Black Holes – Problem Sheet 5

1. For the electrically charged Kerr-Newman black hole directly show that

$$M = \frac{\kappa}{4\pi} A + 2\Omega_H J + \Phi_H Q$$

This is called the Smarr formula (it can also be derived by directly manipulating the Smarr formulae for the conserved charges).

2. The first law of black hole mechanics for uncharged black holes is

$$dM = \frac{\kappa}{8\pi} dA + \Omega_H dJ$$

Use dimensional analysis to put in factors of G and c in order to make this dimensionally correct as a change in energy.

3. Use the second law of black hole mechanics (the Area Theorem) to prove that an uncharged, non-rotating black hole cannot split into two uncharged, non-rotating black holes. Assume that the initial state is stationary and the final state is well approximated by two well separated black holes each approximated by the Schwarzschild metric.

4. Consider the reverse process where two uncharged, non-rotating black holes of masses M_1 and M_2, initially far apart, coalesce into a single non-rotating black hole. This process will give off gravitational radiation. Use energy conservation and the second law to find the maximum amount of work that can be extracted from the system during the process. Show that the maximum efficiency of the process satisfies $\eta \leq 1 - \frac{1}{\sqrt{2}}$ and determine for what initial masses this is achieved.

5. In the LIGO event measure in 2015 it was deduced that two black holes of mass $36M_\odot$ and $29M_\odot$ (and unknown rotation) coalesced to form a rotating black hole of mass $62M_\odot$ and rotation parameter per unit mass of $a/M = 0.68$. Show that this is consistent with the Area law. The merger took place in about 15ms. Work out the power produced during the merger (for comparison the power emitted by the sun is about $3 \times 10^{26} W$).

6. The Penrose process for the Kerr black hole can decrease the mass of the black hole. Use the second law to show that when the mass decreases, the angular momentum must decrease also. Find the maximum amount of energy that can be extracted from a Kerr black hole with mass M and angular momentum J.

7. This question methodically works through the proof of an identity that was used in the proof of the zeroth law. Let $\mathcal{N}$ be a Killing horizon of a Killing vector field ξ with surface gravity κ.

(a) If we know $A = 0$ on $\mathcal{N}$ for some tensor $A_{\mu_1 \dots \mu_p}$ then for an *arbitrary* tensor $B^{\mu_1 \cdots \mu_p}$ we have $A \cdot B \equiv A_{\mu_1 \dots \mu_p} B^{\mu_1 \cdots \mu_p} = 0$ on $\mathcal{N}$. Thus $\mathcal{N}$ is a surface of constant $A \cdot B$ and so $d(A \cdot B)$ (with components $\nabla_\mu (A \cdot B)$) is normal to $\mathcal{N}$ and hence $\xi \wedge d(A \cdot B) = 0$ on $\mathcal{N}$.

(i) Show that $\xi_{[\mu}\nabla_{\nu]}A_{\rho_1\ldots\rho_p} = 0$ on $\mathcal{N}$.

(ii) Taking $A_\mu = \xi^\nu\nabla_\nu\xi_\mu - \kappa\xi_\mu$ use this and the Killing vector Lemma ($\nabla_\mu\nabla_\nu\xi_\rho = R^\sigma{}_{\mu\nu\rho}\xi_\sigma$) to show that

$$\xi_\mu\xi_{[\sigma}\nabla_{\rho]}\kappa + \kappa\xi_{[\sigma}\nabla_{\rho]}\xi_\mu = \left(\xi_{[\sigma}\nabla_{\rho]}\xi^\nu\right)\nabla_\nu\xi_\mu + \xi^\nu\xi_{[\sigma}R^\delta{}_{\rho]\nu\mu}\xi_\delta \qquad \text{on } \mathcal{N} \tag{1}$$

(b) Use Frobenius theorem to show that

$$\xi_\rho\nabla_\mu\xi_\nu = -2\xi_{[\mu}\nabla_{\nu]}\xi_\rho \qquad \text{on } \mathcal{N} \tag{2}$$

Hence show that $\left(\xi_{[\sigma}\nabla_{\rho]}\xi^\nu\right)\nabla_\nu\xi_\mu = \kappa\xi_{[\sigma}\nabla_{\rho]}\xi_\mu$ on $\mathcal{N}$, so equation (1) reduces to

$$\xi_\mu\xi_{[\sigma}\nabla_{\rho]}\kappa = \xi^\nu\xi_{[\sigma}R^\delta{}_{\rho]\nu\mu}\xi_\delta \qquad \text{on } \mathcal{N} \tag{3}$$

(c) Set $A_{\mu\nu\rho} = \xi_\rho\nabla_\mu\xi_\nu + 2\xi_{[\mu}\nabla_{\nu]}\xi_\rho$ and use the result of (a)(i) and equation (2) (repeatedly) to show that

$$\xi_\rho\xi_{[\sigma}\nabla_{\delta]}\nabla_\mu\xi_\nu = -2\left(\xi_{[\sigma}\nabla_{\delta]}\nabla_{[\nu}\xi_{|\rho|}\right)\xi_{\mu]} \qquad \text{on } \mathcal{N}$$

(where the bars on the index $|\rho|$ imply exclude that index from the antisymmetrisation inside the brackets) and hence using the Killing vector identity that

$$\xi_\rho\xi_{[\sigma}R^\gamma{}_{\delta]\mu\nu}\xi_\gamma = 2\xi_{[\sigma}R^\gamma{}_{\delta]\rho[\nu}\xi_{\mu]}\xi_\gamma \qquad \text{on } \mathcal{N}$$

(d) Contract this equation on the indices ρ and δ, show that the LHS vanishes and thus

$$-\xi_{[\mu}R_{\nu]}{}^\gamma\xi_\gamma\xi_\sigma = \xi_{[\mu}R^\gamma{}_{\nu]\rho\sigma}\xi^\rho\xi_\gamma \qquad \text{on } \mathcal{N}$$

By relabelling indices hence show that

$$\xi_{[\sigma}\nabla_{\rho]}\kappa = -\xi_{[\sigma}R_{\rho]}{}^\delta\xi_\delta \qquad \text{on } \mathcal{N}$$

This was the key result used in the lectures to show the zeroth law.