(i)

$$I[x,e] \equiv \frac{1}{2}\int d\lambda \left[e^{-1}\dot{x}^{\mu}\dot{x}^{\nu}g_{\mu\nu} - m^2 e\right]$$

$$\mathcal{L} = \frac{1}{2}\left[e^{-1}\dot{x}^{\mu}\dot{x}^{\nu}g_{\mu\nu} - m^2 e\right]$$

$$\begin{cases} \dfrac{\delta \mathcal{L}}{\delta e} = \dfrac{1}{2}\left[-e^{-2}\dot{x}^{\mu}\dot{x}^{\nu}g_{\mu\nu} - m^2\right] \\ \dfrac{\delta \mathcal{L}}{\delta \dot{e}} = 0 \end{cases}$$

Euler-Lagrange $\Rightarrow \dfrac{\delta \mathcal{L}}{\delta e} = \dfrac{d}{d\lambda}\dfrac{\delta \mathcal{L}}{\delta \dot{e}} = 0$

$$\Rightarrow \quad e^2 = \frac{1}{m^2}\left(-g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}\right)$$

$$\Rightarrow \quad e = \frac{1}{m}\left(-g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}\right)^{1/2} = \frac{1}{m}\frac{d\tau}{d\lambda}$$

$$\frac{\delta \mathcal{L}}{\delta x^{\rho}} = \frac{1}{2}\left[e^{-1}\dot{x}^{\mu}\dot{x}^{\nu}g_{\mu\nu,\rho}\right]$$

$$\frac{\delta \mathcal{L}}{\delta \dot{x}^{\rho}} = \frac{1}{2}\left[2e^{-1}\dot{x}^{\nu}g_{\rho\nu}\right]$$

$$\text{E-L} \Rightarrow \frac{1}{2}e^{-1}\dot{x}^{\mu}\dot{x}^{\nu}g_{\mu\nu,\rho} = \frac{d}{d\lambda}\left(e^{-1}\dot{x}^{\nu}g_{\rho\nu}\right)$$

$$= \frac{d}{d\lambda}\left(e^{-1}\dot{x}^{\nu}\right)g_{\rho\nu} + e^{-1}\dot{x}^{\nu}g_{\rho\nu,\sigma}\dot{x}^{\sigma}$$

$$\Rightarrow \frac{d}{d\lambda}\left(e^{-1}\dot{x}^{\nu}\right)g_{\rho\nu} + \frac{1}{2}e^{-1}\dot{x}^{\nu}\dot{x}^{\sigma}\left(2g_{\rho\nu,\sigma} - g_{\sigma\nu,\rho}\right) = 0$$

$\times\, g^{\rho\mu}$

$$\Rightarrow \frac{d}{d\lambda}\left(e^{-1}\dot{x}^{\mu}\right) + \cancel{\tfrac{1}{2}}\, e^{-1}\dot{x}^{\nu}\dot{x}^{\sigma}\frac{1}{2}g^{\rho\mu}\left(g_{\rho\nu,\sigma} + g_{\rho\sigma,\nu} - g_{\sigma\nu,\rho}\right) = 0$$

$$\Rightarrow \frac{d}{d\lambda}\left(e^{-1}\dot{x}^{\mu}\right) + e^{-1}\dot{x}^{\nu}\dot{x}^{\sigma}\Gamma^{\mu}_{\nu\sigma} = 0$$

$$\Rightarrow \frac{D}{d\lambda}\left(e^{-1}\dot{x}^{\mu}\right) = 0$$

$$\left(\text{recall } \frac{D}{d\lambda}\left(e^{-1}\dot{x}^{\mu}\right) \equiv \dot{x}^{\nu}\nabla_{\nu}\left(e^{-1}\dot{x}^{\mu}\right) = \dot{x}^{\nu}\partial_{\nu}\left(e^{-1}\dot{x}^{\mu}\right) + \dot{x}^{\nu}\Gamma^{\mu}_{\nu\sigma}\dot{x}^{\sigma}e^{-1} = \frac{d}{d\lambda}e^{-1}\dot{x}^{\mu} + \dot{x}^{\nu}\dot{x}^{\sigma}e^{-1}\Gamma^{\mu}_{\nu\sigma}\right)$$

$$\Rightarrow \underbrace{\frac{D}{d\lambda}\left(e^{-1}\right)\dot{x}^{\mu}}_{\left(\frac{d}{d\lambda}e^{-1}\right)\dot{x}^{\mu}} + e^{-1}\left(\frac{D}{d\lambda}\dot{x}^{\mu}\right) = 0$$

$$\Rightarrow \frac{D}{d\lambda}\left(\dot{x}^{\mu}\right) = e^{-1}\frac{de}{d\lambda}\frac{dx^{\mu}}{d\lambda}$$

$$\Rightarrow \frac{D}{d\lambda}\left(\frac{dx^{\mu}}{d\lambda}\right) = \frac{d}{d\lambda}(\ln e)\frac{dx^{\mu}}{d\lambda}$$

$e = \frac{1}{m}\frac{d\tau}{d\lambda}$ so $\frac{d}{d\lambda}\ln e = \frac{d}{d\lambda}\ln\left(\frac{d\tau}{d\lambda}\right)$

Hence
$$\frac{D}{d\lambda}\left(\frac{dx^{\mu}}{d\lambda}\right) = \frac{d}{d\lambda}\ln\left(\frac{d\tau}{d\lambda}\right)\left(\frac{dx^{\mu}}{d\lambda}\right)$$

__Now__ if $e = \text{const} \Leftrightarrow \lambda = a\tau + b$ a, b constant
then we have an affine parameter λ
with
$$\frac{D}{d\lambda}\frac{dx^\mu}{d\lambda} = 0.$$

__Suppose__ ρ is an affine parameter for a geodesic

$$\boxed{\frac{D}{d\rho}\frac{dx^\mu}{d\rho} = 0}$$

Since $\rho = a\tau + b$
we have
$$\frac{D}{d\tau}\frac{dx^\mu}{d\tau} = 0$$

The above result implies that for any other parameter ρ' we have

$$\frac{D}{d\rho'}\left(\frac{dx^\mu}{d\rho'}\right) = \frac{d\ln}{d\rho'}\left(\frac{d\tau}{d\rho'}\right)\left(\frac{dx^\mu}{d\rho'}\right)$$

& since $\frac{d\tau}{d\rho'} = \frac{1}{a}\frac{d\rho}{d\rho'}$
we have

$$\boxed{\frac{D}{d\rho'}\left(\frac{dx^\mu}{d\rho'}\right) = \left[\frac{d\ln}{d\rho'}\left(\frac{d\rho}{d\rho'}\right)\right]\left(\frac{dx^\mu}{d\rho'}\right)}$$

