}\left(1-\frac{2M}{R}\right)^2\left(\frac{2M}{R}-1+\mathcal{E}^2\right)} \qquad \dot{R} = \frac{dR}{dt}$$

We take $0<\mathcal{E}<1$ to have gravitationally bound motion



$2M$ $\qquad R_{Max} \equiv \frac{2M}{1-\mathcal{E}^2}$

$\dot{R}=0$ at R_{Max}: particle starts from rest here

R then decreases & approaches $2M$ at $t \to \infty$

To see this: $t_{2M} \equiv \int_0^{t_{2M}} dt$

$$= -\mathcal{E}\int_{R_{Max}}^{2M} dR \left(1-\frac{2M}{R}\right)^{-1}\left(\frac{2M}{R}-1+\mathcal{E}^2\right)^{-1/2}$$

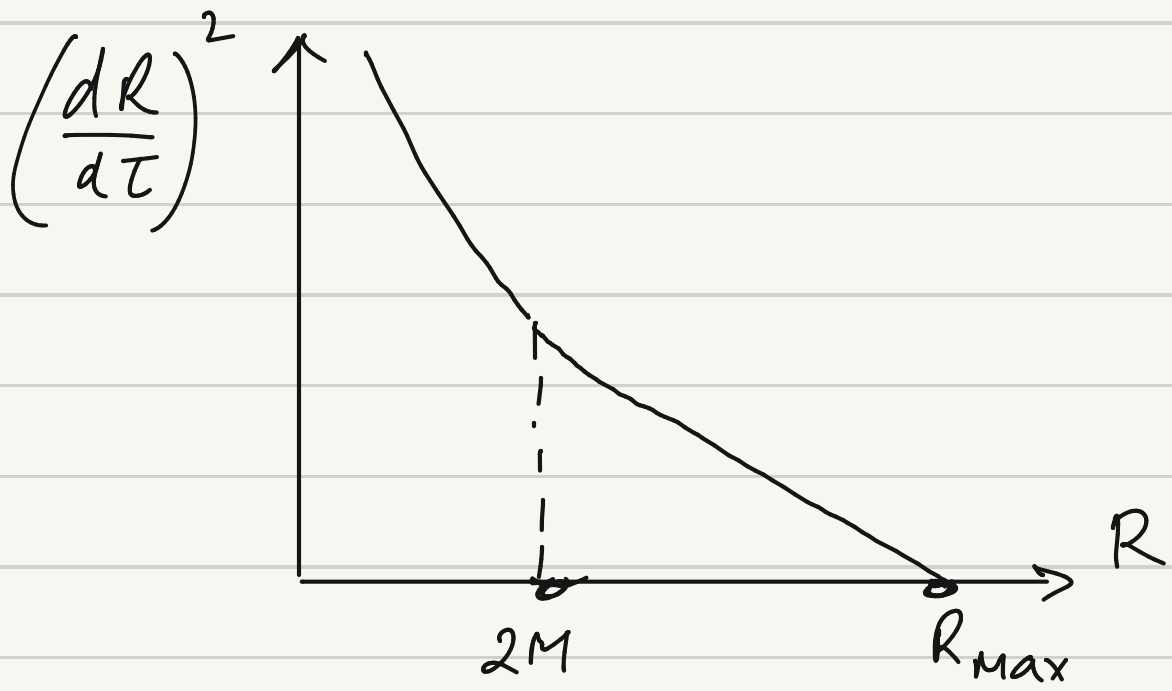
$\to \infty$ (Ex.)

According to an observer at asymptotic ∞, using t as their proper time, they see the surface of the star approach $R \to 2M$ & stay there (a "frozen star")

However, for an observer on the surface of the star we are interested in $R(\tau)$

Since $$\frac{d}{dt} = \left(\frac{dt}{d\tau}\right)^{-1} \frac{d}{d\tau} = \mathcal{E}^{-1}\left(1 - \frac{2M}{R}\right)\frac{d}{d\tau}$$

$$\Rightarrow \left(\frac{dR}{d\tau}\right)^2 = (1-\mathcal{E}^2)\left(\frac{R_{Max}}{R} - 1\right)$$



Proper time to reach $R = 2M$ is finite!

As is proper time to reach $R = 0$:

$$\tau_0 = \int_0^{\tau_0} d\tau = -\int_{R_{Max}}^{0} dR\,(1-\mathcal{E}^2)^{-1/2}\left(\frac{R_{Max}}{R} - 1\right)^{-1/2}$$

$$= \pi M / (1-\mathcal{E}^2)^{3/2}$$

This is correct, but the derivation is rough because coordinates break down at $r = 2M$.

So, we introduce new coordinates – we use them to examine spherical collapse AND at the same time, the geometry of the Schwarzschild geometry.

Eddington – Finkelstein coordinates

Idea: Replace t by a coordinate v adapted to ingoing radial null geodesics.
On such a geodesic:

$$ds^2 = 0 \Rightarrow dt^2 = \left(1 - \frac{2M}{r}\right)^{-2} dr^2 \equiv dr_*^2$$

Take $r_* = r + 2M \ln\left|\frac{r-2M}{2M}\right|$

$$2M < r < \infty \quad \leftrightarrow \quad -\infty < r_* < \infty$$

On a radial null geodesic
$d(t \pm r_*) = 0$

$u \equiv t - r_*$ lines of constant u $\leftrightarrow$ Outgoing radial null geodesics

$v \equiv t + r_*$ lines of v constant $\leftrightarrow$ Ingoing radial null geodesics

$(t, r, \theta, \phi) \to (v, r, \theta, \phi)$ Ingoing E-F Coordinates

$$dS^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2\,dv\,dr + r^2 d\Omega \qquad (\text{Ex.})$$

Initially metric only defined for $r > 2M$. But it can now be analytically continued to $r > 0$! In particular in E-F coord's the metric is non-singular at $r = 2M$

$$g_{\mu\nu}\big|_{r=2M} = \begin{array}{c} \\ v \\ r \\ \theta \\ \phi \end{array} \begin{array}{c} \begin{array}{cccc} v & r & \theta & \phi \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 4M^2 & 0 \\ 0 & 0 & 0 & 4M^2 \sin^2\theta \end{pmatrix} \end{array}$$

is finite & invertible, $\det g|_{r=2M} \neq 0$

The star does indeed collapse through $r = 2M$ down to $r = 0$

Recall $k = \partial_t$ for $r > 2M$

$$= \frac{\partial x^{\mu'}}{\partial t} \frac{\partial}{\partial x'^{\mu}}$$

$$= \frac{\partial}{\partial v}$$

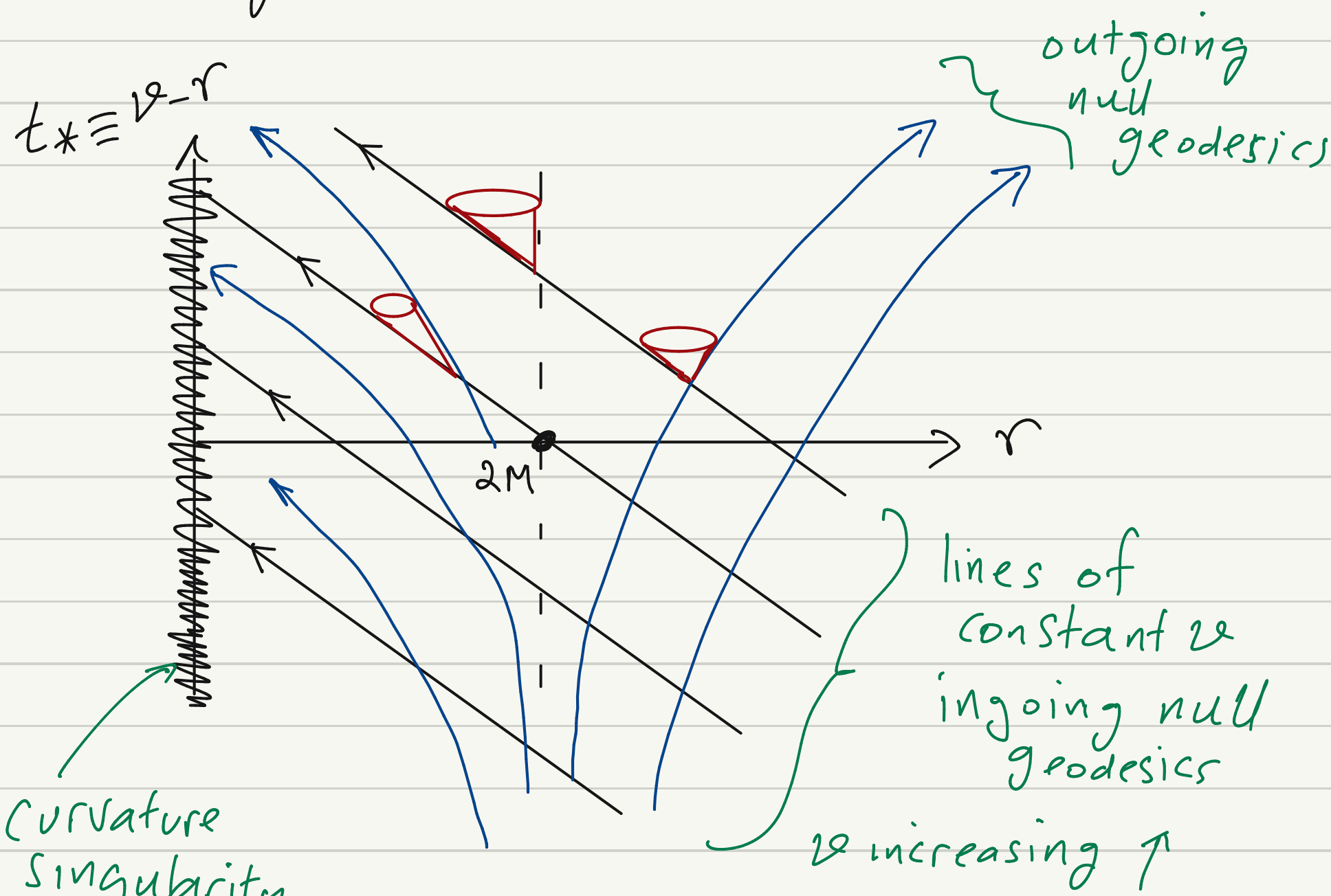
$$x'^{\mu} = \begin{pmatrix} v = t - r_* \\ r = r \\ \theta = \theta \\ \phi = \phi \end{pmatrix}$$

& this extends definition of Killing vector to $0 < r < \infty$

Note: $\|k\|^2 \equiv g_{\mu\nu} k^\mu k^\nu = g_{vv} = -\left(1 - \frac{2M}{r}\right)$

k is	timelike	$r > 2M$
	null	$r = 2M$
	spacelike	$r < 2M$

E-F diagram for Schwartzschild



Ingoing radial null geodesic: $v = \text{const.}$

Outgoing radial null geodesic: $u = t - r_* = \text{const.}$

$\Leftrightarrow v - 2r - 4M \ln\left|\frac{r-2M}{2M}\right| = \text{const.}$

Radial timelike curves have tangent vectors that lie inside the light cone at any point.

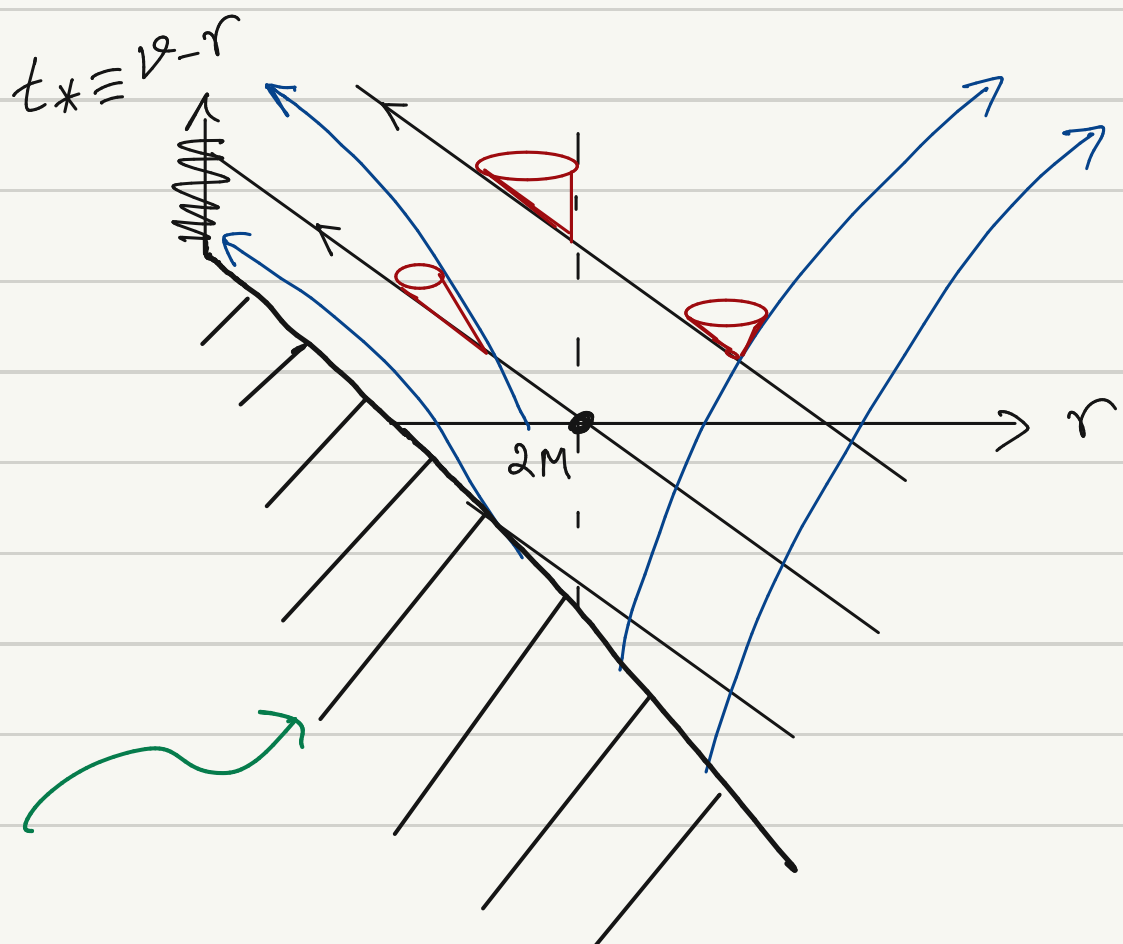
For $r > 2M$ outgoing radial null geodesics have increasing r

For $r < 2M$ Both ingoing & "outgoing" radial null geodesics have decreasing r

$r = 2M$ is a null surface (even though "r = constant")

No radial timelike curves come from $r < 2M$ to a point with $r > 2M$. (see later)
In fact we can relax the radial condition & can conclude that no signal emerges from $r < 2M$ to $r = \infty$: we have a black hole.

E-F diagram for spherical collapse:



Definitions

A vector is causal if it is timelike or null

A curve is causal if it's tangent vector is everywhere causal

Definitions

A spacetime is time orientable if it admits a time orientation - a causal vector field T^μ

Another causal vector X^μ is future directed if it lies in the same light cone as T^μ (i.e. $X^\mu T^\nu g_{\mu\nu} \lesssim 0$) & past directed otherwise.

For schwarzschild (sch. coord's, E.F. coord's)

$r > 2M$ choose $k = \partial_t = \partial_v$ as a time orientation

$r < 2M$ cannot choose ∂_v as it is spacelike

However, can choose $\pm \partial_r$ (E.F. coord's) as these are null.

