Black Holes - Problems 1

(1)

i) $l_P = \sqrt{\frac{G\hbar}{c^3}} \sim 1.6 \times 10^{-35} m$

$M_P = \sqrt{\frac{c\hbar}{G}} \sim 2.2 \times 10^{-8} kg \sim$ rest mass of a flea egg!

$t_P = \sqrt{\frac{G\hbar}{c^5}} = \frac{l_P}{c} \simeq 2.2 \times 10^{-44} s$

$T_P \simeq \frac{M_P c^2}{k_B} \simeq 1.4 \times 10^{32} K$

ii) Age of universe $\sim 14 \times 10^9$ years $\sim 4.4 \times 10^{17} s$
$\sim 8 \times 10^{60} t_P$

Temperature of universe $\sim 2.7K \sim 1.9 \times 10^{-32} T_P$

iii) $\frac{1eV}{c^2} = \frac{1.6 \times 10^{-19} C}{(3 \times 10^{-5} ms^{-1})^2} \sim 1.8 \times 10^{-36} kg$

$\Rightarrow M_{Pl} = 2.2 \times 10^{-8} kg \simeq 1.2 \times 10^{28} \frac{eV}{c^2} \simeq 1.2 \times 10^{19} \frac{GeV}{c^2}$

Mass of Higgs $\sim 125 \frac{GeV}{c^2}$

iv) $ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 + \frac{dr^2}{\left(1 - \frac{2GM}{c^2 r}\right)} + r^2 d\Omega$

Note

$$\frac{GM}{c^2 r} = \left(\frac{M}{M_P}\right)\left(\frac{l_{Pl}}{r}\right)$$

So, Define $\begin{cases} \bar{M} = M/m_p \\ \bar{r} = r/\ell_p \\ \bar{t} = t/t_p \end{cases}$

then

$$ds^2 = \ell_{pe}^2 \left[-\left(1 - \frac{2\bar{M}}{\bar{r}}\right) d\bar{t}^2 + \frac{d\bar{r}^2}{(1 - 2M/\bar{r})} + \bar{r}^2 d\Omega_2 \right]$$

v) $r_{sch} = \frac{2GM}{c^2}$, $M_\odot \sim 2 \times 10^{30}$ kg

For $1 M_\odot$ $r_{sch} \sim 3 \times 10^3$ m $= 3$ km

$10^8 M_\odot$ $r_{sch} \sim 3 \times 10^{11}$ m

Semi major axis of Neptune $\sim 4 \times 10^{12}$ m

$$\bar{\rho} \equiv \frac{M}{\frac{4}{3}\pi r_{sch}^3} = \frac{3}{4\pi \left(\frac{2G}{c^2}\right)^3} \frac{M}{M^3} = \frac{3c^6}{32\pi G^3} \frac{1}{M^2}$$

for $1 M_\odot$ $\bar{\rho} \sim 10^{19}$ kg/m³

this is nuclear density

eg $\frac{\text{Mass of Proton}}{\text{Compton wavelength of proton}} \sim 10^{18} \frac{\text{kg}}{\text{m}^3}$

for $10^8 M_\odot$ $\bar{\rho} \sim$ ~~1000~~ 10^3 kg/m³

~ 1 g/cm³

This is the density of water (!)

(2) i)

$$S_{\mu\nu} A^{\mu\nu} = - S_{\mu\nu} A^{\nu\mu} \qquad (A \text{ is antisymmetric})$$

$$= - S_{\nu\mu} A^{\mu\nu} \qquad (\text{relabel indices})$$

$$= - S_{\mu\nu} A^{\mu\nu} \qquad (S \text{ is symmetric})$$

$$= 0$$

(ii)

$$T_{\mu\nu} = \underbrace{\tfrac{1}{2}\left(T_{\mu\nu} + T_{\nu\mu}\right)}_{T_{(\mu\nu)}} + \underbrace{\tfrac{1}{2}\left(T_{\mu\nu} - T_{\nu\mu}\right)}_{T_{[\mu\nu]}}$$

