Black Holes – Problem Sheet 1

1. Planck units are the units of length, time and mass that can be formed from the fundamental constants of nature G, c and $\hbar$. Planck length $l_p = \sqrt{G\hbar/c^3}$, Plank mass $m_p = \sqrt{c\hbar/G}$, and Planck time $t_p = \sqrt{G\hbar/c^5} = l_p/c$. We can also define a Planck temperature, using Boltzmann's constant k_B, $T_p = m_p c^2/k_B$.

   (i) Calculate these quantities in SI units.

   (ii) Work out the age of the universe and the current temperature of the universe (i.e. of the CMB) in Planck units.

   (iii) In particle physics, masses are often measured in units of eV/c^2. Determine the Planck mass in units of GeV/c^2 and compare to the mass of the Higgs boson.

   (iv) Write out the Schwarzschild metric including factors of the fundamental constants and convert to Planck units.

   (v) In SI units, calculate the Schwarzschild radius, r_{Sch}, for a solar mass black hole, with mass $M = M_\odot$, and for a supermassive black hole in the centre of a galaxy with $M = 10^8 M_\odot$ and compare the latter to the size of the solar system. Define a *heuristic* "mean density" $\bar{\rho}$ of a black hole by viewing the mass as being located in a euclidean ball of volume given by $4/3\pi r_{Sch}^3$ and notice how it scales with mass. Calculate $\bar{\rho}$ for $M = M_\odot$ and $M = 10^8 M_\odot$ and identify other matter with comparable densities.

2. Prove the following simple facts:

   (i) If $S_{\mu\nu} = S_{\nu\mu}$ (symmetric tensor) and $A^{\mu\nu} = -A^{\nu\mu}$ (anti-symmetric tensor), then $S_{\mu\nu}A^{\mu\nu} = 0$.

   (ii) Show that a general tensor $T^{\mu\nu}$ may be split uniquely into $T_{\mu\nu} = T_{(\mu\nu)} + T_{[\mu\nu]}$, with $T_{(\mu\nu)}$ and $T_{[\mu\nu]}$ symmetric and anti-symmetric, respectively, and should be identified.

   (iii) Thus if $S_{\mu\nu}$ is symmetric then $S_{\mu\nu}T^{\mu\nu} = S_{\mu\nu}T^{(\mu\nu)}$

   (iv) Suppose $A_{\mu\rho\sigma} = A_{\mu[\rho\sigma]}$. Show that $A_{[\mu\rho\sigma]} = 1/3(A_{\mu\rho\sigma} + A_{\rho\sigma\mu} + A_{\sigma\mu\rho})$.

3. Let V^μ and W^μ be vector fields, S_μ, T_μ be co-vector fields (also called one-forms) and f a function. The Lie derivative of f and W with respect to V are given

$$(\mathcal{L}_V f) = V^\mu \partial_\mu f$$
$$(\mathcal{L}_V W)^\nu = V^\mu \partial_\mu W^\nu - W^\mu \partial_\mu V^\nu$$

   (i) Using the fact that the Lie derivative commutes with contraction, show that

$$(\mathcal{L}_V T)_\nu = V^\mu \partial_\mu T_\nu + T_\mu \partial_\nu V^\mu$$

   (ii) By consider the tensor with components $A_{\mu\nu} = S_\mu T_\nu$, and the fact that the Lie derivative satisfies the Liebnitz rule, show that

$$(\mathcal{L}_V A)_{\mu\nu} = V^\rho \partial_\rho A_{\mu\nu} + A_{\rho\nu}\partial_\mu V^\rho + A_{\mu\rho}\partial_\nu V^\rho$$

(iii) If we assume that, in addition, we have a metric $g_{\mu\nu}$ and Levi-Civita connection ∇_μ, show that in $(\mathcal{L}_V f)$, $(\mathcal{L}_V W)^\nu$ and $(\mathcal{L}_V T)_\nu$ we can replace the partial derivatives with covariant derivatives.

4. On a manifold with metric and Levi-Civita connection ∇_μ, let V and W both be Killing vectors satisfying $\nabla_{(\mu}V_{\nu)} = \nabla_{(\mu}W_{\nu)} = 0$. Show that $S \equiv \mathcal{L}_V W$ is also a Killing vector. Hint: recall that for an arbitrary vector field we have

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) V^\rho = R_{\mu\nu}{}^\rho{}_\sigma V^\sigma \tag{1}$$

5. On a manifold with metric and Levi-Civita connection ∇_μ, assume that V is a Killing vector, $\nabla_{(\mu}V_{\nu)} = 0$.

(i) By contracting (1) show that

$$\nabla_\mu \nabla^\mu V_\nu = -R_{\nu\sigma} V^\sigma \tag{2}$$

(ii) Prove the "Killing vector Lemma"

$$\nabla_\rho \nabla_\mu V_\nu = -R_{\mu\nu\rho\sigma} V^\sigma \tag{3}$$

Hint: use the Bianchi identity $R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} = 0$. Note that contracting (3) immediately gives (2).

6. Consider the round metric on the two-sphere

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2 \tag{4}$$

(i) Show that the only non-vanishing Christoffel components are $\Gamma^\theta_{\phi\phi}$ and $\Gamma^\phi_{\theta\phi}$ and calculate them.

(ii) Show that the vector field $V^{(1)} = \partial_\phi$ is a Killing vector.

(iii) Show that the vector field $V^{(2)} = \sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi$ is another Killing vector.

(iv) Show that a third Killing vector can be obtained via $V^{(3)} = \mathcal{L}_{V^{(1)}} V^{(2)}$ and determine its components.

(v) Can any other vector fields be obtained in a similar way? Comment on your result.