1. Consider the action
\[ I[x(\lambda), e(\lambda)] = \frac{1}{2} \int d\lambda \left( e^{-1} \dot{x}^{\mu} \dot{x}^{\nu} g_{\mu\nu} - m^2 e \right) \]
where \( \dot{x}^\mu = \frac{dx^\mu}{d\lambda} \). Show that the Euler-Lagrange equations imply
\[ e = m^{-1} \frac{d\tau}{d\lambda}, \quad \frac{D(e^{-1} \dot{x}^\mu)}{d\lambda} \equiv \frac{dx^\nu}{d\lambda} \nabla_\nu (e^{-1} \dot{x}^\mu) = 0 \]
The parameter \( \lambda \) is an affine parameter if \( e = constant \). Show that if \( \rho \) is an arbitrary affine parameter for a geodesic \( x^\mu(\rho) \) satisfying
\[ \frac{D}{d\rho} \frac{dx^\mu}{d\rho} = 0 \]
then for any other parameter \( \rho'(\rho) \), which in general is non-affine, we have
\[ \frac{D}{d\rho'} \frac{dx^\mu}{d\rho'} = \left( \frac{d}{d\rho} \ln \left( \frac{d\rho}{d\rho'} \right) \right) \frac{dx^\mu}{d\rho} \]

2. Consider the action in Q1. Let \( k^\mu \) be a Killing vector, satisfying \( \nabla (k^\mu k^\nu) = 0 \). Show that the shift \( x^\mu \rightarrow x^\mu + \epsilon k^\mu (x) \) induces vanishing variation of the action at leading order in \( \epsilon \).
Show that \( Q = p_\mu k^\mu \), where \( p_\mu \equiv \frac{\delta L}{\delta \dot{x}^\mu} \) is conserved using the equations of motion.

3. Write the Schwarzschild metric in outgoing Eddington-Finklestein coordinates \((u, r, \theta, \phi)\) where \( u = t - r_* \). Show the Killing vector \( k = \partial_t \) in Schwarzschild coordinates is given by \( k = \partial_u \) in outgoing E-F coordinates. Show that a time orientation is given by \( \partial_r \) in outgoing E-F coordinates for \( r > 0 \). Also show that for any future directed causal curve, we must have \( \frac{dr}{d\lambda} > 0 \) if \( r < 2m \), showing we have a white hole. Plot the radial null geodesics in the \((r, u + r)\) plane to further support his conclusion.

4. If we parametrise radial null geodesics of the Schwarzschild metric using the Schwarzschild radial coordinate show that the tangent vector has components \( t^\mu = (\pm (1 - 2m/r)^{-1}, 1, 0, 0) \).
Hence show that \( r \) is an affine parameter for these null geodesics. Also show that if we use \( r_* \) to parametrise the radial null geodesics then \( r_* \) is not an affine parameter.

5. The Weyl tensor in \( n \) spacetime dimensions is defined as
\[ C^\mu_{\nu\rho\sigma} = R^\mu_{\nu\rho\sigma} + \frac{2}{n - 2} \left( \delta^\mu_{[\sigma} R_{\rho\nu]} - g_{\nu[\sigma} R_{\rho\mu]} \right) + \frac{2}{(n - 1)(n - 2)} R \delta^\mu_{\rho\sigma} g_{\nu[\rho} R_{\sigma]\mu]} \]
Show that it has the same symmetries as the Riemann tensor and also that it is trace free:
(i) \( C_{\mu\nu\rho\sigma} = -C_{\mu\nu\sigma\rho} \)
(ii) \( C_{\mu\nu\rho\sigma} = C_{\rho\sigma\mu\nu} \)
6. Consider the Weyl transformed metric $\hat{g}_{\mu\nu} = \Lambda^2 g_{\mu\nu}$.