Clearly $T_{(\mu\nu)}$ is symmetric + $T_{[\mu\nu]}$ is antisymmetric

iii)

$$S_{\mu\nu} T^{\mu\nu} = S_{\mu\nu}\left(T^{(\mu\nu)} + T^{[\mu\nu]}\right)$$

$$= S_{\mu\nu} T^{(\mu\nu)}$$

iv)

$$T_{[\mu\nu\rho]} = \frac{1}{3!}\left(T_{\mu\nu\rho} + T_{\nu\rho\mu} + T_{\rho\mu\nu} - T_{\mu\rho\nu} - T_{\nu\mu\rho} - T_{\rho\nu\mu}\right)$$

$$= \frac{1}{3!}\left(T_{\mu\nu\rho} + T_{\nu\rho\mu} + T_{\rho\mu\nu} + T_{\mu\nu\rho} + T_{\nu\rho\mu} + T_{\rho\nu\mu}\right)$$

$$= \frac{1}{3}\left(T_{\mu\nu\rho} + T_{\nu\rho\mu} + T_{\rho\mu\nu}\right)$$

3

i) Lie derivative commutes with contraction:

$$\mathcal{L}_V (W \cdot T) = (\mathcal{L}_V W)^\mu T_\mu + W^\mu (\mathcal{L}_V T)_\mu$$

$$= (V^\rho \partial_\rho W^\mu - W^\rho \partial_\rho V^\mu) T_\mu + W^\mu (\mathcal{L}_V T)_\mu$$

We also know how Lie derivative acts on functions

$$\mathcal{L}_V (W \cdot T) = V^\rho \partial_\rho (W^\mu T_\mu) = V^\rho (\partial_\rho W^\mu T_\mu + W^\mu \partial_\rho T_\mu)$$

~~E~~quating

$$\cancel{(V^\rho \partial_\rho W^\mu)} T_\mu + V^\rho W^\mu \partial_\rho T_\mu = V^\rho \partial_\rho \cancel{W^\mu} T_\mu - (W^\rho \partial_\rho V^\mu) T_\mu$$
$$+ W^\mu (\mathcal{L}_V T)_\mu$$

$$\Rightarrow W^\mu (\mathcal{L}_V T)_\mu = W^\mu V^\rho \partial_\rho T_\mu + W^\mu \partial_\mu V^\rho T_\rho$$

where we relabelled indices in

$$\Rightarrow \boxed{(\mathcal{L}_V T_\mu) = V^\rho \partial_\rho T_\mu + T_\rho \partial_\mu V^\rho}$$

(ii) $A_{\mu\nu} = S_\mu T_\nu$

$$
\begin{aligned}
(\mathcal{L}_V A)_{\mu\nu} &= (\mathcal{L}_V S)_\mu T_\nu + S_\mu (\mathcal{L}_V T)_\nu \\
&= \left(V^\rho \partial_\rho S_\mu + S_\rho \partial_\mu V^\rho\right) T_\nu \\
&\quad + S_\mu \left(V^\rho \partial_\rho T_\nu + T_\rho \partial_\nu V^\rho \right) \\
&= V^\rho \left(\partial_\rho S_\mu T_\nu + S_\mu \partial_\rho T_\nu \right) \\
&\quad + S_\rho T_\nu \partial_\mu V^\rho + S_\mu T_\rho \partial_\nu V^\rho \\
&= V^\rho \partial_\rho A_{\mu\nu} + A_{\rho\nu} \partial_\mu V^\rho + A_{\mu\rho} \partial_\nu V^\rho
\end{aligned}
$$

(iii) $\mathcal{L}_V f = V^\rho \partial_\rho f$
$= V^\rho \nabla_\rho f$

$$(\mathcal{L}_V W)^\nu = V^\rho \partial_\rho W^\nu - W^\rho \partial_\rho V^\nu$$
$$= V^\rho \left(\nabla_\rho W^\nu - \Gamma^\nu_{\rho\sigma} W^\sigma \right) - W^\rho \left(\nabla_\rho V^\nu - \Gamma^\nu_{\rho\sigma} V^\sigma \right)$$
$$= V^\rho \nabla_\rho W^\nu - W^\rho \nabla_\rho V^\nu$$
$$- V^\rho W^\sigma \Gamma^\nu_{\rho\sigma} + \underbrace{\underbrace{W^\rho V^\sigma \Gamma^\nu_{\rho\sigma}}_{W^\sigma V^\rho \Gamma^\nu_{\sigma\rho}}}_{W^\sigma V^\rho \Gamma^\nu_{\rho\sigma}}$$