(2)

$$\delta I = I[x+\epsilon K, e] - I[x, e]$$

$$= \frac{\epsilon}{2}\int d\lambda \Big[\underbrace{\underbrace{e^{-1}\dot{x}^\mu \dot{K}^\nu g_{\mu\nu} + e^{-1}\dot{x}^\nu \dot{K}^\mu g_{\mu\nu}}_{2e^{-1}\dot{x}^\mu \dot{K}^\nu g_{\mu\nu}}}_{2e^{-1}\dot{x}^\mu K^\nu{}_{,\sigma}\dot{x}^\sigma g_{\mu\nu}} + e^{-1}\dot{x}^\mu \dot{x}^\nu g_{\mu\nu,\rho} K^\rho$$

$$= \frac{\epsilon}{2}\int d\lambda\, e^{-1}\dot{x}^\mu \dot{x}^\sigma \left[2K^\nu{}_{,\sigma} g_{\mu\nu} + g_{\mu\sigma,\rho} K^\rho \right]$$

Now

$$g_{\mu\nu}K^\nu{}_{,\sigma} = g_{\mu\nu}\left(K^\nu{}_{;\sigma} - \Gamma^\nu_{\sigma\rho}K^\rho\right)$$

$$= g_{\mu\nu}K^\nu{}_{;\sigma} - g_{\mu\nu}\frac{1}{2}g^{\nu\lambda}\left(g_{\lambda\sigma,\rho} + g_{\lambda\rho,\sigma} - g_{\sigma\rho,\lambda}\right) \times K^\rho$$

$$= K_{\mu;\sigma} - \frac{1}{2}\left(g_{\mu\sigma,\rho} + g_{\mu\rho,\sigma} - g_{\sigma\rho,\mu}\right)K^\rho$$

$$\Rightarrow \delta I = \frac{\epsilon}{2}\int d\lambda\, e^{-1}\dot{x}^\mu \dot{x}^\sigma \Big[2K_{\mu;\sigma} - \cancel{g_{\mu\sigma,\rho}K^\rho} - g_{\mu\rho,\sigma}K^\rho + g_{\sigma\rho,\mu}K^\rho + \cancel{g_{\mu\sigma,\rho}K^\rho} \Big]$$

$$= \epsilon \int d\lambda\, e^{-1}\dot{x}^\mu \dot{x}^\sigma K_{(\mu;\sigma)}$$

$$+ \frac{\epsilon}{2}\int d\lambda\, e^{-1}\dot{x}^\mu \dot{x}^\sigma \left[2g_{\rho[\sigma,\mu]}K^\rho \right]$$

symmetric on μ & σ antisymmetric on μ & σ.

/

First term vanishes for Killing vector $K_{(\mu;\nu)}=0$

Second vanishes because $S^{\mu\nu} A_{\mu\nu} = 0$ if

$S^{\mu\nu} = S^{\nu\mu}$ & $A_{\mu\nu} = -A_{\nu\mu}$.

$Q = P_\mu K^\mu$

$$\frac{d}{d\lambda} Q = \frac{D}{d\lambda} Q \equiv \dot{x}^\nu \nabla_\nu Q$$

$$= \dot{x}^\nu \nabla_\nu (P_\mu K^\mu)$$

$$= \dot{x}^\nu (\nabla_\nu K^\mu) P_\mu + \dot{x}^\nu K^\mu \nabla_\nu P_\mu$$

$$= \underbrace{\dot{x}^\nu (\nabla_\nu K^\mu) e^{-1} \dot{x}^\rho g_{\mu\rho}}_{\underbrace{\dot{x}^\nu \dot{x}^\rho \nabla_\nu K_\rho e^{-1}}_{\underbrace{\dot{x}^\nu \dot{x}^\rho \nabla_{(\nu} K_{\rho)} e^{-1}}_{=0}}} + K^\mu \underbrace{\frac{D}{d\lambda} (e^{-1} \dot{x}^\rho g_{\mu\rho})}_{=0 \text{ by equations for geodesic}}$$

$$= 0$$

(3) Outgoing Eddington Finklestein

$$\left\{\begin{array}{l} u = t - r_* \\ r = r \\ \theta = \theta \\ \phi = \phi \end{array}\right\} \Rightarrow \quad r_* = r + 2M \ln\left|\frac{r-2M}{2M}\right|$$

$$dr_* = \left(1-\frac{2M}{r}\right)^{-1} dr \quad \text{for } r>2M$$

$$du = dt - \left(1-\frac{2M}{r}\right)^{-1} dr$$

$$\Rightarrow \quad dt = du + \left(1-\frac{2M}{r}\right)^{-1} dr$$

$$-\left(1-\frac{2M}{r}\right) dt^2 = -\left(1-\frac{2M}{r}\right)\left[du^2 + \left(1-\frac{2M}{r}\right)^{-2} dr^2 + 2\left(1-\frac{2M}{r}\right)^{-1} du\,dr\right]$$

$$= -\left(1-\frac{2M}{r}\right) du^2 - \left(1-\frac{2M}{r}\right)^{-1} dr^2 - 2\,du\,dr$$

$$\Rightarrow \quad ds^2 = -\left(1-\frac{2M}{r}\right) dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

$$ds^2 = -\left(1-\frac{2M}{r}\right) du^2 - 2\,du\,dr + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

Next

$$k = \frac{\partial}{\partial t} \quad \text{in Schwartschild coord's}$$

$$= \frac{\partial x^{\mu'}}{\partial t}\frac{\partial}{\partial x^{\mu'}} \quad \text{in O.E-F. coords where } x'^{\mu} = (u, r, \theta, \phi)$$

$$= \underbrace{\frac{\partial u}{\partial t}}_{1}\frac{\partial}{\partial u} + \cancel{\frac{\partial r}{\partial t}}^{0}\partial_r + \cancel{\frac{\partial \theta}{\partial t}}^{0}\partial_\theta + \cancel{\frac{\partial \phi}{\partial t}}^{0}\partial_\phi$$

$$\Rightarrow k = \frac{\partial}{\partial u} \quad \text{in O.E-F coord's.}$$

Time orientation: for $r > 2M$ choose $k = \partial_t$ in Sch. coord's.