Need to choose sign to agree with time orientation for $r > 2M$

$$k \cdot \left(-\frac{\partial}{\partial r}\right) = k^\mu \left(-\frac{\partial}{\partial r}\right)^\nu g_{\mu\nu} = -g_{vr} = -1$$

Hence: Can choose $-\frac{\partial}{\partial r}$ (in E.F. coord's) as global time orientation for $r > 0$.

[Warning: $\frac{\partial}{\partial r_{sch.}} \neq \frac{\partial}{\partial r_{E.F.}}$!

as holding different coordinates constant.

In fact

$$\frac{\partial}{\partial r_{sch.}} = \frac{\partial r_*}{\partial r_{sch}} \frac{\partial}{\partial v} + \frac{\partial}{\partial r_{E.F}} \quad \text{(Ex.)}$$]

Proposition Let $x^\mu(\lambda)$ be a future directed causal curve. Assume $r(\lambda_0) \leq 2M$. Then $r(\lambda) \leq 2M \quad \forall \lambda \geq \lambda_0$

Proof:

Tangent vector $V^\mu = \frac{dx^\mu}{d\lambda}$

Future directed $\Leftrightarrow V \cdot \left(\frac{-\partial}{\partial r_{E.F.}}\right) \leq 0$

$\Leftrightarrow -V^\mu g_{r\mu} \leq 0$

$\Leftrightarrow -V^v \leq 0$

$\Leftrightarrow \frac{dv}{d\lambda} \geq 0$

Also have

$$V^2 = -\left(1-\frac{2M}{r}\right)\left(\frac{dv}{d\lambda}\right)^2 + \frac{2dv}{d\lambda}\frac{dr}{d\lambda} + r^2\left[\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\lambda}\right)^2\right]$$

$$\Rightarrow$$

$$-\frac{2dv}{dr}\frac{dr}{d\lambda} = \underbrace{-V^2 + \left(\frac{2M}{r}-1\right)\left(\frac{dv}{d\lambda}\right)^2 + r^2\left[\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\lambda}\right)^2\right]}_{\geq 0 \text{ if } r \leq 2M} \quad (*)$$

$$\Rightarrow \frac{dr}{d\lambda} \leq 0 \quad \text{for } r \leq 2M$$

Now, if $r < 2M$, $\frac{dr}{d\lambda} < 0$ (If not, $(*)$ with $\frac{dr}{d\lambda} = 0$ $\Rightarrow \dot{v} = \dot{\theta} = \dot{\phi} = 0$ $\Rightarrow V^\mu = 0$, a contradiction)

Hence, if $r(\lambda_0) < 2M \Rightarrow r(\lambda) < 2M \quad \forall \lambda \geq \lambda_0$

The $r(\lambda_0) = 2M$ case can also be shown (e.g. Reall p.20) //

Consider any particle (not necessarily on a geodesic) in region $r < 2M$. Using Schwarzschild coord's.:

$$d\tau^2 = -\left(\frac{2M}{r}-1\right)dt^2 + \left(\frac{2M}{r}-1\right)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

Proper time to reach $r=0$ is maximised if

$$\frac{d\theta}{d\tau} = \frac{d\phi}{d\tau} = \frac{dt}{d\tau} = 0$$

$$\Rightarrow \quad dt = -\left(\frac{2M}{r}-1\right)^{-1/2} dr$$

$$\therefore \int dt = -\int_{2M}^{0} dr \left(\frac{2M}{r}-1\right)^{-1/2} = \pi M$$

$$\simeq 10^{-5}\left(\frac{M}{M_\odot}\right) \text{seconds} \; !$$

Note: this is achieved for a geodesic (i.e. if you fall into a black hole ... just relax!):

$$\frac{dr}{d\tau} = -\left(\frac{2M}{r}-1\right)^{1/2}$$

$$\frac{d}{d\tau}\left(\frac{dr}{d\tau}\right) = \left(\frac{2M}{r}-1\right)^{-1/2} \frac{M}{r^2} \frac{dr}{d\tau} = -M/r^2$$

$$\Gamma^r_{rr}\left(\frac{dr}{d\tau}\right)^2 = \frac{1}{2} g^{rr}\, \partial_r g_{rr} \left(\frac{2M}{r}-1\right) = M/r^2$$

$$\therefore \quad \frac{d}{d\tau}\left(\frac{dx^\mu}{d\tau}\right) + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0 \, .$$

Can also note that $\mathcal{E} \equiv \left(1-\frac{2M}{r}\right)\frac{dt}{d\tau} = 0$

White holes

What happens if we use outgoing E-F coord's instead of ingoing?

$u = t - r_*$ $\quad -\infty < u < \infty$

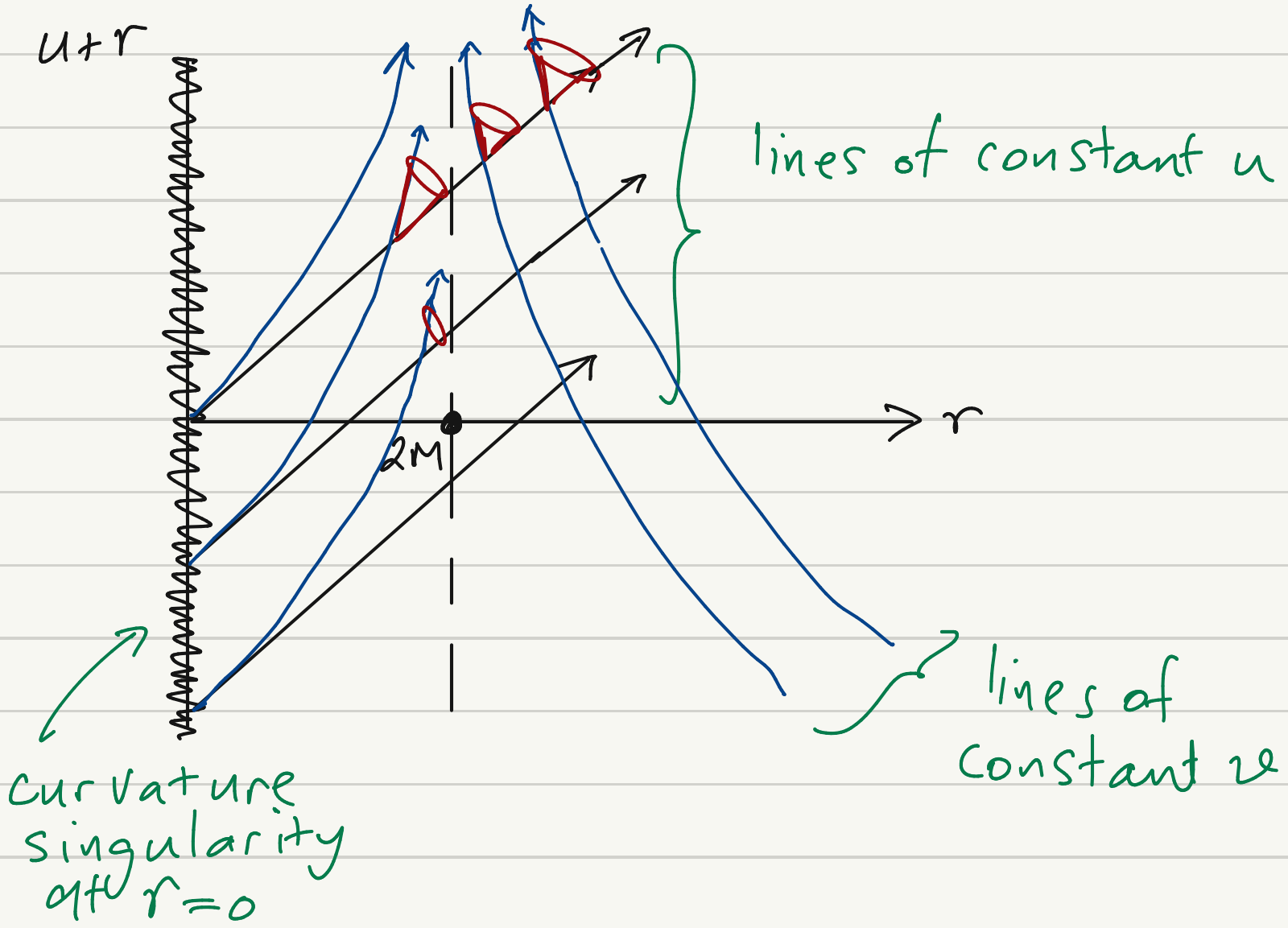
$$ds^2 = -\left(1 - \frac{2M}{r}\right) du^2 - 2\,dr\,du + r^2\,d\Omega \qquad (Ex.)$$

Initially coord's defined for $r > 2M$. However, we can now analytically continue to all $r > 0$.

the region $r < 2M$ is <u>not</u> the same as $r < 2M$

($r < 2M$: outgoing E.F.; $r < 2M$: ingoing E.F.)

This is revealed in (outgoing) E.F. diagram

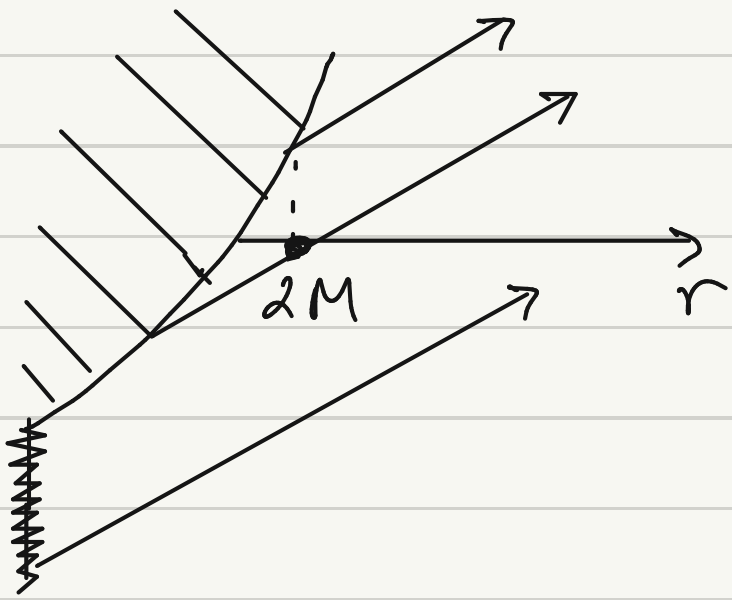


Note ingoing radial null geodesics (v=const) approach $r = 2M$.

Outgoing radial null geodesics (u=const) escape to ∞.

This is a white hole. It is the time reverse of the black hole (note Einstein equations are time reversal invariant).

Can we use this as a model of a kind of exploding star?



Note it starts with a singularity!
Also, black holes are stable - small perturbations are expected to decay. The time reverse of this says that perturbations of white holes will grow & hence are unstable.

$\Rightarrow$ White holes are not physical!

Kruskal - Szekeres (KS) coordinates

Have seen there are two ways to extend Schwarzschild past $r = 2M$. To relate them we introduce new coord's.

First note, for $r > 2M$ we can use (u, v, θ, ϕ)

$$ds^2 = -\left(1 - \frac{2M}{r}\right) du\,dv + r^2 d\Omega \qquad (Ex.)$$

... not good at $r = 2M$.

Define KS coord's for $r > 2M$

$$U = -e^{-u/4M} \qquad V = e^{v/4M} \qquad \begin{matrix} U < 0 \\ V > 0 \end{matrix} \qquad r > 2M$$

Note $\begin{cases} UV = -e^{r_*/2M} = -e^{r/2M}\left(\frac{r}{2M} - 1\right) & \otimes \\ \frac{V}{U} = -e^{t/2M} \end{cases}$

which give t & r in terms of U & V

$$ds^2 = -\frac{32M^3}{r} e^{-r/2M} dU\,dV + r^2 d\Omega$$

with $r = r(U, V)$

Now <u>define</u> $r(U, V)$ for $U \geq 0$ or $V \leq 0$ via $\otimes$

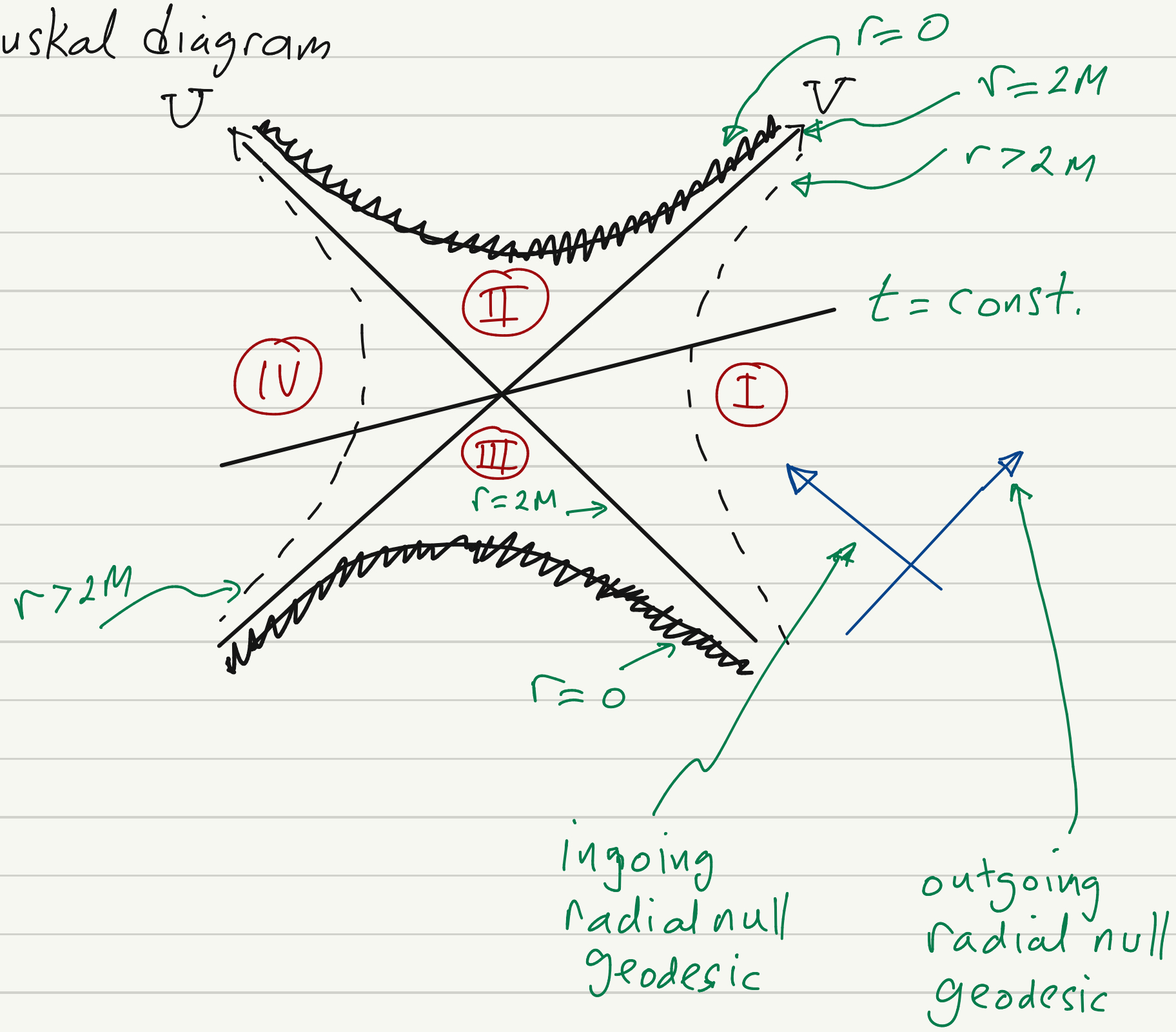
The metric can be analytically extended through
$U=0$ & $V=0$ to new regions

Notice $r=2M$ in ⊗ $\Rightarrow$ $U=0$ or $V=0$

& $r=0$ in ⊗ $\Rightarrow$ $UV=1$

Also, radial null geodesics have U or V = constant.