$\downarrow$ relabel indices

$\downarrow$ $\Gamma^\mu_{\nu\rho} = \Gamma^\mu_{(\nu\rho)}$

$$= V^\rho \nabla_\rho W^\nu - W^\rho \nabla_\rho V^\nu$$

$$(\mathcal{L}_V T)_\nu = V^\rho \partial_\rho T_\nu + T_\rho \partial_\nu V^\rho$$
$$= V^\rho \left(\nabla_\rho T_\nu + \Gamma^\sigma_{\rho\nu} T_\sigma \right) + T_\rho \left(\nabla_\nu V^\rho - \Gamma^\rho_{\nu\sigma} V^\sigma \right)$$
$$= V^\rho \nabla_\rho T_\nu + T_\rho \nabla_\nu V^\rho$$
$$+ V^\rho T_\sigma \Gamma^\sigma_{\rho\nu} \underbrace{\underbrace{- V^\sigma T_\rho \Gamma^\rho_{\nu\sigma}}_{- V^\rho T_\sigma \Gamma^\sigma_{\nu\rho}}}_{- V^\rho T_\sigma \Gamma^\sigma_{\rho\nu}}$$
$$= V^\rho \nabla_\rho T_\nu + T_\rho \nabla_\nu V^\rho$$

//

4

$$S = \mathcal{L}_V W$$

$$S^\mu = V^\rho \nabla_\rho W^\mu - W^\rho \nabla_\rho V^\mu$$

$$S_\mu = V^\rho \nabla_\rho W_\mu - W^\rho \nabla_\rho V_\mu$$

$$\begin{aligned}\nabla_\mu S_\nu = {} & (\nabla_\mu V^\rho)(\nabla_\rho W_\nu) + V^\rho (\nabla_\mu \nabla_\rho W_\nu) \\ & - (\nabla_\mu W^\rho)(\nabla_\rho V_\nu) - W^\rho \nabla_\mu \nabla_\rho V_\nu\end{aligned}$$

$\Rightarrow \nabla_\mu S_\nu + \nabla_\nu S_\mu$

$$\left.\begin{aligned} = {} & (\nabla_\mu V^\rho)(\nabla_\rho W_\nu) && + V^\rho \nabla_\mu \nabla_\rho W_\nu \\ & - (\nabla_\mu W^\rho)(\nabla_\rho V_\nu) && - W^\rho \nabla_\mu \nabla_\rho V_\nu \\ & + \underline{\nabla_\nu V^\rho \nabla_\rho W_\mu} && + V^\rho \nabla_\nu \nabla_\rho W_\mu \\ & - (\nabla_\nu W^\rho)(\nabla_\rho V_\mu) && - W^\rho \nabla_\nu \nabla_\rho V_\mu \end{aligned}\right\} \circledast$$

Consider ~~...~~ terms using killing's equation

$$\nabla_\nu V^\rho \nabla_\rho W_\mu = - \nabla_\nu V^\rho \nabla_\mu W_\rho = - \nabla_\nu V_\rho \nabla_\mu W^\rho$$

$$= + \nabla_\rho V_\nu \nabla_\mu W^\rho$$

Similarly

$$- (\nabla_\nu W^\rho)(\nabla_\rho V_\mu) = + \nabla_\nu W^\rho \nabla_\mu V_\rho = - \nabla_\rho W_\nu \nabla_\mu V^\rho$$

Hence 4 terms in first column in $\circledast$ vanish.

$$\Rightarrow D_\mu S_\nu + D_\nu S_\mu$$

$$= V^\rho D_\mu D_\rho W_\nu - W^\rho D_\mu D_\rho V_\nu + V^\rho D_\nu D_\rho W_\mu - W^\rho D_\nu D_\rho V_\mu$$

$$= V^\rho D_\rho D_\mu W_\nu - W^\rho D_\rho D_\mu V_\nu + V^\rho D_\rho D_\nu W_\mu - W^\rho D_\rho D_\nu V_\mu$$

$$+ V^\rho R_{\mu\rho\nu}{}^\sigma W_\sigma - W^\rho R_{\mu\rho\nu}{}^\sigma V_\sigma + V^\rho R_{\nu\rho\mu}{}^\sigma W_\sigma - W^\rho R_{\nu\rho\mu}{}^\sigma V_\sigma$$