Now in o. E-F coord's $k = \partial_u$ which we cannot use for $r < 2M$ since it is spacelike:

$$\|k\|^2 = k \cdot k = k^u k^u g_{uu} = -\left(1 - \frac{2M}{r}\right) = \frac{2M}{r} - 1 \geq 0$$

for $r < 2M$

Note that $\frac{\partial}{\partial r}$ in o. E-F coord's is null:

$$\left\|\frac{\partial}{\partial r}\right\|^2 = \left(\frac{\partial}{\partial r}\right) \cdot \left(\frac{\partial}{\partial r}\right) = \left(\frac{\partial}{\partial r}\right)^\mu \left(\frac{\partial}{\partial r}\right)^\nu g_{\mu\nu} = g_{rr} = 0$$

& hence is causal.

Also

$$\left(\frac{\partial}{\partial r}\right) \cdot k = \left(\frac{\partial}{\partial r}\right)^\mu k^\nu g_{\mu\nu} = g_{ur} = -1$$

thus $\frac{\partial}{\partial r}$ lies inside light cone of k for $r > 2M$

& hence $\left(\frac{\partial}{\partial r}\right)$ gives same time orientation & k for $r > 2M$

Note that $\left(\frac{\partial}{\partial r}\right)$ is defined ~~[illegible]~~ for $r > 0$

& hence can be used as time orientation for $r > 0$

Let $X^\mu(\lambda)$ be a future directed causal curve

Tangent vector is $V^\mu = \dfrac{dx^\mu}{d\lambda}$

Future directed $\Rightarrow$ $V^\mu \cdot \left(\dfrac{\partial}{\partial r}\right) \le 0$ ← in outgoing E-F coords

$$\Rightarrow V^\mu g_{r\mu} \le 0$$

$$\Rightarrow V^u g_{ru} \le 0 \qquad g_{ru} = -1$$

$$\Rightarrow \frac{du}{d\lambda} \ge 0$$

Also $V^2 = -\left(1-\dfrac{2M}{r}\right)\left(\dfrac{du}{d\lambda}\right)^2 - 2\dfrac{du}{d\lambda}\dfrac{dr}{d\lambda} + r^2\left(\left(\dfrac{d\theta}{d\lambda}\right)^2 + \sin^2\theta\left(\dfrac{d\phi}{d\lambda}\right)^2\right)$

$$\Rightarrow \quad 2\frac{du}{d\lambda}\frac{dr}{d\lambda} = \underbrace{-V^2}_{\ge 0} + \underbrace{\left(\frac{2M}{r}-1\right)\left(\frac{du}{d\lambda}\right)^2}_{\substack{\ge 0 \\ \text{for } r<2M}} + \underbrace{r^2\left(\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\lambda}\right)^2\right)}_{\ge 0} \qquad (*)$$

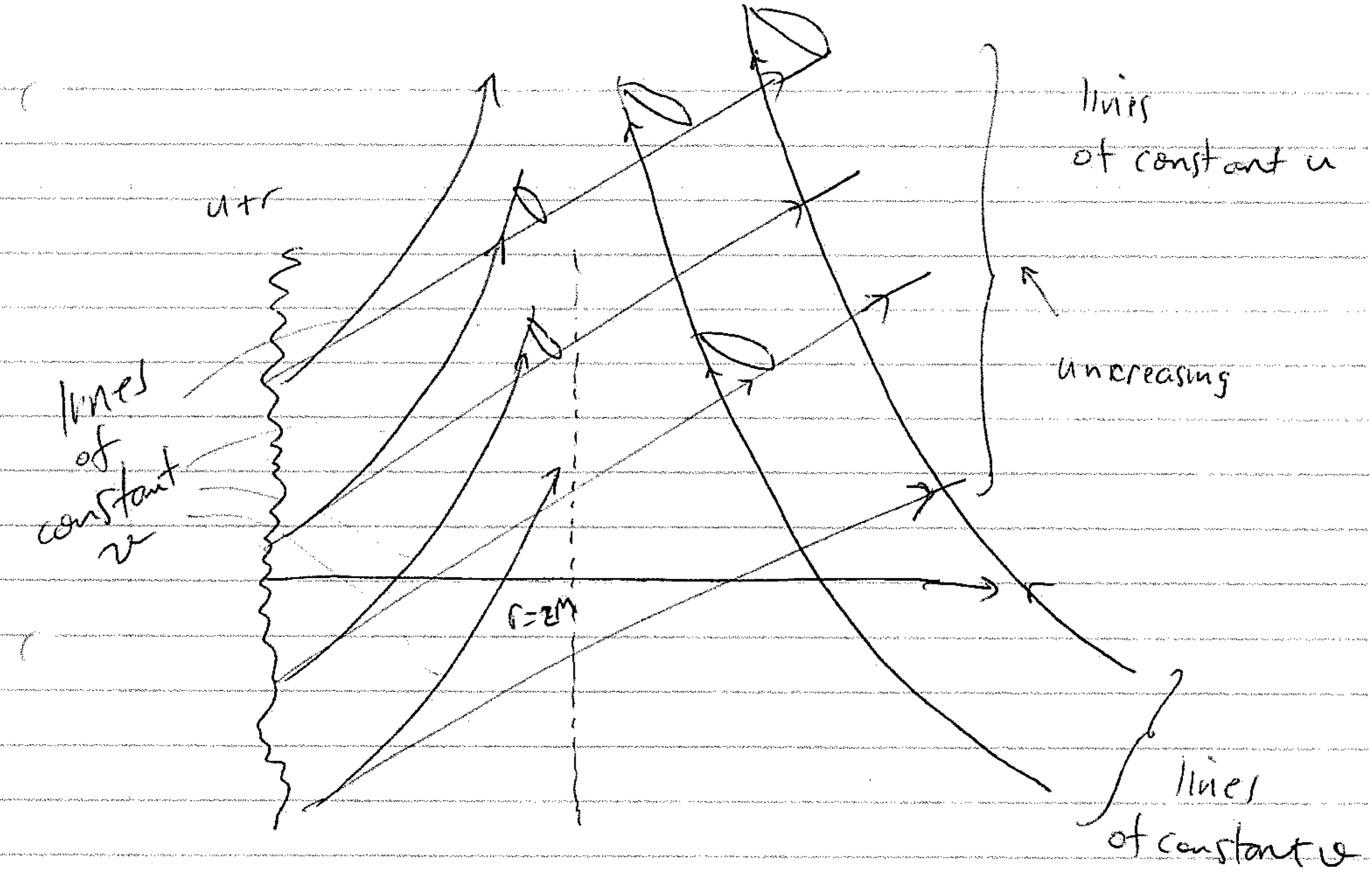
$$\Rightarrow \boxed{\frac{dr}{d\lambda} \ge 0 \quad \text{for } r<2M}$$

If $r < 2M$ have $\dfrac{dr}{d\lambda} > 0$

(if not: $\dfrac{dr}{d\lambda} = 0$, & $(*)$ $\Rightarrow$ $\dfrac{du}{d\lambda} = \dfrac{d\theta}{d\lambda} = \dfrac{d\phi}{d\lambda} = 0$

$\Rightarrow$ $V^\mu = 0$ a contradiction)

~~Hence~~



We see that all light cones are pointing away from $r=0$ in $r<2M$ region ~~[illegible]~~ associated with everything leaving $r<2M$. This is a white hole.

