Black Holes – Problem Sheet 4

Solutions to be deposited into a marked box in Huxley 512, by Mar 5, 4PM, for marking by Matthew Cheung. The Rapid Feedback session will be on Mar 7, 4 pm

1. Varying the action

\[ S = \int d\tau g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \]

with the normalisation condition \( g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -1 \) gives affinely parametrised timelike geodesics with affine parameter the proper time \( \tau \). In Boyer-Lindquist coordinates the Kerr metric has non-zero metric components \( g_{tt}, g_{t\phi}, g_{\phi\phi}, g_{\theta\theta} \) and \( g_{rr} \) all of which are functions of \( r, \theta \). Write down the form of the geodesic equation, identifying two conserved quantities. Consider a particle falling from rest at some \( r = R > r_+ \) and \( \theta = \pi/2 \) (i.e. in the equatorial plane). Show, by considering the explicit form of \( g_{\mu\nu} \), that \( \theta \) remains equal to \( \pi/2 \) for all \( \tau \). Also show \( \phi \) cannot remain constant.

2. A rank \( m \) Killing tensor is a totally symmetric tensor \( K_{\nu_1...\nu_m} = K(\nu_1...\nu_m) \) that satisfies \( \nabla_{(\mu} K_{\nu_1...\nu_m)} = 0 \). If \( V^\mu \equiv \dot{x}^\mu \) is tangent to an affinely parametrised geodesic, show that \( Q = V^{\nu_1} ... V^{\nu_m} K_{\nu_1...\nu_m} \) is constant along the geodesic. Remarkably, in addition to the Killing vectors \( \partial_t \) and \( \partial_\phi \) in Boyer-Lindquist coordinates, the Kerr metric also has an irreducible rank two Killing tensor (ie one that can’t be expressed as \( K_{\mu\nu} = K^1_{(\mu} R^2_{\nu)} \)), which allows one to obtain the geodesics explicitly.

3. For the Reissner-Nordstrom black hole in the coordinates given in the lectures with \( A_t = -Q/r \) and \( A_\phi = -P \cos \theta \), show that

\[ P = \frac{1}{4\pi} \int_{S^2_\infty} *F, \quad \quad Q = \frac{1}{4\pi} \int_{S^2_\infty} *F, \]

and hence \( P, Q \) are the electric and magnetic charge of the black hole, respectively.

4. For the Kerr black hole in Boyer-Lindquist coordinates, show that the Komar integrals give

\[ M = -\frac{1}{8\pi} \int_{S^2_\infty} *dk, \quad \quad J = \frac{1}{16\pi} \int_{S^2_\infty} *dm, \]

where \( k, m \) are the one forms associated with the Killing vectors \( k = \partial_t \) and \( m = \partial_\phi \). Hint: you only have to calculate the components and quantities that contribute to the integral on the two-sphere at infinity.

5. Consider the Reissner Nordström metric in ingoing Eddington-Finklestein coordinates

\[ ds^2 = -\frac{\Delta}{r^2} dv^2 + 2dvdr + r^2 d\Omega \]

where \( \Delta = (r - r_+)(r - r_-) \) and \( r_{\pm} = M \pm \sqrt{M^2 - e^2} \). Show that the outer and inner horizons located at \( r = r_{\pm} \) are null hypersurfaces. Show that they are Killing horizons.
with respect to the stationary Killing vector \( \xi = \partial_v \). By calculating \( \nabla_\mu (\xi^2)|_{r=\pm} \) show that the surface gravity on the two Killing horizons is given by

\[
\kappa = \frac{r_+ - r_-}{2r_+^2}
\]

6. Consider the Kerr metric in Kerr coordinates \((v, r, \theta, \chi)\). Show that the event horizon at \( r = r_+ \) is a Killing horizon for the Killing vector \( \xi = \partial_v + \Omega_H \partial_\chi \), where \( \Omega_H = a/(r_+^2 + a^2) \), and that the surface gravity is

\[
\kappa = \frac{\sqrt{M^4 - J^2}}{2M(M^2 + \sqrt{M^4 - J^2})}
\]

(Note: one can avoid working out the inverse metric by showing \( \xi^2 = 0 \) at \( r = r_+ \) and that \( l_\mu \propto \xi_\mu \) at \( r = r_+ \) where \( l^\mu \) is the normal vector to the hypersurfaces of constant \( r \).)

7. This question illuminates the physical interpretation of the surface gravity \( \kappa \) of a black hole.

Consider a stationary, asymptotically flat spacetime with Killing vector \( k^\mu \) such that \( k^2 \to -1 \) at infinity. Let \( V^2 = -k^2 \) where \( V \) is the gravitational redshift factor. Consider a stationary particle of mass \( m \). It moves on an orbit of \( k^\mu \) and its proper acceleration is \( a^\mu = \frac{D\nu^\mu}{D\tau} \) where \( \nu^\mu = V^{-1}k^\mu \) is the 4-velocity of the particle and \( \frac{D\nu}{D\tau} = \nu^\nu \nabla_\nu \) where \( \tau \) is proper time. Let \( a \equiv (a^\mu a_\mu)^{1/2} \) be the magnitude of the acceleration.

(i) Show that \( a_\mu = \nabla_\mu \ln V \).

(ii) Suppose the particle is kept stationary by an idealised string held by a stationary observer at infinity. Let \( F_\mu = ma_\mu \) be the magnitude of the local force exerted on the particle. Use conservation of energy arguments to show that the magnitude of the force exerted at infinity is \( F_\infty = VF \).

(iii) Show that for Schwarzschild we have that \( a \) and hence \( F \) diverges as \( r \to r_+ \) but that \( V a \) (i.e. \( F_\infty \) per unit mass) equals the surface gravity \( \kappa \) of the event horizon.