First line vanishes via Killing's equations

$$\Rightarrow D_\mu S_\nu + D_\nu S_\mu$$

$$= V^\rho W^\sigma (R_{\mu\rho\nu\sigma} + R_{\nu\rho\mu\sigma}) \underbrace{- W^\rho V^\sigma (R_{\mu\rho\nu\sigma} + R_{\nu\rho\mu\sigma})}_{- W^\sigma V^\rho (R_{\mu\sigma\nu\rho} + R_{\nu\sigma\mu\rho})}$$

$$= 0 \quad //$$

5

i) $\nabla_{(\mu} V_{\nu)} = 0 \Rightarrow g^{\mu\nu} \nabla_{(\mu} V_{\nu)} = 0$

$\Rightarrow g^{\mu\nu} \nabla_\mu V_\nu = 0$

$\Rightarrow \nabla_\mu V^\mu = 0$

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) V^\rho = R_{\mu\nu}{}^\rho{}_\sigma V^\sigma \qquad (*)$$

Contract on μ & ρ

$\Rightarrow \nabla_\mu \nabla_\nu V^\mu = R_{\nu\sigma} V^\sigma$

$\Rightarrow -\nabla_\mu \nabla^\mu V_\nu = R_{\nu\sigma} V^\sigma$

ii) Take $(*)$ (lower index on ρ) & add $(\mu\nu\rho) \to (\nu\rho\mu)$

& subtract $(\mu\nu\rho) \to (\rho\mu\nu)$.

$$\left\{ \begin{aligned} &(\cancel{\nabla_\mu} \nabla_\nu - \nabla_\nu \nabla_\mu) V_\rho \\ &+ (\nabla_\nu \nabla_\rho - \cancel{\nabla_\rho \nabla_\nu}) V_\mu \\ &- (\cancel{\nabla_\rho \nabla_\mu} - \cancel{\nabla_\mu \nabla_\rho}) V_\nu \end{aligned} \right\} = \left\{ \begin{aligned} &R_{\mu\nu\rho\sigma} V^\sigma \\ &+ R_{\nu\rho\mu\sigma} V^\sigma \\ &- R_{\rho\mu\nu\sigma} V^\sigma \end{aligned} \right\}$$

$\Rightarrow$ terms cancel using

e.g. $\nabla_\mu \nabla_\rho V_\nu = \nabla_\mu \nabla_\nu V_\rho$

$$\Rightarrow \quad -D_\nu D_\mu V_\rho + D_\nu D_\rho V_\mu = \left(R_{\mu\nu\rho\sigma} + R_{\nu\rho\mu\sigma} - R_{\rho\mu\nu\sigma}\right) V^\sigma$$

$$\Rightarrow \quad 2 D_\nu D_\rho V_\mu = \left(\underbrace{R_{\mu\nu\rho\sigma} + R_{\mu\sigma\nu\rho}}_{= -R_{\mu\rho\sigma\nu}} - R_{\rho\mu\nu\sigma}\right) V^\sigma$$

$$\Rightarrow \quad 2 D_\nu D_\rho V_\mu = -2 R_{\rho\mu\nu\sigma} V^\sigma$$

$$\Rightarrow \quad \boxed{D_\nu D_\rho V_\mu = - R_{\rho\mu\nu\sigma} V^\sigma}$$

check: contract on ρ & ν $\quad D_\nu D^\nu V_\mu = - R_{\mu\sigma} V^\sigma$ ✓

6

$$ds^2 = d\theta^2 + \sin^2\theta \, d\phi^2 \qquad g_{\theta\theta} = 1 \qquad g^{\theta\theta} = 1$$

$$g_{\phi\phi} = \sin^2\theta \qquad g^{\phi\phi} = 1/\sin^2\theta$$

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} \left(g_{\sigma\nu,\rho} + g_{\sigma\rho,\nu} - g_{\nu\rho,\sigma} \right)$$

$$\cdot \ \Gamma^{\theta}_{\nu\rho} = \frac{1}{2} \left(g_{\theta\nu,\rho} + g_{\theta\rho,\nu} - g_{\nu\rho,\theta} \right)$$

$$\Gamma^{\theta}_{\theta\theta} = 0 \qquad \Gamma^{\theta}_{\theta\phi} = 0 \qquad \Gamma^{\theta}_{\phi\phi} = -\frac{1}{2} 2\sin\theta\cos\theta$$

$$\cdot \ \Gamma^{\phi}_{\mu\nu} = \frac{1}{2}\left(\frac{1}{\sin^2\theta}\right)\left(g_{\phi\nu,\rho} + g_{\phi\rho,\nu} - \cancel{g_{\nu\rho,\phi}} \right)$$