④

tangent vector $t^\mu = \frac{dx^\mu}{dr} = \left(\frac{dt}{dr}, 1, 0, 0\right) \equiv (\dot{t}, 1, 0, 0)$

& null $\Rightarrow ds^2 = 0 \Rightarrow \left(1-\frac{2M}{r}\right)dt^2 = \frac{dr^2}{1-2M/r} \Rightarrow \dot{t} = \pm\left(1-\frac{2M}{r}\right)^{-1}$

Calculate $t^\mu \nabla_\mu t^\nu = \ddot{x}^\nu + \Gamma^\nu_{\lambda\rho}\dot{x}^\lambda \dot{x}^\rho = \frac{D}{dr}\dot{x}^\nu$

$= \ddot{x}^\nu + \Gamma^\nu_{tt}(\dot{t})^2 + 2\Gamma^\nu_{rt}\dot{t} + \Gamma^\nu_{rr}$

Calculate

$\Gamma^t_{tt} = 0$ $\quad \Gamma^t_{tr} = \left(1-\frac{2M}{r}\right)^{-1}\frac{M}{r^2}$ $\quad \Gamma^t_{rr} = 0$

$\Gamma^r_{tt} = \left(1-\frac{2M}{r}\right)\frac{M}{r^2}$ $\quad \Gamma^r_{tr} = 0$ $\quad \Gamma^r_{rr} = -\left(1-\frac{2M}{r}\right)^{-1}\frac{M}{r^2}$

$\Gamma^\theta_{tt} = 0$ $\quad \Gamma^\theta_{tr} = 0$ $\quad \Gamma^\theta_{rr} = 0$

$\Gamma^\phi_{tt} = 0$ $\quad \Gamma^\phi_{tr} = 0$ $\quad \Gamma^\phi_{rr} = 0$

Checking $\nu = t, r, \theta, \phi$ components of $\frac{D}{dr}\dot{x}^\nu$ in turn we see that

$$\frac{D}{dr}\dot{x}^\nu = t^\mu \nabla_\mu t^\nu = 0$$

Which implies that r is an affine parameter for the null geodesics. //

Since r_* is not affinely related to r, i.e. $r_* \neq ar+b$ for constants a & b, r_* is not an affine parameter

Alternatively $\tilde{t}^\mu \equiv \frac{dx^\mu}{dr_*} = \frac{dr}{dr_*}\frac{dx^\mu}{dr} \equiv f t^\mu$

Then $\tilde{t}^\mu \nabla_\mu \tilde{t}^\nu = f t^\mu \nabla_\mu (f t^\nu) = f^2 \underbrace{(t^\mu \nabla_\mu t^\nu)}_{0} + f(t^\mu \nabla_\mu f) t^\nu$

$\Rightarrow \tilde{t}^\mu \nabla_\mu \tilde{t}^\nu = (t^\mu \nabla_\mu f)\tilde{t}^\nu$

$\tilde{t}^\mu \nabla_\mu \tilde{t}^\nu = \left(\frac{df}{dr}\right)\tilde{t}^\nu$ & not affinely parametrised.

(5)

$$C^{\mu}{}_{\nu\rho\sigma} = R^{\mu}{}_{\nu\rho\sigma} + \frac{2}{n-2}\left(\delta^{\mu}{}_{[\sigma} R_{\rho]\nu} - g_{\nu[\sigma} R_{\rho]}{}^{\mu} \right)$$

$$+ \frac{2}{(n-1)(n-2)} R\, \delta^{\mu}_{[\rho} g_{\sigma]\nu}$$

$$\Rightarrow C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + \frac{2}{n-2}\left(g_{\mu[\sigma} R_{\rho]\nu} - g_{\nu[\sigma} R_{\rho]\mu} \right)$$

$$+ \frac{2}{(n-1)(n-2)} R\, g_{\mu[\rho} g_{\sigma]\nu}$$

~~i) Clearly $C_{\mu\nu\rho\sigma} = - C_{\mu\nu\sigma\rho}$~~

i) Clearly $C_{\mu\nu\rho\sigma} = - C_{\mu\nu\sigma\rho}$

~~ii) $C_{\rho\sigma\mu}$~~

ii) $$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + \frac{1}{n-2}\left(g_{\mu\sigma} R_{\rho\nu} - g_{\mu\rho} R_{\sigma\nu} - g_{\nu\sigma} R_{\rho\mu} + g_{\nu\rho} R_{\sigma\mu} \right)$$

$$+ \frac{R}{(n-1)(n-2)} \left(g_{\mu\rho} g_{\sigma\nu} - g_{\mu\sigma} g_{\rho\nu} \right)$$

But

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} + \frac{1}{n-2}\left(g_{\rho\nu} R_{\mu\sigma} - g_{\rho\mu} R_{\nu\sigma} - g_{\sigma\nu} R_{\mu\rho} + g_{\sigma\mu} R_{\nu\rho} \right)$$

$$+ \frac{R}{(n-1)(n-2)} \left(g_{\rho\mu} g_{\nu\sigma} - g_{\rho\nu} g_{\mu\sigma} \right)$$

Using $g_{\mu\nu} = g(\mu,\nu) = g_{\nu\mu}$ & $R_{\mu\nu} = R(\mu,\nu) = R_{\nu\mu}$

We see they are equal

<u>Comment</u>: One gets more skilled at seeing this without writing out the terms.

