Black Holes – Problem Sheet 2

1. Consider the action

$$I[x(\lambda), e(\lambda)] = \int d\lambda\, \mathcal{L} = \frac{1}{2}\int d\lambda \left(e^{-1}\dot{x}^\mu \dot{x}^\nu g_{\mu\nu} - m^2 e\right)$$

where $\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$. Show that the Euler-Lagrange equations imply

$$e = m^{-1}\frac{d\tau}{d\lambda}, \qquad \frac{D(e^{-1}\dot{x}^\mu)}{d\lambda} \equiv \frac{dx^\nu}{d\lambda}\nabla_\nu \left(e^{-1}\dot{x}^\mu\right) = 0$$

The parameter λ is an affine parameter if $e = \textit{constant}$. Show that if ρ is an arbitrary affine parameter for a geodesic $x^\mu(\rho)$ satisfying

$$\frac{D}{d\rho}\frac{dx^\mu}{d\rho} = 0$$

then for any other parameter $\rho'(\rho)$, which in general is non-affine, we have

$$\frac{D}{d\rho'}\frac{dx^\mu}{d\rho'} = \left(\frac{d}{d\rho'}[\ln(\frac{d\rho}{d\rho'})]\right)\frac{dx^\mu}{d\rho'}$$

2. Consider the action in Q1. Let k^μ be a Killing vector, satisfying $\nabla_{(\mu}k_{\nu)} = 0$. Show that the shift $x^\mu \to x^\mu + \epsilon k^\mu(x)$ induces vanishing variation of the action at leading order in ϵ. Show that $Q = p_\mu k^\mu$, where $p_\mu \equiv \frac{\delta \mathcal{L}}{\delta \dot{x}^\mu}$, is conserved using the equations of motion.

3. Write the Schwarzschild metric in outgoing Eddington-Finklestein coordinates (u, r, θ, ϕ) where $u = t - r_*$. Show the Killing vector $k = \partial_t$ in Schwarzschild coordinates is given by $k = \partial_u$ in outgoing E-F coordinates. Show that a time orientation is given by ∂_r in outgoing E-F coordinates for $r > 0$. Also show that for any future directed causal curve, we must have $\frac{dr}{d\lambda} > 0$ if $r < 2m$, showing we have a white hole. Plot the radial null geodesics in the $(r, u + r)$ plane to further support his conclusion.

4. If we parametrise radial null geodesics of the Schwarzschild metric using the Schwarzschild radial coordinate show that the tangent vector has components $t^\mu = (\pm(1-2m/r)^{-1}, 1, 0, 0)$. Hence show that r is an affine parameter for these null geodesics. Also show that if we use r_* to parametrise the radial null geodesics then r_* is not an affine parameter.

5. The Weyl tensor in n spacetime dimensions is defined as

$$C^\mu{}_{\nu\rho\sigma} = R^\mu{}_{\nu\rho\sigma} + \frac{2}{n-2}\left(\delta^\mu_{[\sigma}R_{\rho]\nu} - g_{\nu[\sigma}R_{\rho]}{}^\mu\right) + \frac{2}{(n-1)(n-2)}R\delta^\mu_{[\rho}g_{\sigma]\nu}$$

Show that it has the same symmetries as the Riemann tensor and also that it is trace free:

(i) $C_{\mu\nu\rho\sigma} = -C_{\mu\nu\sigma\rho}$

(ii) $C_{\mu\nu\rho\sigma} = C_{\rho\sigma\mu\nu}$

(iii) $C^\mu{}_{[\nu\rho\sigma]} = 0$

(iv) $C^\mu{}_{\nu\mu\sigma} = 0$

6. Consider the Weyl transformed metric $\hat{g}_{\mu\nu} = \Lambda^2 g_{\mu\nu}$.

   (i) Show that the Christoffel symbols are related via $\hat{\Gamma}^{\mu}_{\nu\rho} = \Gamma^{\mu}_{\nu\rho} + S^{\mu}_{\nu\rho}$ where

   $$S^{\mu}_{\nu\rho} = \nabla_\rho \ln\Lambda \delta^{\mu}_{\nu} + \nabla_\nu \ln\Lambda \delta^{\mu}_{\rho} - \nabla^{\mu}\ln\Lambda g_{\nu\rho}$$