$$\Gamma^{\phi}_{\theta\theta} = 0 \qquad \Gamma^{\phi}_{\phi\phi} = 0 \qquad \Gamma^{\phi}_{\theta\phi} = \frac{1}{2}\frac{1}{\sin^2\theta}\partial_\theta(\sin^2\theta)$$

$$= \cot\theta$$

$$\boxed{V^1 = \partial_\phi}$$

since $\partial_\phi (g_{\mu\nu}) = 0 \Rightarrow \partial_\phi$ Killing

$$\boxed{V^2 = \sin\phi \, \partial_\theta + \cot\theta \cos\phi \, \partial_\phi}$$

$$(V^2)^\theta = \sin\phi \qquad (V^2)_\theta = \sin\phi$$

$$(V^2)^\phi = \cot\theta\cos\phi \qquad (V^2)_\phi = \sin\theta\cos\theta\cos\phi$$

$$= \frac{1}{2}\sin 2\theta \cos\phi$$

$$\nabla_\mu V^{(2)}_\nu = \partial_\mu V^2_\nu - \Gamma^{\rho}_{\mu\nu} (V^2)_\rho$$

$$= \partial_\mu (V^2)_\nu - \Gamma^{\theta}_{\mu\nu}\sin\phi - \Gamma^{\phi}_{\mu\nu}\sin\theta\cos\theta\cos\phi$$

$$\nabla_\theta (V^2)_\theta = 0$$

$$\nabla_\theta (V^2)_\phi = \cos 2\theta \, c\phi - \cot\theta \sin\theta \cos\theta \cos\phi$$
$$= (\cos 2\theta - \cos^2\theta)\cos\phi$$
$$= (\cos^2\theta - 1)\cos\phi$$

$$\nabla_\phi (V^2)_\theta = \cos\phi - \cot\theta \sin\cos\theta \cos\phi$$
$$= (1 - \cos^2\theta)\cos\phi$$

$$\nabla_\phi (V^2)_\phi = -\sin\theta \cos\theta \sin\phi - (-\sin\theta\cos\theta)\sin\phi = 0$$

$$\Rightarrow \nabla_{(\mu} V^2_{\nu)} \equiv \tfrac{1}{2}\, \nabla_\mu (V^2)_\nu + \nabla_\nu (V^2)_\mu = 0$$

$\Rightarrow (V^2)^\mu$ is a Killing vector.

$$V^3 \equiv \mathcal{L}_{(V^1)}(V^2) \equiv [V^1, V^2]$$
$$= [\partial_\phi, \sin\phi \, \partial_\theta + \cot\theta \cos\phi \, \partial_\phi]$$
$$= \cos\phi \, \partial_\theta - \cot\theta \sin\phi \, \partial_\phi$$

From previous result V^3 must be Killing

<u>Note</u>

$$\left.\begin{aligned} \mathcal{L}_{V^1} V^2 &= V^3 \\ \mathcal{L}_{V^1} V^3 &= -V^2 \\ \mathcal{L}_{V^2} V^3 &= V^1 \end{aligned}\right\}$$

Notice this is the Lie algebra for SO(3).

To see the last one:

$$[V^2, V^3] = [\sin\phi\,\partial_\theta + \cot\theta\cos\phi\,\partial_\phi,\ \cos\phi\,\partial_\theta - \cot\theta\sin\phi\,\partial_\phi]$$

$$= -\partial_\theta(\cot\theta)\sin^2\phi\,\partial_\phi + \cot\theta\cos\phi\,\partial_\phi(\cos\phi)\,\partial_\theta$$

$$-(\cot\theta)^2\cos\phi\,\partial_\phi(\sin\phi)\,\partial_\phi - \cos^2\phi\,\partial_\theta(\cot\theta)\,\partial_\phi + (\cot\theta)^2\sin\phi\,\partial_\phi(\cos\phi)\,\partial_\phi$$

$$+ (\cot\theta)\sin\phi\,\partial_\phi(\sin\phi)\,\partial_\theta$$

$$= \left[\frac{1}{\sin^2\theta}\sin^2\phi - (\cot\theta)^2\cos^2\phi + \cos^2\phi\frac{1}{\sin^2\theta} - (\cot\theta)^2\sin^2\phi\right]\partial_\phi$$

$$+ \left[\cot\theta\cos\phi\sin\phi + \cot\theta\sin\phi\cos\phi\right]\partial_\theta$$

$$= \partial_\phi$$

$$= V^3$$