iii) $C^\mu{}_{\nu\rho\sigma} + C^\mu{}_{\sigma\nu\rho} + C^\mu{}_{\rho\sigma\nu}$

$$= \overbrace{\left[R^\mu{}_{\nu\rho\sigma} + R^\mu{}_{\sigma\nu\rho} + R^\mu{}_{\rho\sigma\nu}\right]}^{=0}$$

$$+ \frac{1}{n-2}\left\{ \delta^\mu_\sigma R_{\rho\nu} - \delta^\mu_\rho R_{\sigma\nu} - g_{\nu\sigma} R_\rho{}^\mu + g_{\nu\rho} R_\sigma{}^\mu \right\}$$

$$+ \frac{1}{(n-1)(n-2)} R\left(\cancel{\delta^\mu_\rho g_{\sigma\nu}} - \cancel{\delta^\mu_\sigma g_{\rho\nu}} \right)$$

$$+ \frac{1}{n-2}\left\{ \delta^\mu_\rho R_{\nu\sigma} - \delta^\mu_\nu R_{\rho\sigma} - g_{\sigma\rho} R_\nu{}^\mu + g_{\sigma\nu} R_\rho{}^\mu \right\}$$

$$+ \frac{1}{(n-1)(n-2)} R\left(\cancel{\delta^\mu_\nu g_{\rho\sigma}} - \cancel{\delta^\mu_\rho g_{\nu\sigma}} \right)$$

$$+ \frac{1}{n-2}\left\{ \delta^\mu_\nu R_{\sigma\rho} - \delta^\mu_\sigma R_{\nu\rho} - g_{\rho\nu} R_\sigma{}^\mu + g_{\rho\sigma} R_\nu{}^\mu \right\}$$

$$+ \frac{1}{(n-1)(n-2)} R\left(\cancel{\delta^\mu_\sigma g_{\nu\rho}} - \cancel{\delta^\mu_\nu g_{\sigma\rho}} \right)$$

One can see all terms cancel.

(iv)

$$C^\mu{}_{\nu\mu\sigma} = R_{\nu\sigma} + \frac{1}{n-2}\left\{ \delta^\mu_\sigma R_{\mu\nu} - \delta^\mu_\mu R_{\sigma\nu} - g_{\nu\sigma} R_\mu{}^\mu + g_{\nu\mu} R_\sigma{}^\mu \right\} + \frac{R}{(n-1)(n-2)}\left\{ \delta^\mu_\mu g_{\sigma\nu} - \delta^\mu_\sigma g_{\mu\nu} \right\}$$

$$= R_{\nu\sigma} + \frac{1}{n-2}\left\{ R_{\sigma\nu} - n R_{\sigma\nu} - g_{\nu\sigma} R + R_{\sigma\nu} \right\} + \frac{R}{(n-1)(n-2)}(n-1)\, g_{\sigma\nu}$$

$$= 0.$$

6

$$\hat{g}_{\mu\nu} = \Lambda^2 g_{\mu\nu}$$

$$\Rightarrow \hat{g}^{\mu\nu} = \Lambda^{-2} g^{\mu\nu}$$

i) $$\hat{\Gamma}^{\mu}_{\nu\rho} = \frac{1}{2}\hat{g}^{\mu\sigma}\left(\hat{g}_{\sigma\nu,\rho} + \hat{g}_{\sigma\rho,\nu} - \hat{g}_{\nu\rho,\sigma}\right)$$

$$= \frac{1}{2}\Lambda^{-2} g^{\mu\sigma}\left(\partial_\rho(\Lambda^2 g_{\sigma\nu}) + \partial_\nu(\Lambda^2 g_{\sigma\rho}) - \partial_\sigma(\Lambda^2 g_{\nu\rho})\right)$$

$$= \Gamma^{\mu}_{\nu\rho} + \frac{1}{2}\Lambda^{-1} g^{\mu\sigma}\, 2\left[\partial_\rho\Lambda\, g_{\sigma\nu} + \partial_\nu\Lambda\, g_{\sigma\rho} - \partial_\sigma\Lambda\, g_{\nu\rho}\right]$$

$$= \Gamma^{\mu}_{\nu\rho} + \underbrace{\Lambda^{-1}\left[\partial_\rho\Lambda\,\delta^{\mu}_{\nu} + \partial_\nu\Lambda\,\delta^{\mu}_{\rho} - \partial_\sigma\Lambda\, g^{\mu\sigma} g_{\nu\rho}\right]}_{S^{\mu}_{\nu\rho}}$$

(ii) $$\hat{R}^{\mu}{}_{\nu\rho\sigma} = \partial_\rho\hat{\Gamma}^{\mu}{}_{\nu\sigma} + \hat{\Gamma}^{\mu}_{\rho\delta}\hat{\Gamma}^{\delta}_{\nu\sigma} - (\rho\leftrightarrow\sigma)$$

$$= \partial_\rho\Gamma^{\mu}_{\nu\sigma} + \partial_\rho S^{\mu}_{\nu\sigma} + \Gamma^{\mu}_{\rho\delta}\Gamma^{\delta}_{\nu\sigma}$$

$$+ \Gamma^{\mu}_{\rho\delta} S^{\delta}_{\nu\sigma} + S^{\mu}_{\rho\delta}\Gamma^{\delta}_{\nu\sigma} + S^{\mu}_{\rho\delta} S^{\delta}_{\nu\sigma} - (\rho\leftrightarrow\sigma)$$

Now $$\nabla_\rho S^{\mu}_{\nu\sigma} = \partial_\rho S^{\mu}_{\nu\sigma} + \Gamma^{\mu}_{\rho\delta} S^{\delta}_{\nu\sigma} - \Gamma^{\delta}_{\rho\nu} S^{\mu}_{\delta\sigma}$$

$$- \Gamma^{\delta}_{\rho\sigma} S^{\mu}_{\nu\delta}$$

$$\hat{R}^{\mu}{}_{\nu\rho\sigma} = R^{\mu}{}_{\nu\rho\sigma}$$

$$+ \Big\{ \nabla_\rho S^{\mu}_{\nu\sigma} + \Gamma^{\delta}_{\rho\nu} S^{\mu}_{\delta\sigma} + \Gamma^{\delta}_{\rho\sigma} S^{\mu}_{\nu\delta}$$

$$+ S^{\mu}_{\rho\delta}\Gamma^{\delta}_{\nu\sigma} + S^{\mu}_{\rho\delta} S^{\delta}_{\nu\sigma} \quad - (\rho\leftrightarrow\sigma)\Big\}$$