Kruskal diagram



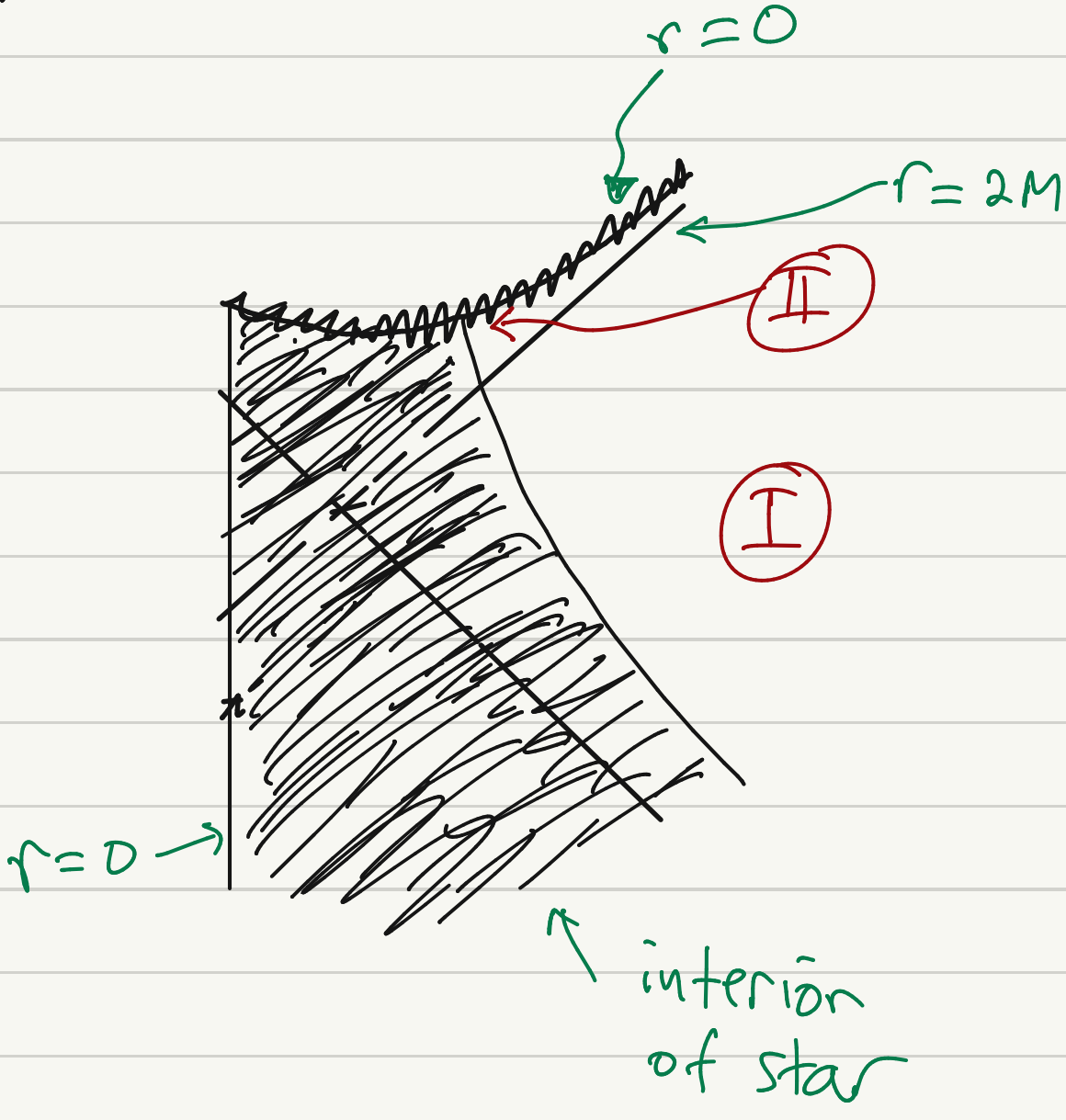
Regions I & II covered by ingoing E-F coord's.

I & III " " outgoing " "

Region III is new! (it can be obtained from region I by taking $(U, V) \to -(U, V)$)

We have constructed the maximally extended <u>eternal</u> Schwarzschild black hole

Regions III & <u>IV</u> are not relevant for gravitational collapse:



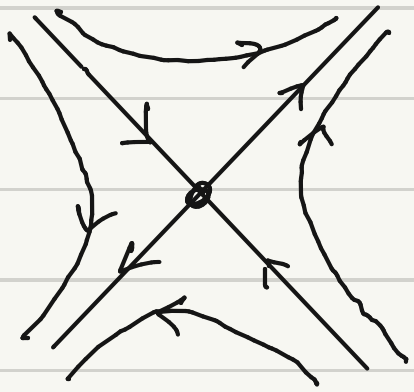
Notice that falling into a black hole, the singularity <u>happens</u>.

$k = \partial_t$ in region I

$= \frac{1}{4M}(V\partial_V - U\partial_U)$ defined everywhere

$\|k\|^2 = -\left(1 - \frac{2M}{r}\right) \Rightarrow$ null on $U = 0$ or $V = 0$

Orbits of k:



Also note $k = 0$ on "bifurcation 2-sphere" at $U = V = 0$

Einstein-Rosen bridge

Consider a slice of constant t – a line through the origin
In region I, the induced metric on this 3-d slice is

$$ds^2 = \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega \qquad r > 2M$$

change radial coordinate: $d\rho = \left(1-\frac{2M}{r}\right)^{-1/2} dr$

$$\Rightarrow \rho = \left(1-\frac{2M}{r}\right)^{1/2} r + M \log\left(\frac{r}{M} - 1 + \frac{r}{M}\left(1-\frac{2M}{r}\right)^{1/2}\right)$$

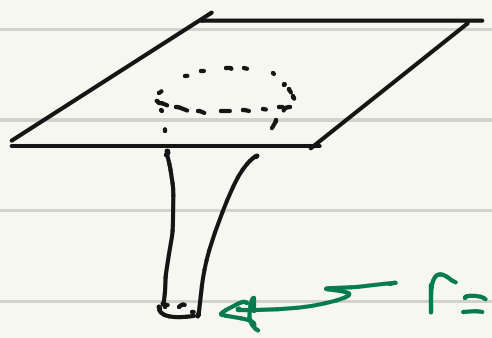
$\rho = 0$ at $r = 2M$

&

$$dS^2 = d\rho^2 + r^2(\rho)\, d\Omega$$

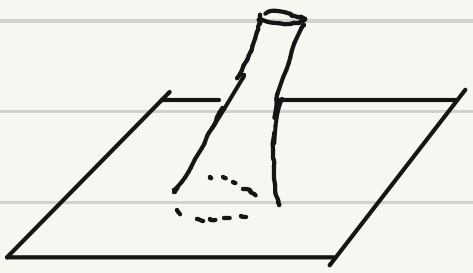
& at $r = 2M$ $\quad dS^2 = 4M^2 d\Omega$

I:

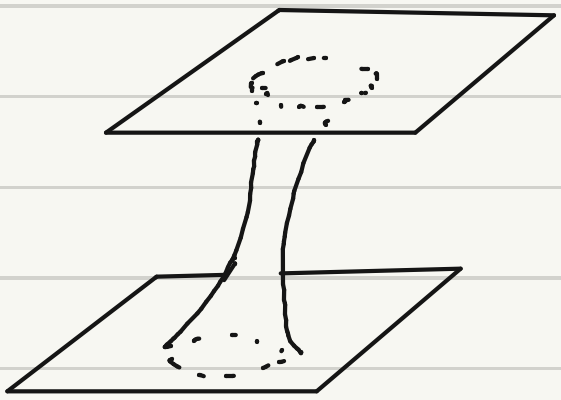


Minimal 2-sphere

IV:



in region IV there is an identical slice. They join precisely at the bifurcation 2-sphere at $U=V=0$ forming an E-R bridge.



Topology $\mathbb{R} \times S^2$
An example of a worm hole
It is (clearly!) non-traversible.

Penrose Diagrams

A powerful way to represent the causal structure of spacetime.

Definition: A conformal transformation is a map from a spacetime (M, g) to another $(M, \hat{g})$ with

$$\hat{g}_{\mu\nu}(x) = \Lambda^2(x)\, g_{\mu\nu}(x)$$

$\Lambda(x)$ smooth & $\Lambda(x) \neq 0 \quad \forall x \in M$

Conformal transformations preserve causal structure.
Let V be a vector

$$g_{\mu\nu} V^\mu V^\nu \begin{pmatrix} >0 \\ <0 \\ =0 \end{pmatrix} \Rightarrow \hat{g}_{\mu\nu} V^\mu V^\nu \begin{pmatrix} >0 \\ <0 \\ =0 \end{pmatrix} \quad \text{since } \Lambda^2 > 0$$

Curvature is <u>not</u> preserved! - explore on problem sheet

Idea of Penrose diagram

1) Do a coordinate transformation on (M, g) so that infinity is brought to a finite coordinate location.
2) Do a conformal transformation $\rightarrow (M, \hat{g})$ with $\hat{g}$ regular on the boundary.
3) Add points at ∞ to $M \longrightarrow (\hat{M}, \hat{g})$ which is called the conformal compactification of (M, g)

Ex. Minkowski spacetime in $D = 1+1$

$$ds^2 = -dt^2 + dx^2 \qquad -\infty < t, x < +\infty$$

$$\left.\begin{aligned} u &= t - x \\ v &= t + x \end{aligned}\right\} \quad ds^2 = -du\,dv \qquad -\infty < u, v < \infty$$

$$\left.\begin{aligned} u &= \tan\tilde{u} \\ v &= \tan\tilde{v} \end{aligned}\right\} \qquad -\pi/2 < \tilde{u}, \tilde{v} < \pi/2$$

The range is open since $u, v = \pm\infty$ are <u>not</u> points of M

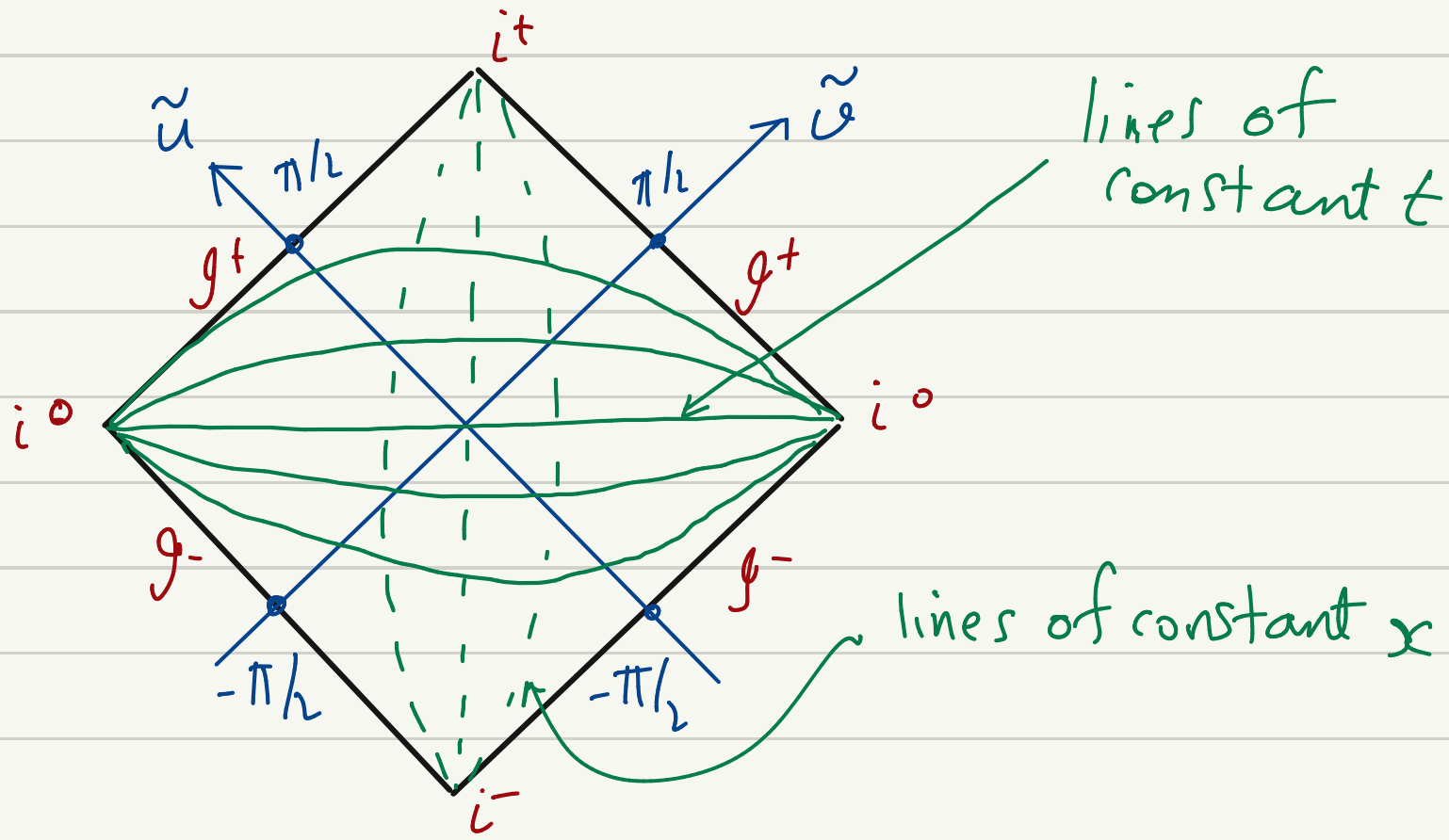
$$ds^2 = \frac{1}{(\cos\tilde{u}\cos\tilde{v})^2}\left[-d\tilde{u}\,d\tilde{v}\right]$$

Conformal compactification $(\hat{M}, \hat{g})$

- choose $\Lambda = \cos\tilde{u}\cos\tilde{v}$

$$\Rightarrow \quad d\hat{s}^2 = -d\tilde{u}\,d\tilde{v}$$

- <u>and</u> $\quad -\pi/2 \le \tilde{u}, \tilde{v} \le \pi/2$



- $(\tilde{u}, \tilde{v}) = (\pi/2, \pi/2) \leftrightarrow \begin{matrix} t \to \infty \\ x \text{ finite} \end{matrix}$ future temporal ∞ i^+
- $(\tilde{u}, \tilde{v}) = (-\pi/2, -\pi/2) \leftrightarrow \begin{matrix} t \to -\infty \\ x \text{ finite} \end{matrix}$ Past temporal ∞ i^-
- $(\tilde{u}, \tilde{v}) = (-\pi/2, \pi/2) \leftrightarrow t$ finite, $x \to \begin{matrix} +\infty \\ -\infty \end{matrix}$ spacelike ∞ i^0

  $(+\pi/2, -\pi/2)$

- $\tilde{u} = \pi/2 \leftrightarrow u \to \infty$, $|\tilde{v}| < \pi/2$, v finite

  $\tilde{v} = \pi/2 \leftrightarrow v \to \infty$, $|\tilde{u}| < \pi/2$, u finite

  } Future null ∞ $\mathscr{I}^+$ "Scri plus"

- $\tilde{u} = -\pi/2 \leftrightarrow u \to -\infty$, $|\tilde{v}| < \pi/2$, v finite

  $\tilde{v} = -\pi/2 \leftrightarrow v \to -\infty$, $|\tilde{u}| < \pi/2$, u finite

  } Past null ∞ $\mathscr{I}^-$ "Scri minus"

Null geodesics start at $\mathscr{I}^-$ & end at $\mathscr{I}^+$

Timelike geodesics start at i^- & end at i^+

For a massive particle to end at $\mathscr{I}^+$ it would need to be continuously accelerated.