(i) Show that the Christoffel symbols are related via $\hat{\Gamma}_{\nu\rho}^\mu = \Gamma_{\nu\rho}^\mu + S_{\nu\rho}^\mu$ where

$$S_{\nu\rho}^\mu = \nabla_{\rho} \ln \Lambda \delta_{\nu}^\mu + \nabla_{\nu} \ln \Lambda \delta_{\rho}^\mu - \nabla_{\mu} \ln \Lambda g_{\nu\rho}$$

(ii) Next calculate the Riemann tensor to show

$$\hat{R}^\mu_{\nu\rho\sigma} = R^\mu_{\nu\rho\sigma} + 2 \delta_{[\nu}^\mu \nabla_{\rho]} \ln \Lambda - 2 g_{\nu[\sigma} \nabla_{\rho]} \nabla^{\mu} \ln \Lambda$$

where “$\rho \leftrightarrow \sigma$” means subtract the two terms in the bracket obtained by interchanging the $\rho$ and $\sigma$ indices. By writing out the two terms in the brackets, and identifying terms that are symmetric in the $\rho, \sigma$ indices (which will vanish when one does the subtraction of the terms $\rho \leftrightarrow \sigma$), show that one can eventually write

$$\hat{R}^\mu_{\nu\rho\sigma} = R^\mu_{\nu\rho\sigma} + 2 \delta_{[\nu}^\mu \nabla_{\rho]} \ln \Lambda - 2 g_{\nu[\sigma} \nabla_{\rho]} (\nabla \ln \Lambda)^2 - 2 g_{\nu[\sigma} \nabla_{\rho]} \ln \Lambda \nabla^{\mu} \ln \Lambda$$

(iii) By contracting this expression show that in $n$ spacetime dimensions

$$\hat{R}_{\nu\sigma} = R_{\nu\sigma} - (n - 2) \nabla_{\nu} \nabla_{\sigma} \ln \Lambda - g_{\nu\sigma} \nabla^2 \ln \Lambda + (n - 2) \nabla_{\nu} \ln \Lambda \nabla_{\sigma} \ln \Lambda - (n - 2) g_{\nu\sigma} (\nabla \ln \Lambda)^2$$

$$\hat{R} = \Lambda^{-2} \left[ R - 2(n - 1) \nabla^2 \ln \Lambda - (n - 2)(n - 1)(\nabla \ln \Lambda)^2 \right]$$

(iv) Finally show that the Weyl tensor is invariant under conformal transformations $\hat{C}^\mu_{\nu\rho\sigma} = C^\mu_{\nu\rho\sigma}$. Thus a conformally flat metric of the form $ds^2 = \Lambda^2 ds^2 (\text{flat})$ has vanishing Weyl tensor (the converse is also true for $n \geq 4$).

7. Show that an affinely parametrised null geodesic with respect to the metric $g_{\mu\nu}$ is also a null geodesic with respect to $\hat{g}_{\mu\nu} = \Lambda^2 g_{\mu\nu}$, but it is not affinely parametrised. Hence, using the result of question 1, show that if $\lambda$ is an affine parameter of the null geodesic with respect to $g_{\mu\nu}$ and $\hat{\lambda}$ is an affine parameter of the null geodesic with respect to $\hat{g}_{\mu\nu}$ then we have $d\hat{\lambda}/d\lambda = c\Lambda^2$ for a constant $c$.

8. A conformal Killing vector is one for which

$$(\mathcal{L}_k g)_{\mu\nu} = \alpha g_{\mu\nu}$$

for some non-zero function $\alpha$.

(i) Show that this is equivalent to $\nabla_{(\mu} k_{\nu)} = \frac{1}{n} (\nabla_{\rho} k^\rho) g_{\mu\nu}$, where $n$ is the dimension of spacetime.

(ii) Show that if $k$ is a Killing vector of some metric $g_{\mu\nu}$ then it a conformal Killing vector for the class of metrics $\hat{g}_{\mu\nu} = \Lambda^2 g_{\mu\nu}$.

(iii) Consider an affinely parametrised geodesic with tangent vector $V^\mu = \frac{dx^\mu}{d\lambda}$. Show that if $k^\mu$ is a conformal Killing vector, show that $V^\mu k_\mu$ is conserved along the geodesic provided that it is null.