Notice that term with $\Gamma^{\delta}_{\rho\sigma}$ is symmetric on ρ & σ

so vanishes when subtract $(\rho\leftrightarrow\sigma)$

Similarly

$$\left(\Gamma^{\delta}_{\rho\nu} S^{\mu}_{\delta\sigma} + \Gamma^{\delta}_{\sigma\nu} S^{\mu}_{\rho\delta}\right)$$

is also symmetric under inter change of ρ & σ

$$\Rightarrow \hat{R}^{\mu}{}_{\nu\rho\sigma} = R^{\mu}{}_{\nu\rho\sigma}$$

$$+\Big\{\nabla_\rho S^{\mu}_{\sigma\nu} + S^{\mu}_{\delta\rho} S^{\delta}_{\sigma\nu} - \rho\leftrightarrow\sigma\Big\}$$

$$= \cancel{2R^{\mu}{}_{\nu\rho\sigma} + 2\nabla_{[\rho} S^{\mu}_{\sigma]\nu} + 2S^{\mu}_{\delta[\rho} S^{\delta}_{\sigma]\nu}}$$

Now

$$\nabla_\rho S^{\mu}_{\sigma\nu} = \nabla_\rho\Big[\nabla_\sigma \ln\Lambda\, \delta^{\mu}_{\nu} + \nabla_\nu \ln\Lambda\, \delta^{\mu}_{\sigma} - \nabla_\delta \ln\Lambda\, g^{\delta\mu} g_{\sigma\nu}$$

$$= \underbrace{\nabla_\rho\nabla_\sigma \ln\Lambda\, \delta^{\mu}_{\nu}}_{\substack{\text{observe this is}\\ \text{symmetric on } \rho\,\&\,\sigma}} + (\nabla_\rho\nabla_\nu \ln\Lambda)\delta^{\mu}_{\sigma} - (\nabla_\rho\nabla_\delta \ln\Lambda)\, g^{\delta\mu} g_{\sigma\nu}$$

Also

$$S^{\mu}_{\delta\rho} S^{\delta}_{\sigma\nu} = \left[\nabla_\delta \ln\Lambda\, \delta^{\mu}_{\rho} + \nabla_\rho \ln\Lambda\, \delta^{\mu}_{\delta} - \nabla^{\mu}\ln\Lambda\, g_{\delta\rho}\right]$$

$$\times \left[\nabla_\sigma \ln\Lambda\, \delta^{\delta}_{\nu} + \nabla_\nu \ln\Lambda\, \delta^{\delta}_{\sigma} - \nabla^{\delta}\ln\Lambda\, g_{\sigma\nu}\right]$$

$$= \nabla_\nu \ln\Lambda\, \nabla_\sigma \ln\Lambda\, \delta^{\mu}_{\rho} + \nabla_\sigma \ln\Lambda\, \nabla_\nu \ln\Lambda\, \delta^{\mu}_{\rho} - (\nabla\ln\Lambda)^2 \delta^{\mu}_{\rho} g_{\sigma\nu}$$

Same

$$+ \underbrace{\nabla_\rho \ln\Lambda\, \nabla_\sigma \ln\Lambda\, \delta^{\mu}_{\nu}}_{\text{symmetric on } \rho \,\&\, \sigma} + \nabla_\rho \ln\Lambda\, \nabla_\nu \ln\Lambda\, \delta^{\mu}_{\sigma} - \cancel{\nabla_\rho \ln\Lambda\, \nabla^{\mu}\ln\Lambda\, g_{\sigma\nu}}$$

$$- \nabla^{\mu}\ln\Lambda\, \nabla_\sigma \ln\Lambda\, g_{\rho\nu} - \underbrace{\nabla^{\mu}\ln\Lambda\, \nabla_\nu \ln\Lambda\, g_{\rho\sigma}}_{\text{symmetric on } \rho \,\&\, \sigma} + \cancel{\nabla^{\mu}\ln\Lambda\, \nabla_\rho \ln\Lambda\, g_{\sigma\nu}}$$

symmetric on ρ & σ

$$= \nabla_\nu \ln\Lambda\, \nabla_\sigma \ln\Lambda\, \delta^{\mu}_{\rho} - (\nabla\ln\Lambda)^2 \delta^{\mu}_{\rho} g_{\sigma\nu}$$

$$- \nabla^{\mu}\ln\Lambda\, \nabla_\sigma \ln\Lambda\, g_{\rho\nu}$$

+ (terms that are symmetric on ρ & σ)

Hence

$$\hat{R}^{\mu}{}_{\nu\rho\sigma} = R^{\mu}{}_{\nu\rho\sigma}$$

$$+ \left\{ \delta^{\mu}_{\sigma} \nabla_\rho \nabla_\nu \ln\Lambda - g_{\nu\sigma} \nabla_\rho \nabla^{\mu}\ln\Lambda - (\rho \leftrightarrow \sigma) \right\}$$

$$+ \left\{ \delta^{\mu}_{\rho} \nabla_\sigma \ln\Lambda\, \nabla_\nu \ln\Lambda - \delta^{\mu}_{\rho} g_{\sigma\nu} (\nabla\ln\Lambda)^2 \right.$$

$$\left. - g_{\nu\rho} \nabla_\sigma \ln\Lambda\, \nabla^{\mu}\ln\Lambda - (\rho \leftrightarrow \sigma) \right\}$$

$$\Rightarrow \hat{R}^{\mu}{}_{\nu\rho\sigma} = R^{\mu}{}_{\nu\rho\sigma}$$

$$+ 2\, \delta^{\mu}_{[\sigma} \nabla_{\rho]} \nabla_{\nu} \ln\Lambda - 2\, g_{\nu[\sigma} \nabla_{\rho]} \nabla^{\mu} \ln\Lambda$$

$$+ 2\, \delta^{\mu}_{[\rho} \nabla_{\sigma]} \ln\Lambda \nabla_{\nu} \ln\Lambda - 2\, \delta^{\mu}_{[\rho} g_{\sigma]\nu} (\nabla \ln\Lambda)^2$$

$$- 2\, g_{\nu[\rho} \nabla_{\sigma]} \ln\Lambda (\nabla^{\mu} \ln\Lambda)$$

(iii)

$$\hat{R}_{\nu\sigma} = \hat{R}^{\mu}{}_{\nu\mu\sigma}$$

$$= R_{\nu\sigma} + \{ \delta^{\mu}_{\sigma} \nabla_{\mu} \nabla_{\nu} \ln\Lambda - g_{\nu\sigma} \nabla_{\mu} \nabla^{\mu} \ln\Lambda$$

$$- \delta^{\mu}_{\mu} \nabla_{\sigma} \nabla_{\nu} \ln\Lambda + g_{\nu\mu} \nabla_{\sigma} \nabla^{\mu} \ln\Lambda \}$$

$$+ \{ \delta^{\mu}_{\mu} \nabla_{\sigma} \ln\Lambda \nabla_{\nu} \ln\Lambda - \delta^{\mu}_{\mu} g_{\sigma\nu} (\nabla \ln\Lambda)^2 - g_{\nu\mu} \nabla_{\sigma} \ln\Lambda \nabla^{\mu} \ln\Lambda$$