Ex. Minkowski spacetime in $d = 3+1$

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

$$= -dt^2 + dr^2 + r^2 d\Omega$$

↓ choose arbitrary pt. as origin of polar coords.

First,

$u = t - r$

$v = t + r$

$$-\infty < u \leq v < \infty$$

↑ since $r \geq 0$

$$ds^2 = -du\,dv + \frac{(v-u)^2}{4} d\Omega$$

Next,

$u = \tan\tilde{u}$

$v = \tan\tilde{v}$

$$-\pi/2 < \tilde{u} \leq \tilde{v} < \pi/2$$

↖ ↗ note: $u, v = \pm\infty$ are not points in M

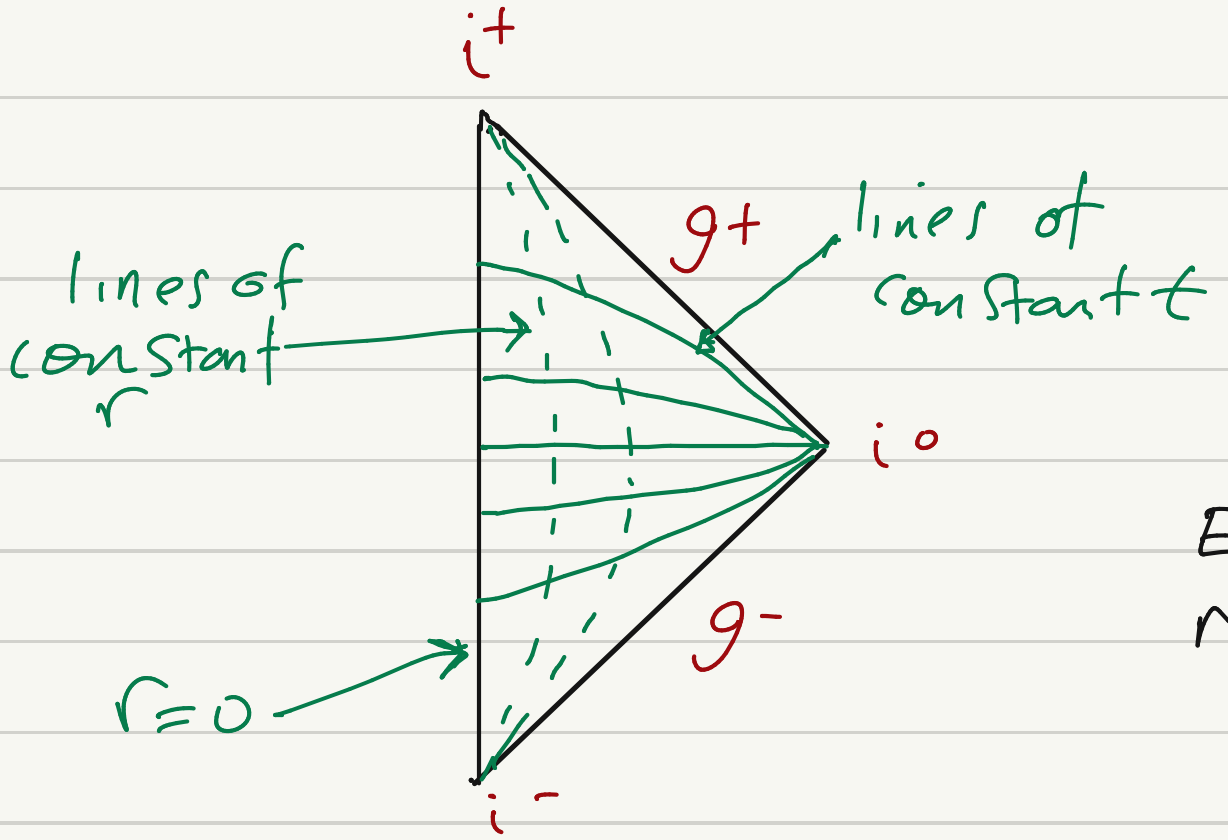
$$ds^2 = \frac{1}{(2\cos\tilde{u}\cos\tilde{v})^2}\left[-4 d\tilde{u}\,d\tilde{v} + \sin^2(\tilde{v}-\tilde{u})\, d\Omega\right]$$

Conformal compactification $(\hat{M}, \hat{g})$:

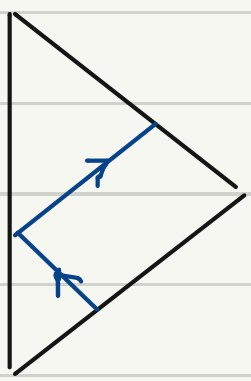
Take $\Lambda = 2\cos\tilde{u}\cos\tilde{v}$

$$d\hat{s}^2 = -4\, d\tilde{u}\, d\tilde{v} + \sin^2(\tilde{v}-\tilde{u})\, d\Omega$$

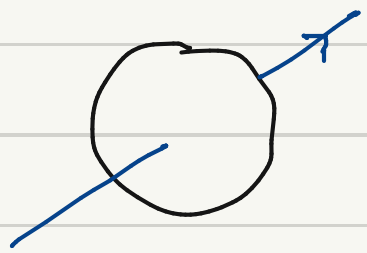
and $\quad -\pi/2 \leq \tilde{u} \leq \tilde{v} \leq \pi/2$



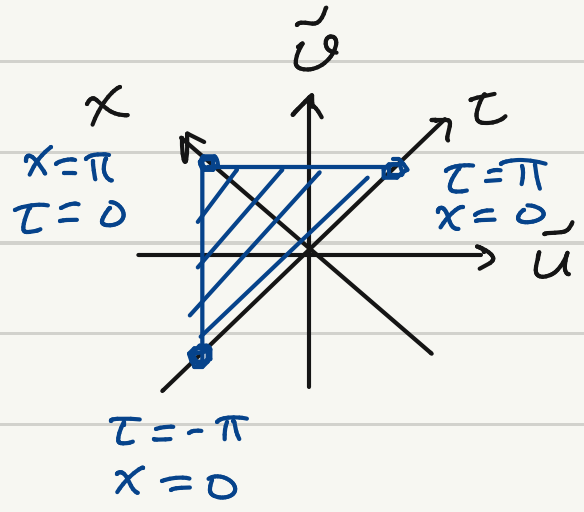
Every point represents an S^2



A radial null line



Another picture: $\begin{cases} \tau = \tilde{v} + \tilde{u} \\ \chi = \tilde{v} - \tilde{u} \end{cases}$

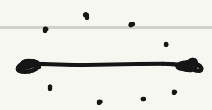


$\tilde{v} \geq \tilde{u} \Rightarrow -\pi \leq \tau \leq \pi$

$0 \leq \chi \leq \pi$

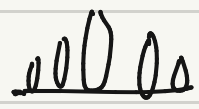
$$d\hat{s}^2 = -dt^2 + \underbrace{d\chi^2 + \sin^2\chi\, d\Omega}_{\text{round } S^3 \text{ metric}}$$

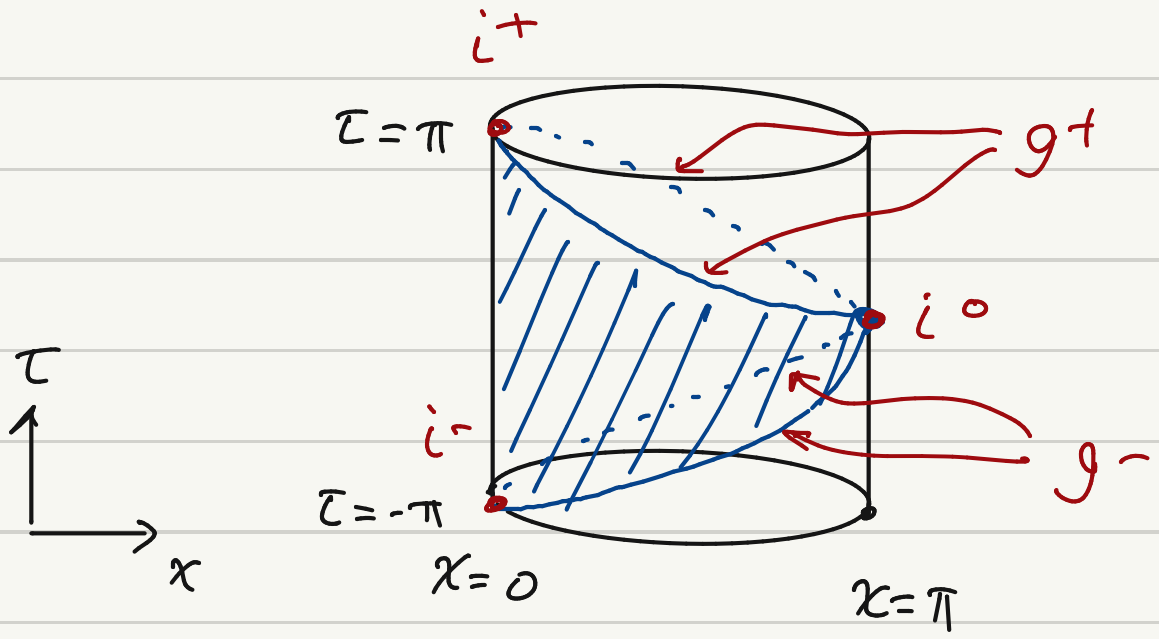
eg S^1

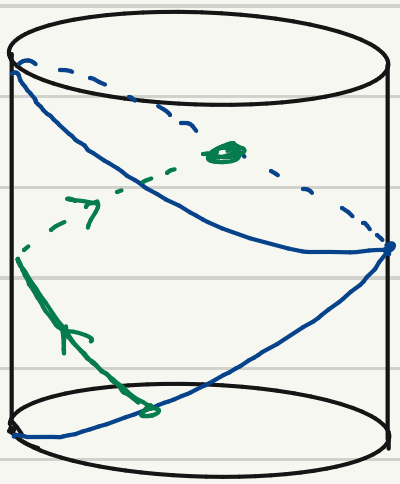


Einstein Static Universe metric

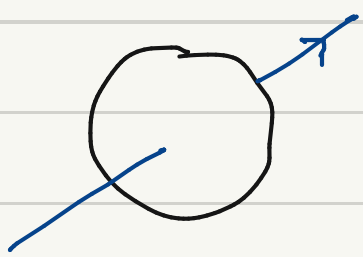
eg S^2:







Radial null lines end up on antipode of S^2



Ex. Kruskal spacetime

In region I $\quad ds^2 = -\left(1-\frac{2M}{r}\right)dudv + r^2 d\Omega$

$u = \tan\tilde{u}$
$v = \tan\tilde{v}$
$\quad -\pi/2 < \tilde{u}, \tilde{v} < \pi/2$

$$ds^2 = \frac{1}{(2\cos\tilde{u}\cos\tilde{v})^2}\left[-4\left(1-\frac{2M}{r}\right)d\tilde{u}d\tilde{v} + r^2\cos^2\tilde{u}\cos^2\tilde{v}d\Omega\right]$$

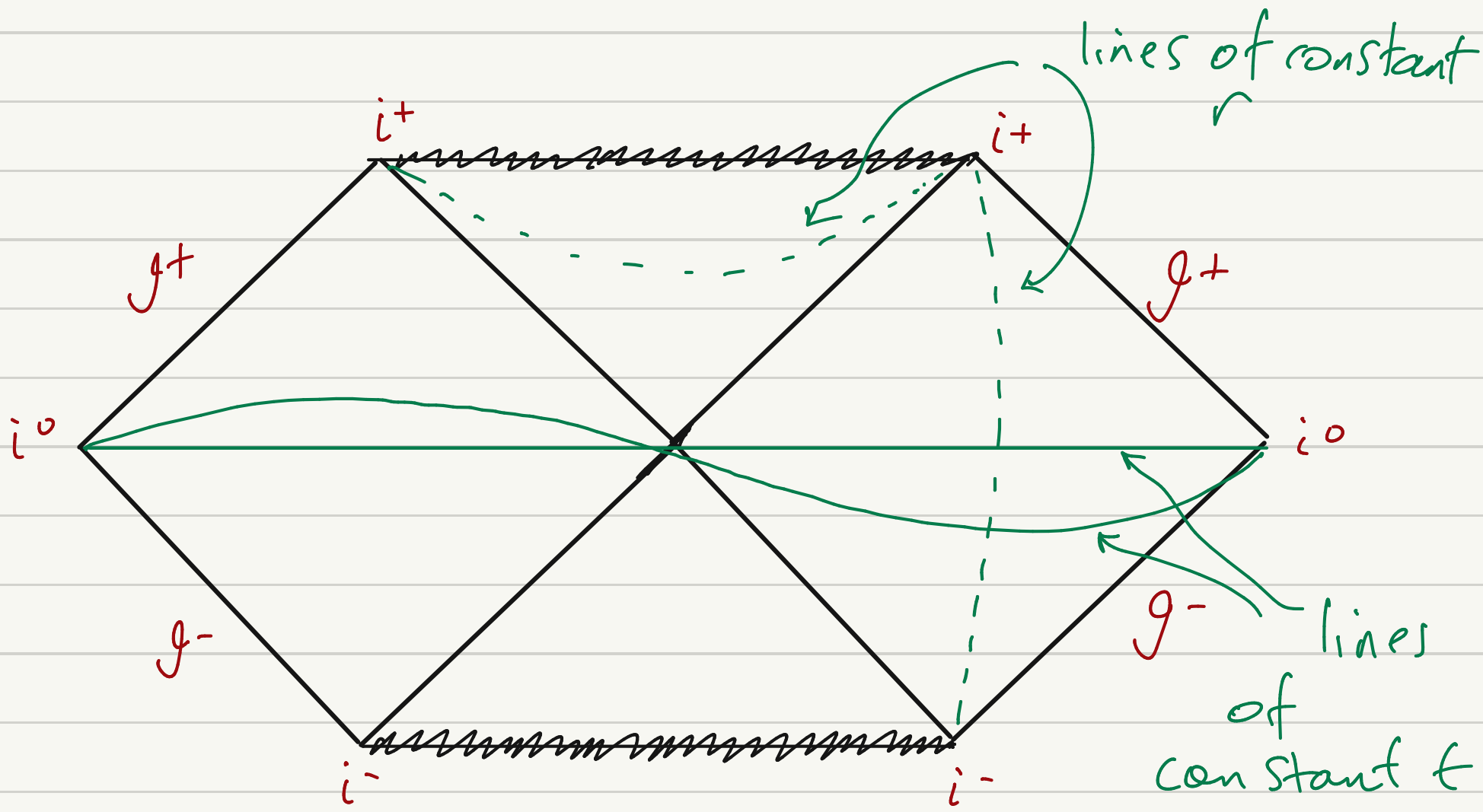
$$r_* = \frac{1}{2}(v-u) = \frac{\sin(\tilde{v}-\tilde{u})}{2\cos\tilde{u}\cos\tilde{v}} \qquad \text{recall } -\infty < r_* < \infty$$

$$d\hat{s}^2 = (2\cos\tilde{u}\cos\tilde{v})^2 ds^2$$

$$= -4\left(1-\frac{2M}{r}\right) d\tilde{u}\,d\tilde{v} + \left(\frac{r}{r_*}\right)^2 \sin^2(\tilde{v}-\tilde{u})\, d\Omega$$

This is an example of an asymptotically flat spacetime. As $r \to \infty$ (for any t) we approach Minkowski spacetime. Thus, we can add i^0, $\mathcal{I}^\pm$ as before.

