Black Holes – Problem Sheet 3

1. Show that changing coordinates from the Schwarzschild coordinates (t, r, θ, ϕ) to the Kruskal coordinates (U, V, θ, ϕ) the Schwarzschild metric in region I (i.e. for $r > 2M$) takes the form
$$ds^2 = -\frac{32M^3}{r}e^{-r/2M}dUdV + r^2 d\Omega$$
with $r = r(U, V)$. Also show that in region I the Killing vector $k = \partial_t$ takes the form in the Kruskal coordinates:
$$k = \frac{1}{4M}(V\partial_V - U\partial_U)$$
Calculate the norm squared of k in the Kruskal coordinates and check that it agrees with the calculation in Schwarzschild coordinates.

2. Consider a radially infalling spaceship, following a geodesic, in the Schwarzschild spacetime. Show that light emitted by the spaceship and seen by a distant observer at $r = \infty$ is redshifted and the redshift increases exponentially fast (in proper time of the observer) so that signals from the particle essentially disappear in time of order $4M$. Hint: Draw the Penrose diagram. Draw the trajectory of the spaceship falling into the black hole and the observer. When the spaceship is at radius R close to $2M$ it sends two photons separated by proper time $\Delta\tau$. Draw the trajectories of the photons. Calculate the separation in EF coordinates Δu of the two pulses. How is this related to the frequency of light seen by the observer?

3. A round, unit radius sphere, S^{N-1}, is parametrised by coordinates μ^I, $I = 1, \ldots, N$ with $\mu^I\mu^I = 1$ and has metric $ds^2 = d\mu^I d\mu^I \equiv d\Omega_{N-1}$. Show that we can parametrise the unit round sphere S^N via $X^0 = \cos\theta$ and $X^I = \sin\theta\mu^I$. What is the range of the coordinate θ? Show that
$$d\Omega_N = d\theta^2 + \sin^2\theta d\Omega_{N-1} \tag{1}$$
(We used this result in discussing the conformal compactification of Minkoswki spacetime.)

4. Consider the derivation of the conformal compactification of Minkowski spacetime in four spacetime dimensions. Show that $i^0, i^\pm$ are all points and that $\mathscr{I}^\pm$ are topologically $\mathbb{R} \times S^2$.

5. Consider the spacetime associated with the spherically symmetric collapse of a star to form a black hole. As an observer crosses the event horizon can they still see the star? Can they still see their feet? If their friend crosses the event horizon later can they meet? Use a Penrose diagram to support your answers.

6. We have $\partial_\mu g = g g^{\rho\sigma}\partial_\mu g_{\rho\sigma}$, true for any invertible matrix (a mnemonic is "det= exp-trace-log"). Show that $(-g)^{-1/2}\partial_\mu(-g)^{1/2} = \Gamma^\rho_{\rho\mu}$. Hence show that for a vector V^μ, a scalar (function) ϕ and a two-form $F_{\mu\nu}$ we have the useful results

(i) $\nabla_\mu V^\mu = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}V^\mu\right)$

(ii) $\nabla^2\phi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right)$

(iii) $\nabla_\mu F^{\mu\nu} = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}F^{\mu\nu}\right)$

(iv) Use the last result to show that Maxwells equations are solved for the Reissner-Nordström black hole solution.

(v) If $S^{\mu\nu} = S^{(\mu\nu)}$ show $\nabla_\mu S^\mu{}_\nu = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}S^\mu{}_\nu\right) - \frac{1}{2}\partial_\nu(g_{\lambda\mu})S^{\lambda\mu}$

7. Consider the Schwarzschild metric in ingoing Eddington-Finklestein coordinates (v, r, θ, ϕ). Introduce new coordinates $(\tilde{t}, r, \theta, \phi)$ with $\tilde{t} = v - r$. Having expressed the metric in these coordinates, make one more change of coordinates from spatial polar coordinates to standard spatial cartesian coordinates: $x + iy = r\sin\theta e^{i\phi}$, $z = r\cos\theta$. Show that the components of the metric are then given in the coordinates $(\tilde{t}, x, y, z)$

$$g_{\mu\nu} = \eta_{\mu\nu} + fk_\mu k_\nu \tag{2}$$

where $\eta_{\mu\nu}$ is the Minkowski metric, $f = 2M/r$ and $k_\mu = (1, x/r, y/r, z/r)$. Show that with respect to either g or η that k^μ is a null vector and is tangent to affinely parametrised geodesics: $k^\rho\nabla_\rho k_\mu = 0$. With a bit more effort one can show that the Kerr metric can also be put into this so-called "Kerr-Schild" form.

8. Consider a p-form, A, and a q-form, B. Let d be the exterior derivative. Show

(i) $A \wedge B = (-1)^{pq} B \wedge A$

(ii) $ddA = 0$

(iii) $d(A \wedge B) = (dA) \wedge B + (-1)^p A \wedge dB$

If we also have a metric we have the Hodge star operator $*$. Show that

(iv) $*(*A) = \pm(-1)^{p(n-p)}A$

(v) $(*d*A)_{\mu_1\ldots\mu_{p-1}} = \pm(-1)^{p(n-p)}\nabla^\nu A_{\mu_1\ldots\mu_{p-1}nu}$

9. Consider Einstein Maxwell theory with Lagrangian $\mathcal{L} = \sqrt{-g}(R - F_{\mu\nu}F^{\mu\nu})$. Derive the Einstein equations $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - 2(F_{\mu\sigma}F_\nu{}^\sigma - \frac{1}{4}g_{\mu\nu}F^2) = 0$.

(i) Show $\delta g_{\mu\rho} = -g_{\mu\nu}g_{\rho\sigma}\delta g^{\nu\sigma}$

(ii) Show $\delta\Gamma^\mu_{\nu\rho} = \frac{1}{2}g^{\mu\sigma}(\delta g_{\sigma\nu;\rho} + \delta g_{\sigma\rho;\nu} - \delta g_{\nu\rho;\sigma})$ (which is a tensor).

(iii) Show $\delta R = R_{\mu\nu}\delta g^{\mu\nu} + \nabla^\mu V_\mu$, where $V_\mu = \nabla^\nu\delta g_{\nu\mu} - g^{\nu\sigma}\nabla_\mu\delta g_{\nu\sigma}$.

(iv) Show $\delta\sqrt{-g} = -\frac{1}{2}\delta\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$.

(v) Hence show $\delta\mathcal{L} = \sqrt{-g}[R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - 2(F_{\mu\sigma}F_\nu{}^\sigma - \frac{1}{4}g_{\mu\nu}F^2)]\delta g^{\mu\nu} + \sqrt{-g}\nabla^\mu V_\mu$. The last term does not contribute after integrating, and hence varying the action with respect to $g_{\mu\nu}$ gives the Einstein equations above.