$$- \delta^{\mu}_{\sigma} \nabla_{\mu} \ln\Lambda \nabla_{\nu} \ln\Lambda + \delta^{\mu}_{\sigma} g_{\mu\nu} (\nabla \ln\Lambda)^2 + g_{\nu\sigma} \nabla_{\mu} \ln\Lambda \nabla^{\mu} \ln\Lambda \}$$

$$= R_{\nu\sigma} + \{ \nabla_{\sigma} \nabla_{\nu} \ln\Lambda - g_{\nu\sigma} \nabla^2 \ln\Lambda - n \nabla_{\sigma} \nabla_{\nu} \ln\Lambda$$

$$+ \nabla_{\sigma} \nabla_{\nu} \ln\Lambda \}$$

$$+ \{ n (\nabla_{\sigma} \ln\Lambda)(\nabla_{\nu} \ln\Lambda) - n\, g_{\sigma\nu} (\nabla \ln\Lambda)^2 - \nabla_{\sigma} \ln\Lambda \nabla_{\nu} \ln\Lambda$$

$$- \nabla_{\sigma} \ln\Lambda \nabla_{\nu} \ln\Lambda + g_{\sigma\nu} (\nabla \ln\Lambda)^2 + g_{\nu\sigma} (\nabla \ln\Lambda)^2 \}$$

$$= R_{\nu\sigma} - (n-2) \nabla_{\nu} \nabla_{\sigma} \ln\Lambda - g_{\nu\sigma} \nabla^2 \ln\Lambda$$

$$+ (n-2) \nabla_{\nu} \ln\Lambda \nabla_{\sigma} \ln\Lambda - (n-2) g_{\sigma\nu} (\nabla \ln\Lambda)^2$$

Finally

$$\hat{R} \equiv \hat{g}^{\nu\sigma} \hat{R}_{\nu\sigma} = \Lambda^{-2} g^{\nu\sigma} \hat{R}_{\nu\sigma}$$

$$= \Lambda^{-2} \Big\{ R - (n-2) \nabla^2 \ln\Lambda - n \nabla^2 \ln\Lambda$$

$$+ (n-2) (\nabla \ln\Lambda)^2 - n(n-2) (\nabla \ln\Lambda)^2 \Big\}$$

$$\Rightarrow \hat{R} = \Lambda^{-2} \Big\{ R - 2(n-1) \nabla^2 \ln\Lambda - (n-2)(n-1) (\nabla \ln\Lambda)^2 \Big\}$$

(iv)

First note

$$\tilde{g}_{\mu\sigma} \tilde{R}_{\rho}{}^{\mu} = \tilde{g}_{\nu\sigma} \tilde{g}^{\mu\lambda} \tilde{R}_{\rho\lambda}$$
$$= g_{\nu\sigma} g^{\mu\lambda} \tilde{R}_{\rho\lambda}$$

$$\tilde{C}^{\mu}{}_{\nu\rho\sigma} = C^{\mu}{}_{\nu\rho\sigma}$$

$$+ 2\delta^{\mu}_{[\sigma} \nabla_{\rho]} \nabla_\nu \ln\Lambda - 2 g_{\nu[\sigma} \nabla_{\rho]} \nabla^{\mu} \ln\Lambda + 2\delta^{\mu}_{[\rho} \nabla_{\sigma]} \ln\Lambda \, \nabla_\nu \ln\Lambda$$

$$\underline{-2\delta^{\mu}_{[\rho} g_{\sigma]\nu} (\nabla \ln\Lambda)^2} - 2 g_{\nu[\rho} \nabla_{\sigma]} \ln\Lambda (\nabla^{\mu} \ln\Lambda)$$

$$+ \frac{2}{n-2} \left\{ + \delta^{\mu}_{[\sigma} \tilde{R}_{\rho]\nu} - \tilde{g}_{\nu[\sigma} \tilde{R}_{\rho]}{}^{\mu} \right\}$$

$$+ \frac{2}{(n-1)(n-2)} \delta^{\mu}{}_{[\rho} g_{\sigma]\nu} \left(-2(n-1)\nabla^2 \ln\Lambda - \underline{(n-2)(n-1)(\nabla \ln\Lambda)^2} \right)$$

$$= C^{\mu}{}_{\nu\rho\sigma} + \cancel{2\delta^{\mu}_{[\sigma} \nabla_{\rho]} \nabla_\nu \ln\Lambda} \cancel{- 2 g_{\nu[\sigma} \nabla_{\rho]} \nabla^{\mu} \ln\Lambda}$$

$$+ 2\delta^{\mu}_{[\rho} \nabla_{\sigma]} \ln\Lambda \nabla_\nu \ln\Lambda - 4\delta^{\mu}_{[\rho} g_{\sigma]\nu} (\nabla \ln\Lambda)^2 - 2 g_{\nu[\rho} \nabla_{\sigma]} \ln\Lambda (\nabla^{\mu} \ln\Lambda)$$

$$- \frac{4}{n-2} \delta^{\mu}{}_{[\rho} g_{\sigma]\nu} \nabla^2 \ln\Lambda$$

$$+ \frac{2}{(n-2)} \delta^{\mu}_{[\sigma} \left\{ \cancel{-(n-2) \nabla_{\rho]\nu} \ln\Lambda} - g_{\rho]\nu} \nabla^2 \ln\Lambda \right.$$
$$\left. + (n-2) \nabla_{\rho]} \ln\Lambda \nabla_\nu \ln\Lambda - (n-2) g_{\rho]\nu} (\nabla \ln\Lambda)^2 \right\}$$

$$- \frac{2}{(n-2)} g_{\nu[\sigma} \left\{ \cancel{-(n-2) \nabla_{\rho]}{}^{\mu} \ln\Lambda} - g_{\rho]}{}^{\mu} \nabla^2 \ln\Lambda \right.$$
$$\left. + (n-2) \nabla_{\rho]} \ln\Lambda \nabla^{\mu} \ln\Lambda - (n-2) g_{\rho]}{}^{\mu} (\nabla \ln\Lambda)^2 \right\}$$

$$= C^{\mu}{}_{\nu\rho\sigma} + 2\,\delta^{\mu}_{[\rho} \nabla_{\sigma]} \ln\Lambda\, \nabla_\nu \ln\Lambda - 4\,\delta^{\mu}_{[\rho}\, g_{\sigma]\nu} (\nabla \ln\Lambda)^2$$