Near $r = 2M$ introduce Kruskal-Szekeres coord's. Then define $U = \tan\tilde{U}$, $V = \tan\tilde{V}$ with $-\pi/2 < \tilde{U}, \tilde{V} < \pi/2$ etc. to deduce:



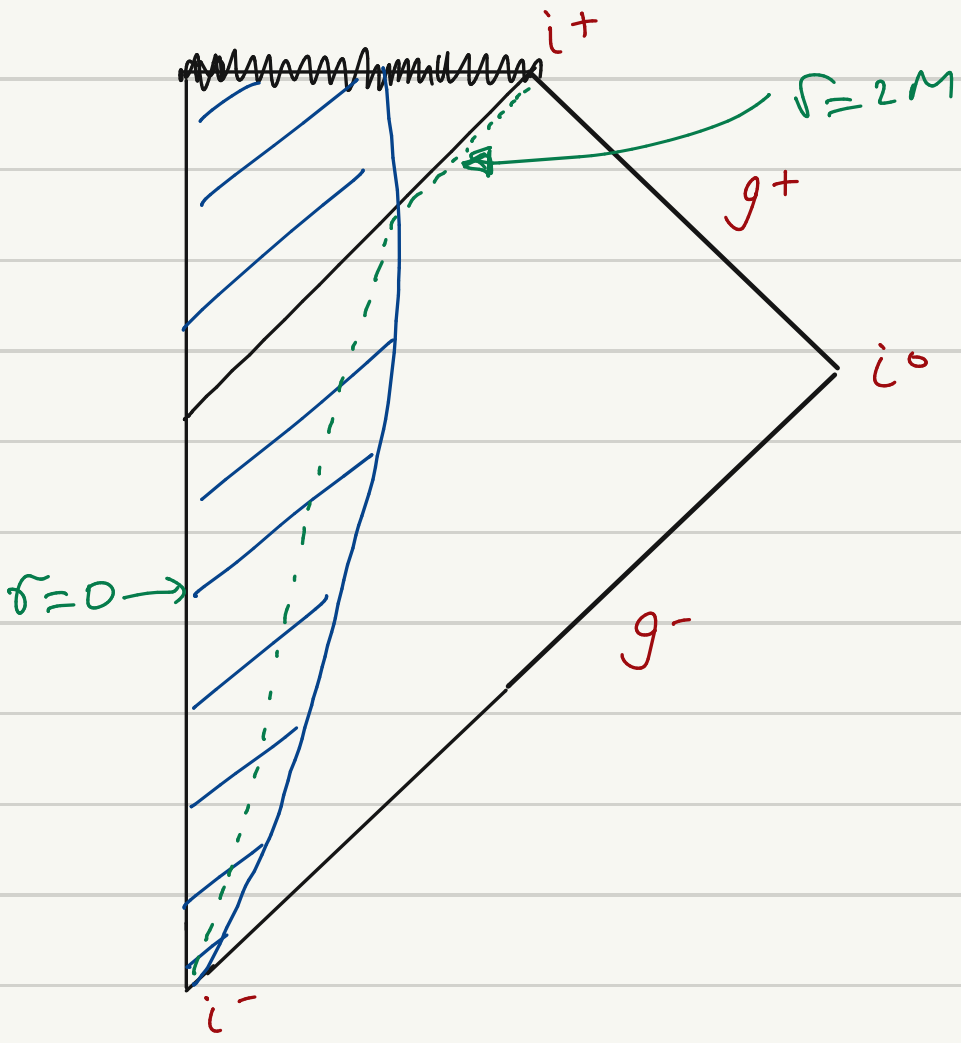
Notes: 1) Note $r=0 \Leftrightarrow UV=1 \Leftrightarrow \tan\tilde{U}\tan\tilde{V}=1$

$\Leftrightarrow \cos(\tilde{U}+\tilde{V})=0$

$\Leftrightarrow \tilde{U}+\tilde{V}=\pm\pi/2$ as in the figure

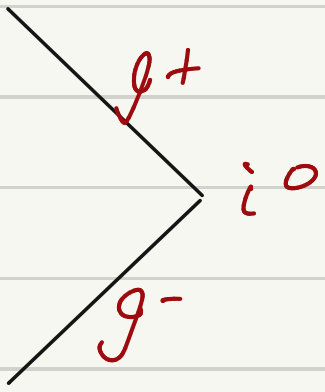
2) All $r=$ const. surfaces meet at i^+, including the singular $r=0$ surface. Thus i^+ is singular and cannot be added.
Same for i^-

Penrose diagram for collapsing star



Asymptotically flat spacetime - "looks like Minkowski at ∞"

The Kruskal example with singular $i^\pm$ means that we can only demand that asymptotically flat has same structure at null infinity, $\mathcal{I}^\pm$, and spacelike infinity i^0



is called an asymptotically flat end.

subtlety: need to be careful about smoothness of i^0 - it isn't smooth for Kruskal. See Wald.

Comments: 1) $\exists$ vectors which approach Killing vectors near i^0

2) Symmetries of $\mathcal{I}^\pm$ known as the Bondi-Metzner-Sachs (BMS) group.

Black holes & event horizons

Defns. * Chronological $\left(\begin{matrix}\text{future}\\ \text{Past}\end{matrix}\right)$ of a point x $\left(\begin{matrix}I^+(x)\\ I^-(x)\end{matrix}\right)$ is

the set of points which can be reached from x by a $\left(\begin{matrix}\text{future}\\ \text{past}\end{matrix}\right)$ directed timelike curve

* Causal $\left(\begin{matrix}\text{future}\\ \text{Past}\end{matrix}\right)$ of a point x $\left(\begin{matrix}J^+(x)\\ J^-(x)\end{matrix}\right)$ is

the set of points which can be reached from x by a $\left(\begin{matrix}\text{future}\\ \text{past}\end{matrix}\right)$ directed timelike or null (i.e. causal) curve

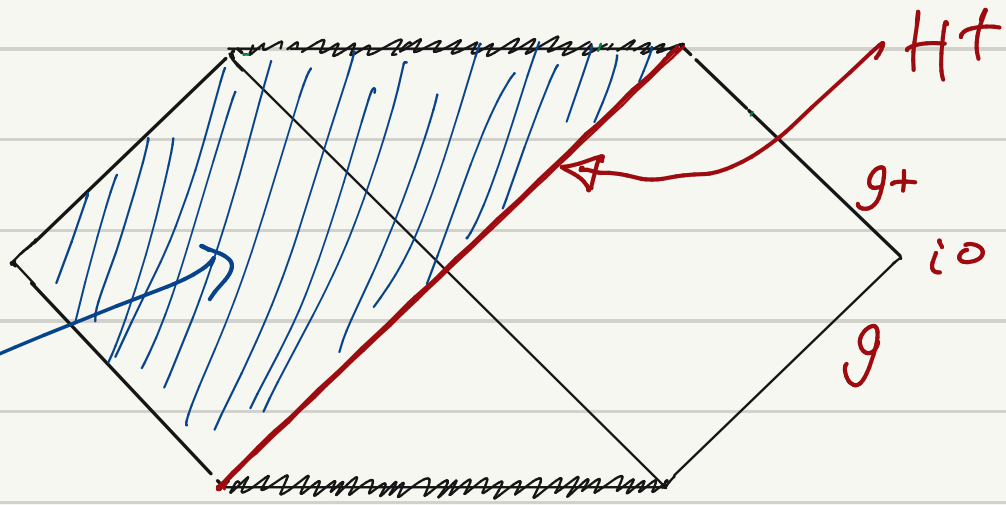
clearly $I^\pm(x) \subset J^\pm(x)$

For a set $U \subset M$ define $I^\pm(U) = \bigcup_i I^\pm(x_i) \;\forall x_i \in U$

$J^\pm(U) = \bigcup_i J^\pm(x_i) \;\forall x_i \in U$

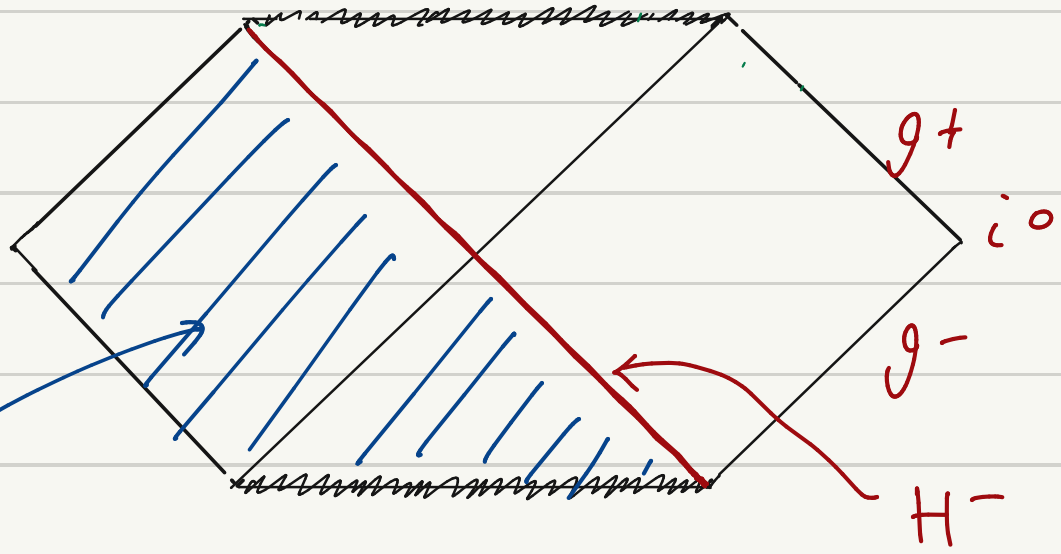
For a given asymptotically flat region, define the future event horizon H^+ to be the boundary ($\dot{J}^-(\mathcal{I}^+)$) of $J^-(\mathcal{I}^+)$ of the black hole region to be $M / J^-(\mathcal{I}^+)$

↑ define later



For a given asymptotically flat region, define the past event horizon H^- to be the boundary ($\dot{J}^+(\mathcal{I}^-)$) of $J^+(\mathcal{I}^-)$ of the white hole region to be $M / J^+(\mathcal{I}^-)$

↑ define later



Singularities

Coordinate singularities can be removed by coordinate transformations. Not physical. e.g. $r = 2M$ in Kruskal.

Curvature singularities - e.g. some scalar constructed from $R^{\mu}{}_{\nu\rho\sigma}$ diverges as in Kruskal at $r=0$. Also possible to have singularities in which no scalar diverges, yet $\nexists$ coordinates in which all components of $R^{\mu}{}_{\nu\rho\sigma}$ remain finite $\longrightarrow$ infinite tidal forces.

Conical singularity: e.g. $dr^2 + \lambda^2 r^2 d\phi^2$ $\quad 0 < r < \infty$, $\quad 0 \le \phi < 2\pi$

$\lambda = 1$: change coordinates to get $dx^2 + dy^2$

$\lambda \neq 1$: $\quad \phi' \equiv \lambda\phi \quad 0 \le \phi' < 2\pi\lambda$

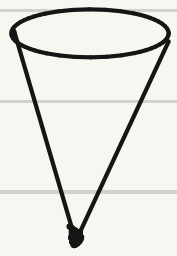
$dr^2 + r^2 d\phi'^2$

consider a circle of radius ε near $r = 0$

$$\frac{\text{circumference}}{\text{radius}} = \frac{2\pi\lambda\varepsilon}{\varepsilon} = 2\pi\lambda \neq 2\pi \text{ as } \varepsilon \to 0$$

$\Rightarrow$ singular at $r=0$ since neighbourhood of $r=0$ is not locally $\mathbb{R}^2$

eg. $\lambda < 1$



More generally, use the idea that a singularity is captured by having geodesics that can't be extended to arbitrarily large affine parameter.

Defⁿ: A geodesic is complete if an affine parameter exists extending to $\pm\infty$.

A spacetime is geodesically complete if all maximally extended causal geodesics are complete.

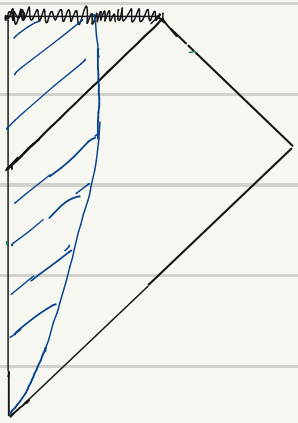
Examples: $\mathbb{M}$ (Minkowski Spacetime) complete

$\mathbb{M} - \{pt.\}$ - is geodesically incomplete

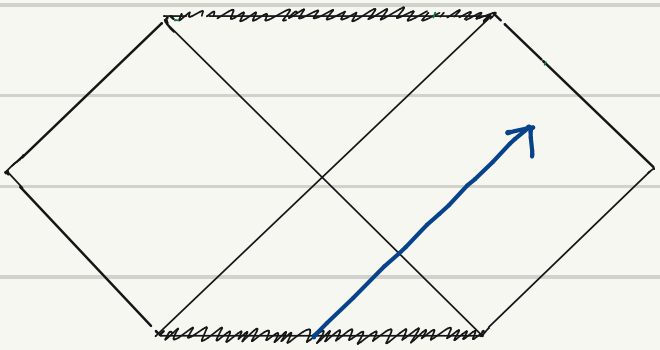
Cone - incomplete

Kruskal - incomplete.

Note



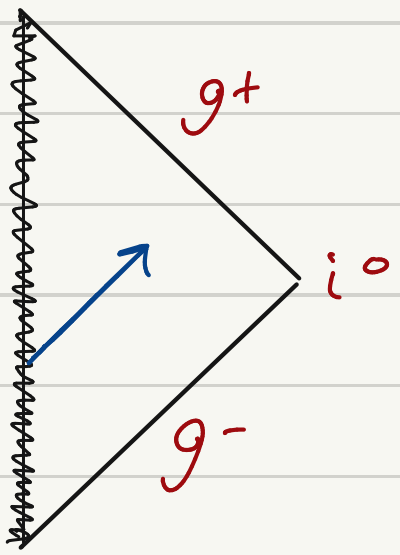
singularity is clothed by the horizon.



singularity is naked: it can be seen at $\mathcal{I}^+$

Negative mass Schwarzschild

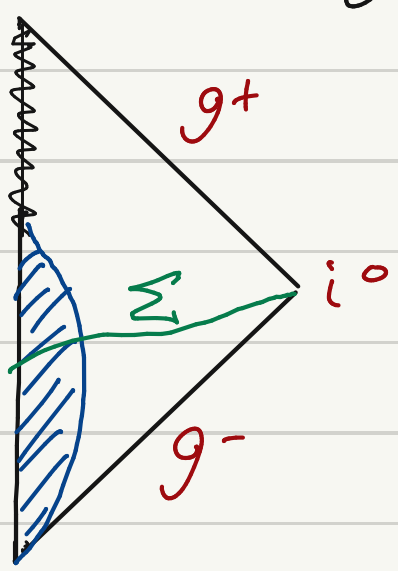
$$ds^2 = -\left(1 + \frac{2|M|}{r}\right) dt^2 + \left(1 + \frac{2|M|}{r}\right)^{-1} dr^2 + r^2 d\Omega$$



Naked singularity.

