Black Holes – Problem Sheet 4

1. Varying the action
$$S = \int d\tau g_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$$
with the normalisation condition $g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -1$ gives affinely parametrised timelike geodesics with affine parameter the proper time τ. In Boyer-Lindquist coordinates the Kerr metric has non-zero metric components g_{tt}, $g_{t\phi}$, $g_{\phi\phi}$, $g_{\theta\theta}$ and g_{rr} all of which are functions of r, θ. Write down the form of the geodesic equation, identifying two conserved quantities. Consider a particle falling from rest at some $r = R > r_+$ and $\theta = \pi/2$ (i.e. in the equatorial plane). Show, by considering the explicit form of $g_{\mu\nu}$, that θ remains equal to $\pi/2$ for all τ. Also show ϕ cannot remain constant.

2. A rank m Killing tensor is a totally symmetric tensor $K_{\nu_1\ldots\nu_m} = K_{(\nu_1\ldots\nu_m)}$ that satisfies $\nabla_{(\mu}K_{\nu_1\ldots\nu_m)} = 0$. If $V^\mu \equiv \dot{x}^\mu$ is tangent to an affinely parametrised geodesic, show that $Q = V^{\nu_1}\ldots V^{\nu_m}K_{\nu_1\ldots\nu_m}$ is constant along the geodesic. Remarkably, in addition to the Killing vectors ∂_t and ∂_ϕ in Boyer-Lindquist coordinates, the Kerr metric also has an irreducible rank two Killing tensor (ie one that can't be expressed as $K_{\mu\nu} = K^1_{(\mu}K^2_{\nu)}$), which allows one to obtain the geodesics explicitly.

3. For the Reissner-Nordstrom black hole in the coordinates given in the lectures with $A_t = -Q/r$ and $A_\phi = -P\cos\theta$, show that
$$P = \frac{1}{4\pi}\int_{S^2_\infty} F, \qquad Q = \frac{1}{4\pi}\int_{S^2_\infty} *F,$$
and hence P, Q are the electric and magnetic charge of the black hole, respectively.

4. For the Kerr black hole in Boyer-Lindquist coordinates, show that the Komar integrals give
$$M = -\frac{1}{8\pi}\int_{S^2_\infty} *dk, \qquad J = \frac{1}{16\pi}\int_{S^2_\infty} *dm,$$
where k, m are the one forms associated with the Killing vectors $k = \partial_t$ and $m = \partial_\phi$. Hint: you only have to calculate the components and quantities that contribute to the integral on the two-sphere at infinity.

5. Consider the Reissner Nordström metric in ingoing Eddington-Finklestein coordinates
$$ds^2 = -\frac{\Delta}{r^2}dv^2 + 2dvdr + r^2 d\Omega$$
where $\Delta = (r - r_+)(r - r_-)$ and $r_\pm = M \pm \sqrt{M^2 - e^2}$. Show that the outer and inner horizons located at $r = r_\pm$ are null hypersurfaces. Show that they are Killing horizons with respect to the stationary Killing vector $\xi = \partial_v$. By calculating $\nabla_\mu(\xi^2)|_{r=r_\pm}$ show that the surface gravity on the two Killing horizons is given by
$$\kappa = \frac{r_\pm - r_\mp}{2r_\pm^2}$$

6. Consider the Kerr metric in Kerr coordinates (v, r, θ, χ). Show that the event horizon at $r = r_+$ is a Killing horizon for the Killing vector $\xi = \partial_v + \Omega_H \partial_\chi$, where $\Omega_H = a/(r_+^2 + a^2)$, and that the surface gravity is

$$\kappa = \frac{\sqrt{M^4 - J^2}}{2M(M^2 + \sqrt{M^4 - J^2})}$$

(Note: one can avoid working out the inverse metric by showing $\xi^2 = 0$ at $r = r_+$ and that $l_\mu \propto \xi_\mu$ at $r = r_+$ where l^μ is the normal vector to the hypersurfaces of constant r.)

7. This questions illuminates the physical interpretation of the surface gravity κ of a black hole.

   Consider a stationary, asymptotically flat spacetime with Killing vector k^μ such that $k^2 \to -1$ at infinity. Let $V^2 = -k^2$ where V is the gravitational redshift factor. Consider a stationary particle of mass m. It moves on an orbit of k and its proper acceleration is $a^\mu = \frac{D}{d\tau} v^\mu$ where $v^\mu = V^{-1} k^\mu$ is the 4-velocity of the particle and $\frac{D}{d\tau} = v^\nu \nabla_\nu$ where τ is proper time. Let $a \equiv (a^\mu a_\mu)^{1/2}$ be the magnitude of the acceleration.

   (i) Show that $a_\mu = \nabla_\mu \ln V$.

   (ii) Suppose the particle is kept stationary by an idealised string held by a stationary observer at infinity. Let $F_\mu = m a_\mu$ be the magnitude of the *local* force exerted on the particle. Use conservation of energy arguments to show that the magnitude of the force exerted at infinity is $F_\infty = VF$.

   (iii) Show that for Schwarzschild we have that a and hence F diverges as $r \to r_+$ but that Va (i.e. F_∞ per unit mass) equals the surface gravity κ of the event horizon.