$$- 2\, g_{\nu[\rho} \nabla_{\sigma]} \ln\Lambda\, (\nabla^{\mu} \ln\Lambda) - \frac{4}{n-2}\, \delta^{\mu}_{[\rho}\, g_{\sigma]\nu}\, \nabla^2 \ln\Lambda$$

$$- \frac{2}{(n-2)}\, \delta^{\mu}_{[\sigma}\, g_{\rho]\nu}\, \nabla^2 \ln\Lambda + 2\,\delta^{\mu}_{[\sigma} \nabla_{\rho]} \ln\Lambda\, \nabla_\nu \ln\Lambda - 2\,\delta^{\mu}_{[\sigma}\, g_{\rho]\nu} (\nabla \ln\Lambda)^2$$

$$+ \frac{2}{(n-2)}\, g_{\nu[\sigma}\, g_{\rho]}{}^{\mu}\, \nabla^2 \ln\Lambda - 2\, g_{\nu[\sigma} \nabla_{\rho]} \ln\Lambda\, \nabla^{\mu} \ln\Lambda + 2\, g_{\nu[\sigma}\, g_{\rho]}{}^{\mu} (\nabla \ln\Lambda)^2$$

where e.g. 3 terms with γ cancel together
2 terms with ℓ " " etc.

$$= C^{\mu}{}_{\nu\rho\sigma} \quad //.$$

(7)

Consider affinely parametrised geodesic with tangent vector V^μ:

$$V^\nu \nabla_\nu V^\mu = 0$$

We can transform this to the conformally scaled metric

$$V^\nu \left[\partial_\nu V^\mu + \Gamma^\mu_{\nu\rho} V^\rho \right] = 0$$

$$\Rightarrow V^\nu \left[\partial_\nu V^\mu + \hat{\Gamma}^\mu_{\nu\rho} V^\rho - S^\mu_{\nu\rho} V^\rho \right] = 0$$

$$\Rightarrow V^\nu \hat{\nabla}_\nu V^\mu - V^\nu V^\rho \left[\nabla_\rho \ln\Lambda \, \delta^\mu_\nu + \nabla_\nu \ln\Lambda \, \delta^\mu_\rho - \nabla^\mu \ln\Lambda \, g_{\nu\rho} \right] =$$

$$\Rightarrow V^\nu \hat{\nabla}_\nu V^\mu = 2 V^\mu \left(V^\rho \nabla_\rho \ln\Lambda \right) + \left(\nabla^\mu \ln\Lambda \right)(V \cdot V)$$

If $V \cdot V = 0$ the last term on RHS vanishes and we see that the affinely parametrised geodesic is mapped to a non affinely parametrised geodesic:

$$\boxed{V^\nu \hat{\nabla}_\nu V^\mu = V^\mu V^\rho \nabla_\rho \ln\Lambda^2}$$

If we write the affinely parametrised geodesic for $g_{\mu\nu}$ as
$V^\mu = \frac{dx^\mu}{d\lambda}$ then

We have $\frac{\hat{D}}{d\lambda}\left(\frac{dx^\mu}{d\lambda}\right) = \left(\frac{dx^\mu}{d\lambda}\right) \frac{d}{d\lambda}\left(\ln\Lambda^2\right)$

~~which implies that if we choose $d\lambda^2 =$~~

We want to find $\hat{\lambda}(\lambda)$ to write this in the form

$$\frac{\hat{D}}{d\hat{\lambda}}\left(\frac{dx^{\mu}}{d\hat{\lambda}}\right) = 0$$

From previous question 1 we should choose

$$\ln \Lambda^2 = \ln\left(\frac{d\hat{\lambda}}{d\lambda}\right) + \text{const}$$

$$\text{or} \quad c\Lambda^2 = \frac{d\hat{\lambda}}{d\lambda} \qquad //$$

(8)

i) $$(\mathcal{L}_K g)_{\mu\nu} = K^\rho \cancel{\nabla_\rho g_{\mu\nu}}^{\,0} + g_{\rho\nu}\nabla_\mu K^\rho + g_{\mu\rho}\nabla_\nu K^\rho$$

$$= \nabla_\mu K_\nu + \nabla_\nu K_\mu$$

$$= 2\nabla_{(\mu}K_{\nu)}$$

Hence $(\mathcal{L}_K g)_{\mu\nu} = \alpha g_{\mu\nu}$

$$\Leftrightarrow \quad 2\nabla_{(\mu}K_{\nu)} = \alpha g_{\mu\nu}$$

Multiply by $g^{\mu\nu}$ & use $g^{\mu\nu}g_{\mu\nu} = \delta^\mu_\mu = n$

$$\Rightarrow \quad 2\nabla_\mu K^\mu = \alpha n$$

$$\Rightarrow \quad \alpha = \frac{2}{n}(\nabla\cdot K)$$

Hence, have $\boxed{\nabla_{(\mu}K_{\nu)} = \frac{1}{n}(\nabla\cdot K)\, g_{\mu\nu}}$

ii) $$\hat{g}_{\mu\nu} = \Lambda^2 g_{\mu\nu}$$

$$(\mathcal{L}_K \hat{g})_{\mu\nu} = \mathcal{L}_K(\Lambda^2 g)_{\mu\nu} = (\mathcal{L}_K \Lambda^2)\, g_{\mu\nu} + \Lambda^2(\mathcal{L}_K g)_{\mu\nu}$$

by Leibnitz rule.

Also have $\mathcal{L}_K g = 0$

Hence

$$(\mathcal{L}_K \hat{g})_{\mu\nu} = (\mathcal{L}_K \Lambda^2)\, g_{\mu\nu}$$ Konformal killing vector

Note: if $\mathcal{L}_K \Lambda^2 = 0$ then $\mathcal{L}_K \hat{g} = 0$ & it is Killing.

(ii) Consider an affinely parametrised geodesic with tangent vector $V^\mu = \frac{dx^\mu}{d\lambda}$

$$\frac{d}{d\lambda}(V^\mu k_\mu) = V^\rho \nabla_\rho (V^\mu k_\mu)$$

$$= \underbrace{(V^\rho \nabla_\rho V^\mu)}_{\substack{0 \\ \text{since geodesic}}} k_\mu + V^\rho V^\mu \nabla_\rho k_\mu$$

$$= V^\rho V^\mu \nabla_{(\rho} k_{\mu)}$$

$$= V^\rho V^\mu \frac{1}{n} (g_{\rho\mu}) \text{ (Q.1c)}$$

$$= 0$$

for $\underline{\underline{\text{null}}}$ geodesics with $V \cdot V = 0$.