Can naked singularities form?

e.g.



If this happened, the future would cease to be predictable from data given on Σ!

Cosmic censorship Roughly: Naked singularities cannot form from generic gravitational collapse in asymptotically flat spacetime that is non-singular on some initial spacelike hypersurface. (Penrose)

Comments: 1) Need to exclude fine tuned situations
e.g. Scalar field + spherical symmetry.
2) Excludes cosmological singularities

3) Need some assumptions on nature of matter ("dominant energy condition" see later.

4) ∃ a "strong" & a "weak" form of cosmic censorship which are logically distinct.

5) Unproven - major goal in classical G.R.

Singularity Theorems — Penrose + Hawking c. 1965

Powerful results that show singularities (not necessarily curvature singularities) will occur in certain situations, including gravitational collapse.

Combined with cosmic censorship we expect that these will always be clothed by event horizons.

Black holes provide a beautiful & consistent picture for the final state of gravitational collapse.

Moreover, G.R. predicts that black holes (& the big bang) are places to look for new physics ⟶ quantum gravity

Reissner - Nordstrom Black Holes

Einstein - Maxwell theory

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left[R - F_{\mu\nu}F^{\mu\nu}\right]$$

(notice normalisation)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$
$$= \nabla_\mu A_\nu - \nabla_\nu A_\mu \quad (Ex.)$$

Also have set $G = c = 1$

Equations of motion:

$$\begin{cases} R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu} = 2\left\{F_\mu{}^\rho F_{\nu\rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right\} \\ \nabla_\mu F^{\mu\nu} = 0 \end{cases}$$

Generalisation of Birkhoff:

The unique spherically symmetric solution of E. Maxwell (with non constant radius function) is the R-N solution

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right)^{-1}dr^2 + r^2 d\Omega$$

$A_t = -\frac{Q}{r}$ $\qquad F_{rt} = \frac{Q}{r^2}$ $\qquad Q$ electric charge

$A_\phi = -P\cos\theta$ $\qquad F_{\theta\phi} = P\sin\theta$ $\qquad P$ Magnetic charge

$e = \sqrt{Q^2 + P^2}$

static: $k = \partial_t$ Killing

Asymptotically flat as $r \to \infty$

To analyse, define $\Delta = r^2 - 2Mr + e^2$

$\equiv (r - r_+)(r - r_-)$

$$r_\pm = M \pm \sqrt{M^2 - e^2}$$

what happens at $r = r_\pm$?

Singular at $r = 0$: $\begin{cases} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \to \infty \\ F_{\mu\nu} F^{\mu\nu} \to \infty \quad \text{easy calc'n.!} \end{cases}$

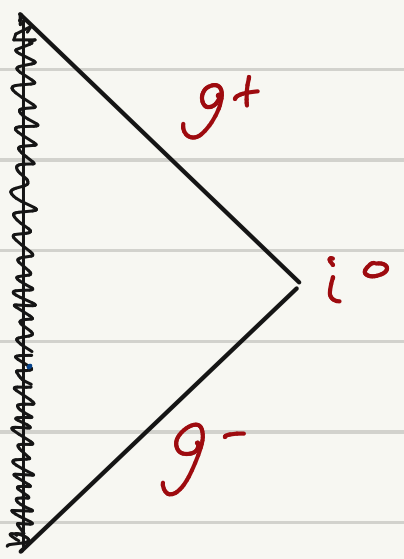
3 Cases: 1) super extremal $e > M$ No Coord. Sing'gs.

2) sub extremal $e < M$ $r_+ \neq r_-$

3) extremal $e = M$ $r_+ = r_-$

1) $\underline{e > M}$: (t, r, θ, ϕ) good $\forall\, r > 0$

$r = 0$ is a timelike singularity as for -ve mass Schwarzschild.



Non-Physical - it cannot appear as end point of collapse (assuming cosmic censorship)

Note: an electron has $e > M$ but it is quantum mechanical.

2) $e < M$

$\Delta = 0$ at $r = r_+, r_-$

Proceed as we did for Schwarzschild. Start with $r > r_+$ & define

$$dr_* = \frac{r^2}{\Delta} dr \Rightarrow r_* = r + \frac{1}{2\kappa_+} \log\left|\frac{r-r_+}{r_+}\right| + \frac{1}{2\kappa_-} \log\left|\frac{r-r_-}{r_-}\right| + \text{const.}$$

where $\kappa_\pm = \dfrac{r_\pm - r_\mp}{2 r_\pm^2}$

Define $u = t - r_*$

$v = t + r_*$

- Ingoing E-F coordinates (v, r, θ, ϕ) w. $r > r_+$

$$ds^2 = -\frac{\Delta}{r^2} dv^2 + 2 dv dr + r^2 d\Omega \quad \text{(Ex.)}$$

Smooth $\forall r > 0$, hence can extend to $0 < r < r_+$

- Outgoing E-F coordinates (u, r, θ, ϕ) w. $r > r_+$

$$ds^2 = -\frac{\Delta}{r^2} du^2 - 2 du dr + r^2 d\Omega \quad \text{(Ex.)}$$

& can extend to a <u>different</u> region $0 < r < r_+$

Kruskal coordinates:

$$\begin{cases} U^{\pm} = -e^{-\kappa_{\pm} u} \\ V^{\pm} = \pm e^{\kappa_{\pm} v} \end{cases}$$

In region $r > r_+$ use (U^+, V^+, θ, ϕ)

$$ds^2 = -\frac{r_+ r_-}{\kappa_+^2 r^2} e^{-2\kappa_+ r} \left(\frac{r - r_-}{r_-}\right)^{1 + \frac{\kappa_+}{|\kappa_-|}} dU^+ dV^+ + r^2 d\Omega$$

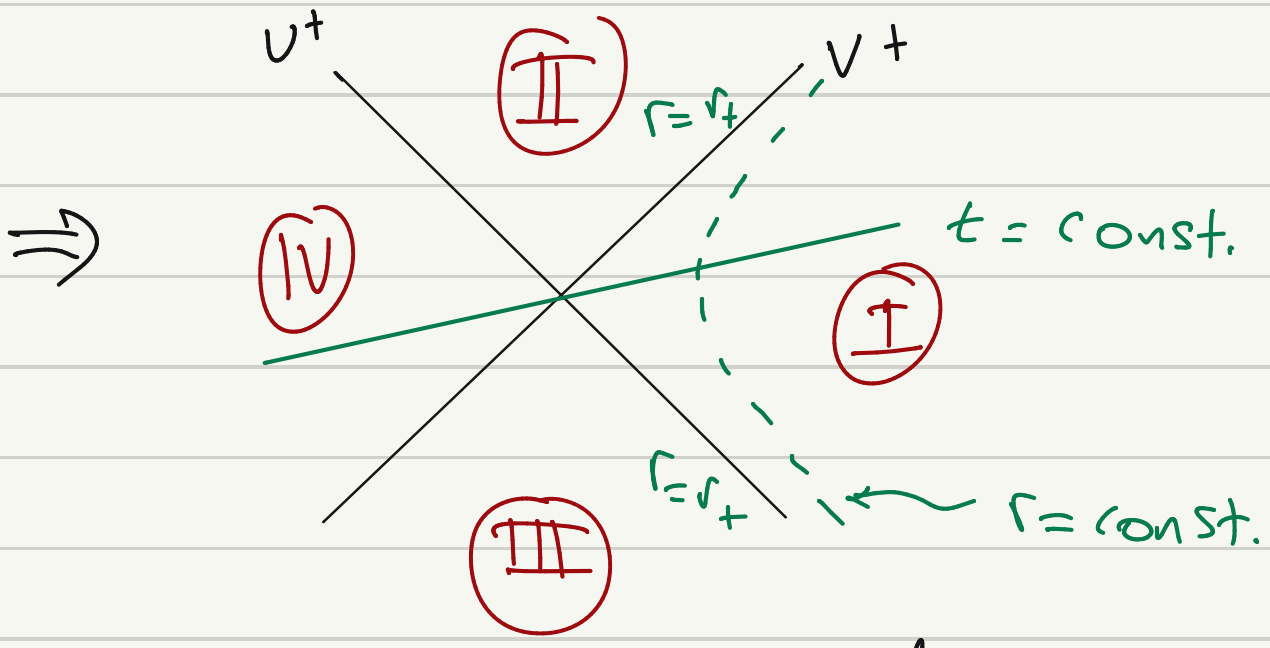
$r(U^+, V^+)$ defined via

$$-U^+ V^+ = e^{2\kappa_+ r} \left(\frac{r - r_+}{r_+}\right) \left(\frac{r_-}{r - r_-}\right)^{\kappa_+/|\kappa_-|}$$

Note RHS is monotonically increasing for $r > r_-$

Initially $U^+ < 0$, $V^+ > 0$ $\Leftrightarrow r > r_+$

but now we can have $U^+ \gtrless 0$ or $V^+ \lesseqgtr 0$



Again $\exists$ E. Rosen bridge for t = const. slices

Also, $\exists$ two null surfaces generated by Killing vector k & they intersect on bifurcation S^2 at $U^+ = V^+ = 0$

Major difference to Kruskal case is that here there are no singularities in regions II & III since $r(U^+, V^+) > r_-$ ($r(U^+, V^+) = r_- \Leftrightarrow U^+ V^+ = \infty$)

Now we know from I.E.F. coordinates that we can extend to $r < r_-$. Hence, we must be able to further extend. Said differently: radial null geodesics reach $r = r_-$ in finite affine parameter & hence they will reach $U^+ V^+ = \infty$ in finite affine parameter. So we need even more coordinates.

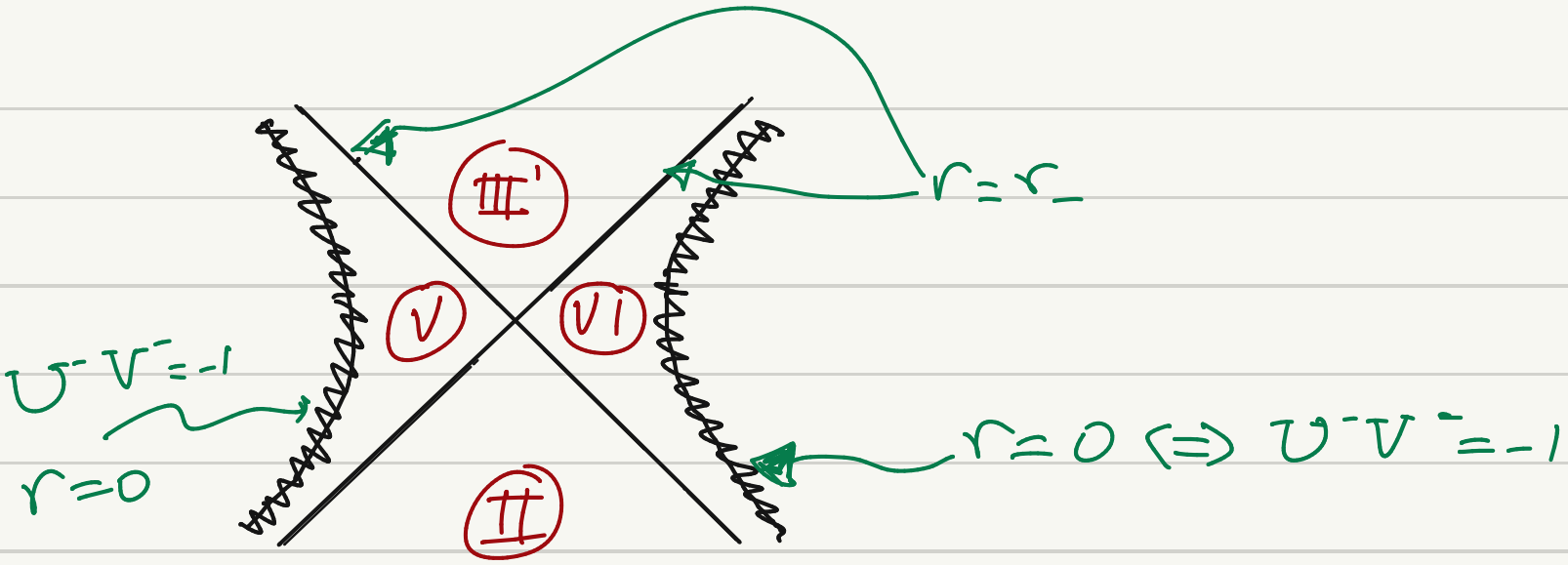
Start in region II & use (v, r, θ, ϕ). Then define u in region II as follows. First define $t = v - r_*$ in region II. Then in region II we can use (t, r, θ, ϕ) & metric is in the original R-N form but now $r_- < r < r_+$. Now define $u = t - r_* = v - 2r_*$ & we have got u, v in region II. Now we can use $U^- < 0$ & $V^- < 0$ in region II:

$$ds^2 = -\frac{r_+ r_-}{K_-^2 r^2} e^{2|K_-|r} \left(\frac{r_+ - r}{r_+}\right)^{1 + \frac{|K_-|}{K_+}} dU^- dV^- + r^2 d\Omega$$

with $r(U^-, V^-) < r_+$ given by

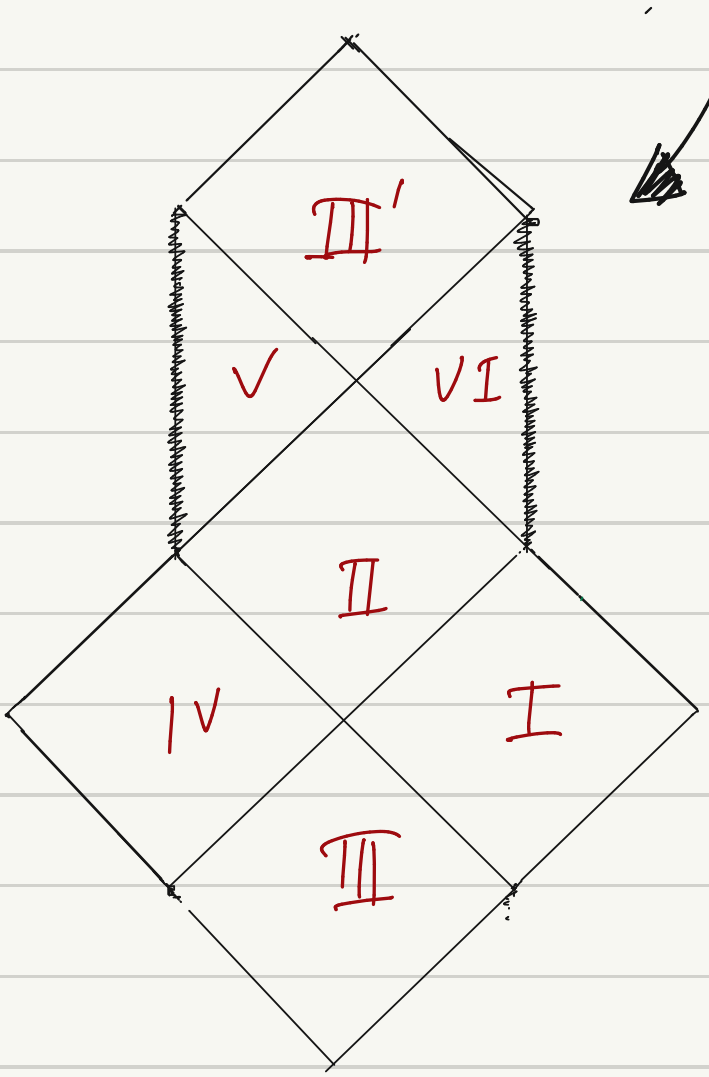
$$U^- V^- = e^{-2|K_-|r} \frac{r - r_-}{r_-} \left(\frac{r_+}{r_+ - r}\right)^{|K_-|/K_+}$$