   (ii) Next calculate the Riemann tensor to show

   $$\hat{R}^{\mu}{}_{\nu\rho\sigma} = R^{\mu}{}_{\nu\rho\sigma} + \left(\nabla_\rho S^{\mu}_{\sigma\nu} + S^{\mu}_{\delta\rho}S^{\delta}_{\sigma\nu} - \rho \leftrightarrow \sigma\right)$$

   where "$-\rho \leftrightarrow \sigma$" means subtract the two terms in the bracket obtained by interchanging the ρ and σ indices. By writing out the two terms in the brackets, and identifying terms that are symmetric in the ρ, σ indices (which will vanish when one does the subtraction of the terms $\rho \leftrightarrow \sigma$), show that one can eventually write

   $$\begin{aligned}\hat{R}^{\mu}{}_{\nu\rho\sigma} =& R^{\mu}{}_{\nu\rho\sigma} + 2\delta^{\mu}_{[\sigma}\nabla_{\rho]}\nabla_\nu \ln\Lambda - 2g_{\nu[\sigma}\nabla_{\rho]}\nabla^{\mu}\ln\Lambda \\ &+ 2\delta^{\mu}_{[\rho}\nabla_{\sigma]}\ln\Lambda\nabla_\nu\ln\Lambda - 2\delta^{\mu}_{[\rho}g_{\sigma]\nu}(\nabla\ln\Lambda)^2 - 2g_{\nu[\rho}\nabla_{\sigma]}\ln\Lambda\nabla^{\mu}\ln\Lambda\end{aligned}$$

   (iii) By contracting this expression show that in n spacetime dimensions

   $$\hat{R}_{\nu\sigma} = R_{\nu\sigma} - (n-2)\nabla_\nu\nabla_\sigma \ln\Lambda - g_{\nu\sigma}\nabla^2\ln\Lambda + (n-2)\nabla_\nu\ln\Lambda\nabla_\sigma\ln\Lambda - (n-2)g_{\nu\sigma}(\nabla\ln\Lambda)^2$$
   $$\hat{R} = \Lambda^{-2}\left[R - 2(n-1)\nabla^2\ln\Lambda - (n-2)(n-1)(\nabla\ln\Lambda)^2\right]$$

   (iv) Finally show that the Weyl tensor is invariant under conformal transformations $\hat{C}^{\mu}{}_{\nu\rho\sigma} = C^{\mu}{}_{\nu\rho\sigma}$. Thus a conformally flat metric of the form $d\hat{s}^2 = \Lambda^2 ds^2(flat)$ has vanishing Weyl tensor (the converse is also true for $n \geq 4$).

7. Show that an affinely parametrised null geodesic with respect to the metric $g_{\mu\nu}$ is also a null geodesic with respect to $\hat{g}_{\mu\nu} = \Lambda^2 g_{\mu\nu}$, but it is not affinely parametrised. Hence, using the result of question 1, show that if λ is an affine parameter of the null geodesic with respect to $g_{\mu\nu}$ and $\hat{\lambda}$ is an affine parameter of the null geodesic with respect to $\hat{g}_{\mu\nu}$ then we have $d\hat{\lambda}/d\lambda = c\Lambda^2$ for a constant c.

8. A conformal Killing vector is one for which

   $$(\mathcal{L}_k g)_{\mu\nu} = \alpha g_{\mu\nu}$$

   for some non-zero function α.

   (i) Show that this is equivalent to $\nabla_{(\mu}k_{\nu)} = \frac{1}{n}(\nabla_\rho k^\rho)g_{\mu\nu}$, where n is the dimension of spacetime.

   (ii) Show that if k is a Killing vector of some metric $g_{\mu\nu}$ then it a conformal Killing vector for the class of metrics $\hat{g}_{\mu\nu} = \Lambda^2 g_{\mu\nu}$.

   (iii) Consider an affinely parametrised geodesic with tangent vector $V^\mu = \frac{dx^\mu}{d\lambda}$. Show that if k^μ is a conformal Killing vector, show that $V^\mu k_\mu$ is conserved along the geodesic provided that it is null.