This can now be continued to $U^- \geq 0$ or $V^- \geq 0$



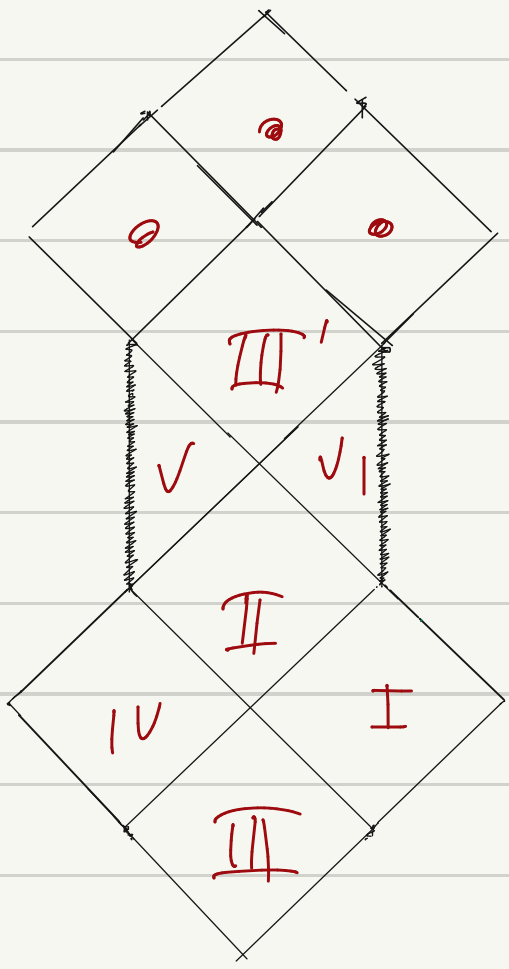
Notice $r=0$ singularity is now timelike

Thus we have deduced:



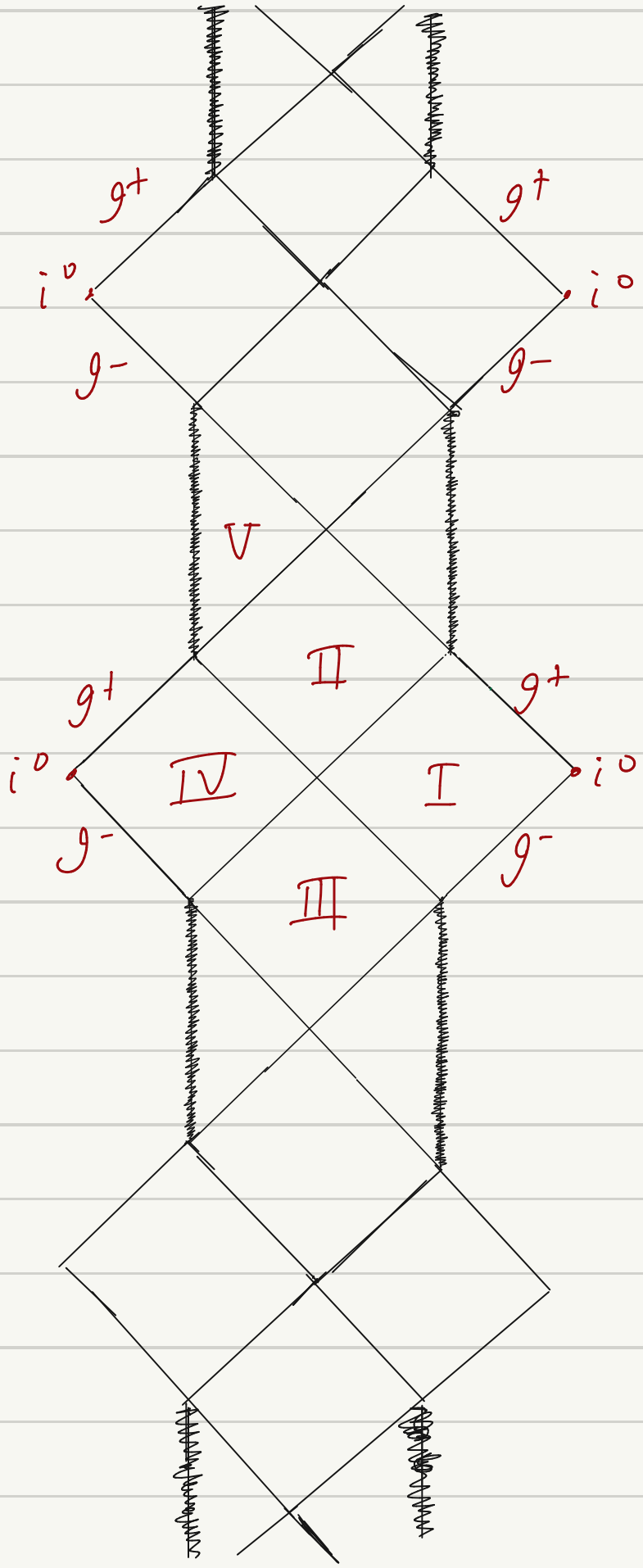
Next: the time reversal of III' is II, which is connected to I, II & IV

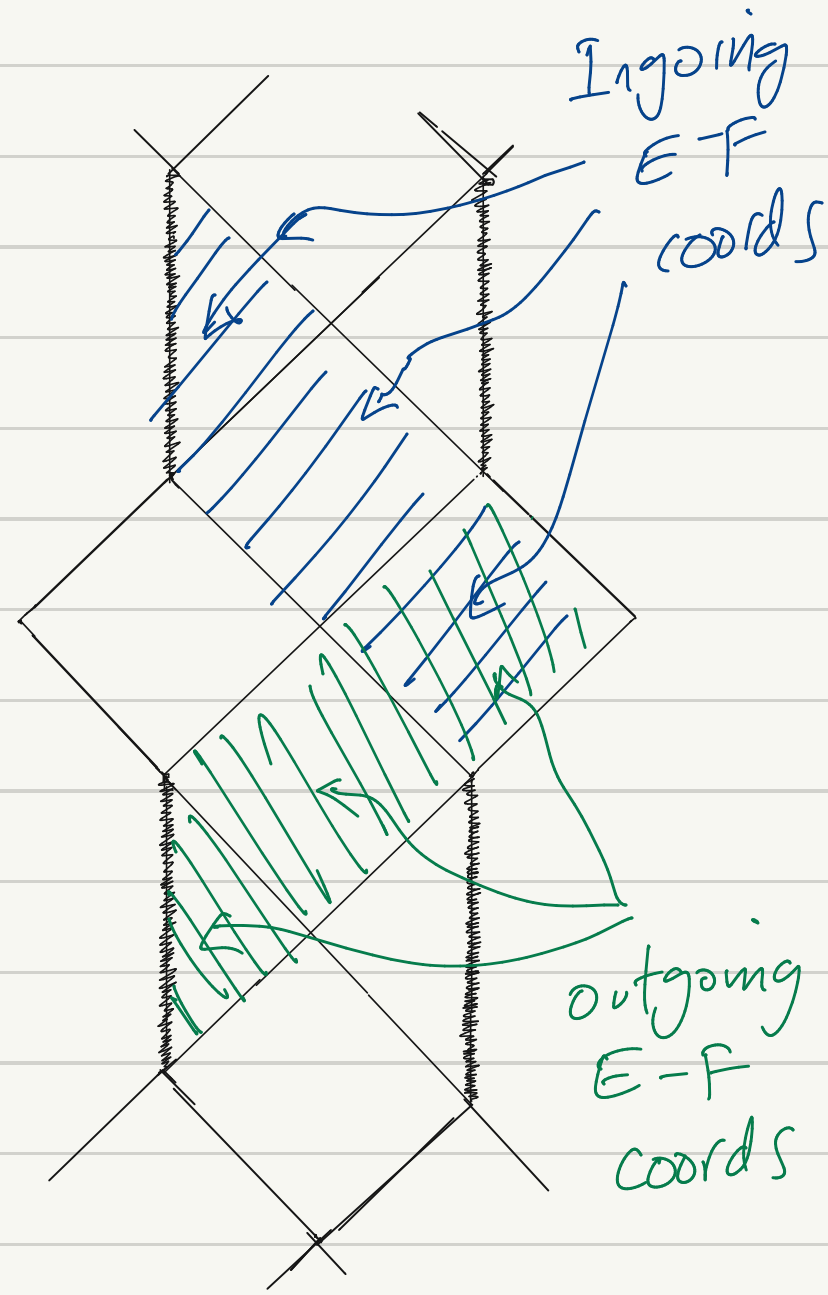
$\Rightarrow$ III' must be connected to something similar & we deduce three more regions

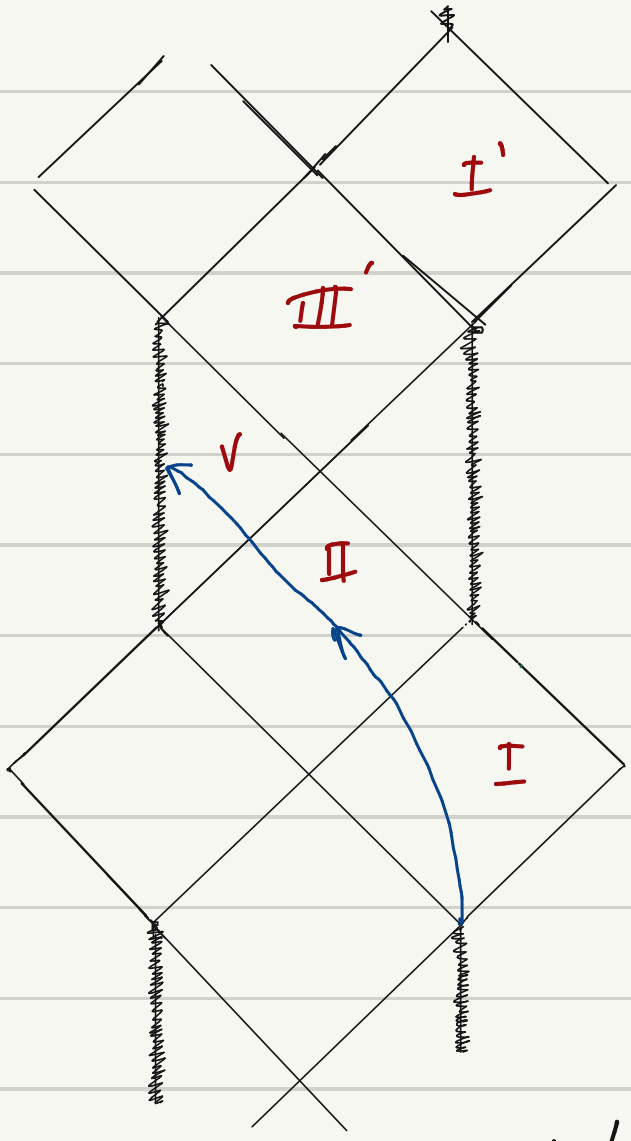


And we can keep going in both directions!

We discover an infinite number of new asymptotically flat regions.







This curve is an ingoing radial null geodesic - we might think of it as a collapsing shell of charged matter. Naively it hits $r=0$ in region V

But suppose it accelerates & goes to region I'!?

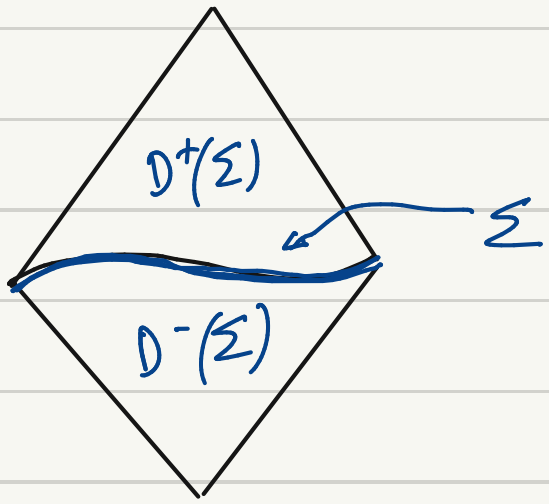
It must cross a Cauchy horizon

Let (M,g) be a timelike orientable spacetime. A partial Cauchy surface Σ is a hypersurface for which no two points are connected by a causal curve in M.

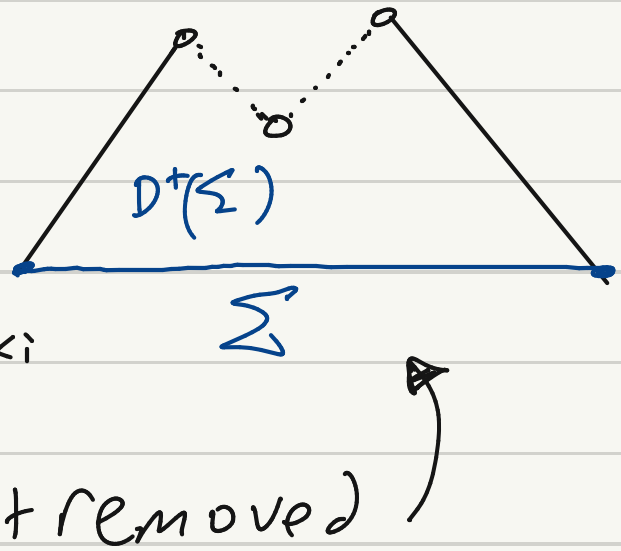
The future domain of dependence of Σ, $D^+(\Sigma)$ is the set of $p \in M$ such that every past inextendible causal curve through p intersects Σ

(causal curve is past inextendible if it has no past endpoint.)

the <u>past domain of dependence</u> $D^-(\Sigma)$ is defined in a similar way.



but note in Minkowski spacetime with point removed

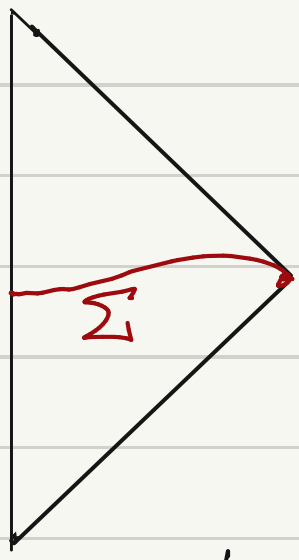


<u>Domain of dependence</u> $D(\Sigma) = D^+(\Sigma) \cup D^-(\Sigma)$

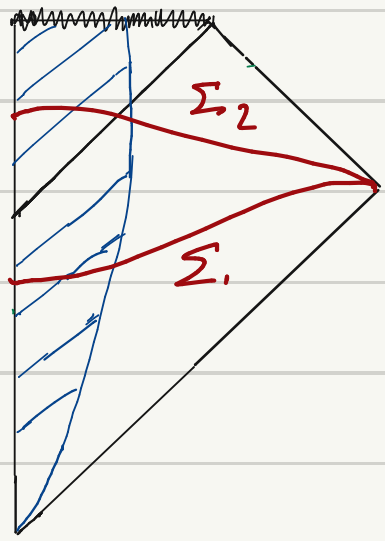
$D(\Sigma)$ is the region of spacetime where we can determine what happens from data given on Σ eg Solutions of hyperbolic p.d.e.'s are uniquely determined in $D(\Sigma)$ by initial data on Σ

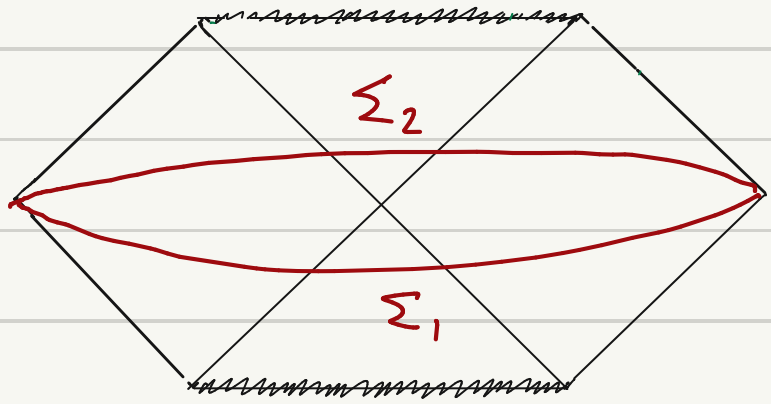
A spacetime M is <u>globally hyperbolic</u> if it has a <u>Cauchy surface</u> i.e. a partial Cauchy surface with $D(\Sigma) = M$

Examples

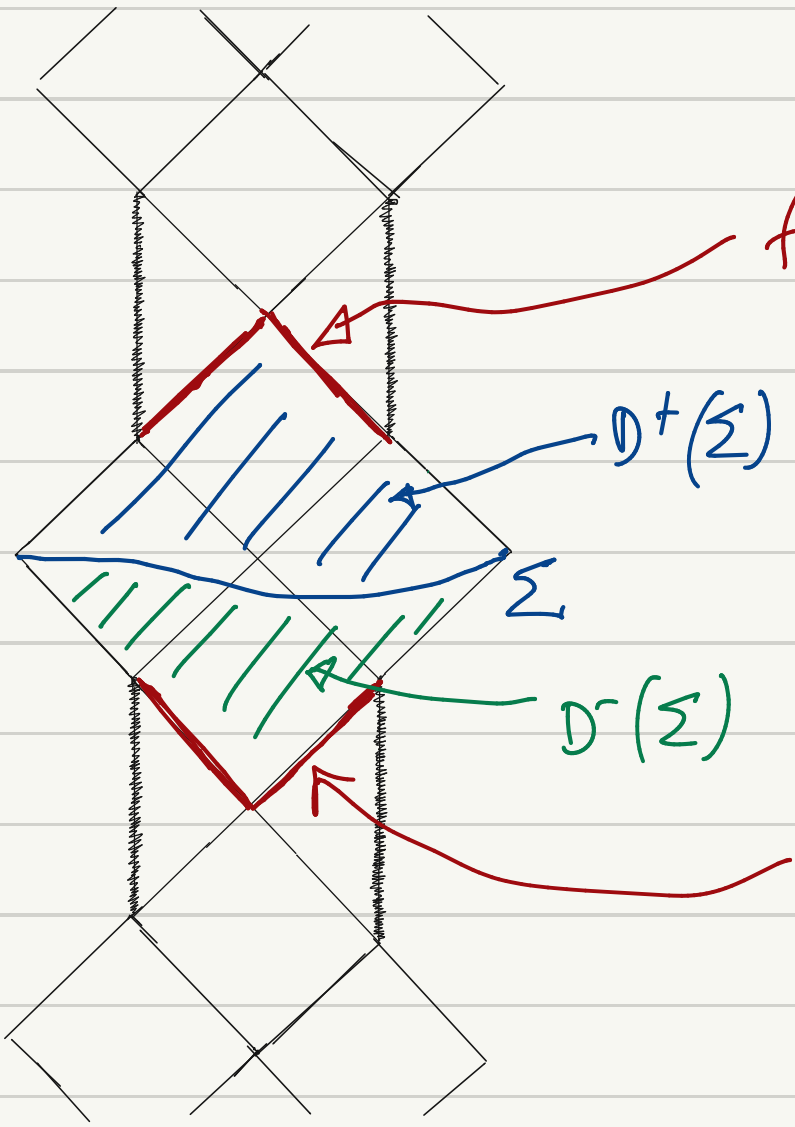


Minkowski

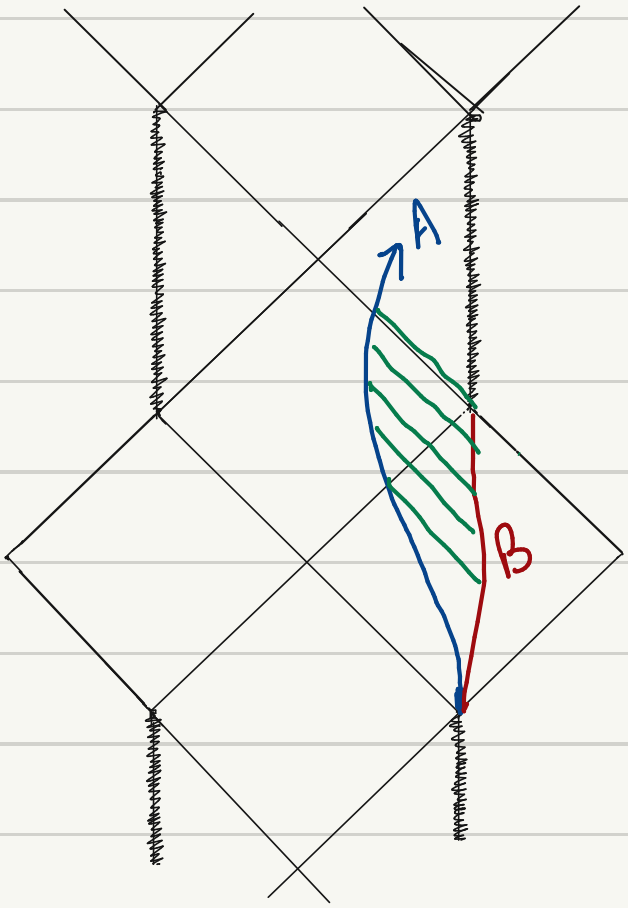




If M is <u>not</u> globally hyperbolic, then for any Σ $D^+(\Sigma)$ and/or $D^-(\Sigma)$ will have a boundary called the future, past Cauchy horizon, respectively.



The existence of Cauchy horizons makes most of this Penrose unphysical. Recall we assumed spherical symmetry & analyticity. If we relax these we can extend beyond Cauchy horizon in infinitely many different ways given data on Σ



Also:

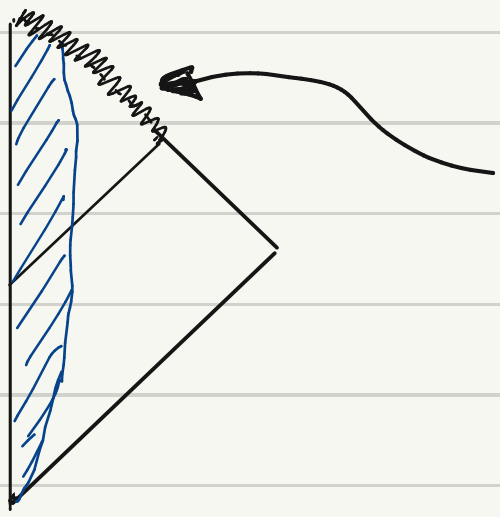
B sends light signals to A

A must receive <u>all</u> these signals in a finite proper time. $\Rightarrow$ Signals from B undergo infinite blueShift.

Hence a small perturbation in region I (eg small frequency) will have enormous energy (eg ∞ frequency) at Cauchy horizon as measured by A.

One possibility is that for realistic setups the $r=r_-$ Cauchy horizon is replaced by a singularity & if so the resulting spacetime would be globally hyperbolic!

Possible Penrose diagram for collapsing charged matter



null? spacelike?

3) Extreme Reissner-Nordstrom: $\underline{e = M}$

Now

$r_+ = r_- = M$

$$ds^2 = -\left(1-\frac{M}{r}\right)^2 dt^2 + \left(1-\frac{M}{r}\right)^{-2} dr^2 + r^2 d\Omega$$

Start with $r > M$: $\quad dr_* = \dfrac{dr}{(1-M/r)}$

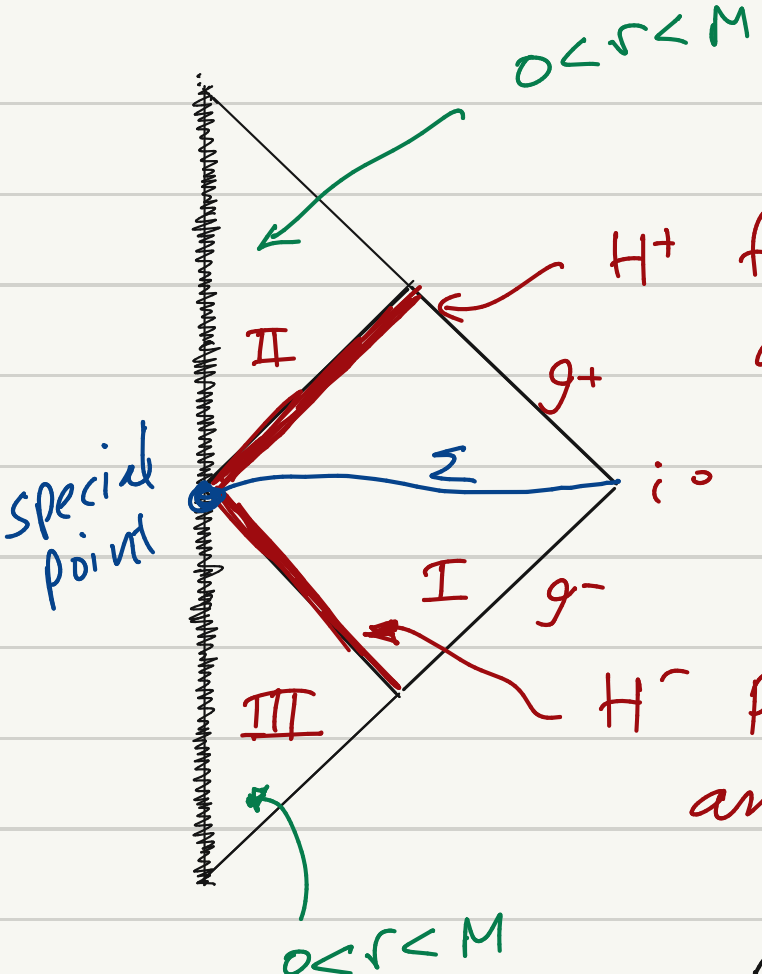
$$\rightarrow r_* = r + 2M \log\left|\frac{r-M}{M}\right| - \frac{M^2}{r-M}$$

$v = t + r_*$

$$ds^2 = -\left(1-\frac{M}{r}\right)^2 dv^2 + 2\,dv\,dr + r^2 d\Omega$$

and we can now extend to $0 < r < M$ to find black hole.

Instead use $u = t - r_*$ to extend to find white hole.



Regions II & III are isometric $\Rightarrow$ new asymptotically flat regions to future of II & past of III. Hence:



Notes

1) $k = \partial_t = \partial_v$ Out. E-F
$= \partial_u$ In. E-F

$\|k\|^2 < 0 \quad r \neq M$
$= 0 \quad r = M$

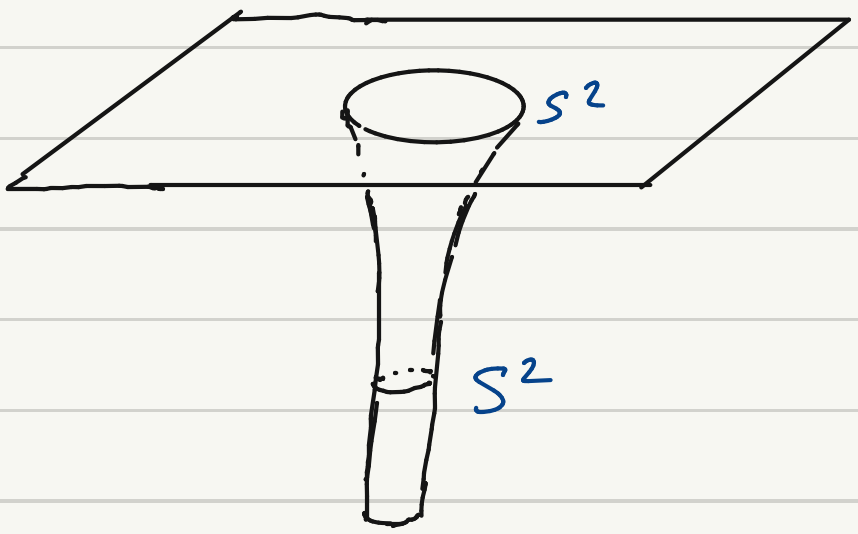
i.e. is timelike everywhere except when $r = M$ where it is null.

2) Novel feauture: the $t=$ const. surface is not an E-R. bridge connecting two asymptotically flat regions, but an infinite throat:

$t=$ const, $(\theta, \phi)=$ const.

proper length from r_0 to $r=M$ is

$$\int_M^{r_0} \frac{dr}{1-M/r} \to \infty$$



3) There exists multi_extremal R-N black holes Majumdar - Papapetrou (1947)

Set $P=0$ so $e=Q$. Introduce new radial coordinate $\rho = r-M$ for extreme R-N:

$$\begin{cases} ds^2 = -H^{-2}dt^2 + H^2(d\rho^2+\rho^2 d\Omega) \\ A = -H^{-1}dt + dt \end{cases}$$

Ex.

↖ gauge transformation.

with $H = 1 + M/\rho$

Special case of

$$\begin{cases} ds^2 = -H^{-2}\,dt^2 + H^2\,(d\underset{\sim}{x}\cdot d\underset{\sim}{x}) \\ A = -H^{-1}\,dt \end{cases}$$

with $H = H(\underset{\sim}{x})$ & $\partial_i \partial_i H(\underset{\sim}{x}) = 0$

which one can show solves Einstein-Maxwell equations of motion. (Ex. show $D_\mu F^{\mu\nu} = 0$)

Choose

$$H = 1 + \sum_{i=1}^{N} \frac{M_i}{|\underset{\sim}{x} - \underset{\sim}{x}_i|}$$

* Static solution
* Describes N extremal R-N black holes with mass (= charge) $= M_i$ at "positions" $\underset{\sim}{X}_i$ (In fact $\underset{\sim}{x} = \underset{\sim}{x}_i$ is an S^2 not a point in the spacetime)
* $\exists$ a cancellation of gravitational attraction & electromagnetic repulsion precisely for $e = M$
* Can embed these in $D=4$ supergravity & the solutions are "supersymmetric"!
  $\Leftrightarrow \exists$ killing spinors ε satisfying

$$D_\mu \varepsilon + F_{\rho\sigma}\,\gamma^{\rho\sigma}\,\gamma_\mu\,\varepsilon = 0$$