1

Prob sheet 3.

Recall $u = t - r_*$ $\qquad du = dt - dr_*$

$v = t + r_*$ $\qquad dv = dt + dr_*$

$$du\,dv = dt^2 - (dr_*)^2$$

$$= dt^2 - \left(1-\frac{2M}{r}\right)^{-2} dr^2$$

$$\Rightarrow ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega$$

$$\boxed{ds^2 = -\left(1-\frac{2M}{r}\right)du\,dv + r^2 d\Omega}$$

(note as an aside $g_{uv} = -\frac{1}{2}\left(1-\frac{2M}{r}\right)$
↑ note the 1/2 !)

$$U = -e^{-u/4M} \Rightarrow dU = e^{-u/4M}\left(\frac{du}{4M}\right) = \frac{1}{4M}U\,du$$

$$V = e^{v/4M} \Rightarrow dV = \frac{1}{4M}V\,dv$$

$$\Rightarrow ds^2 = -\left(1-\frac{2M}{r}\right)\left(\frac{dU\,4M}{U}\right)\frac{4M}{V}dV + r^2 d\Omega$$

$$= -16M^2\left(1-\frac{2M}{r}\right)\frac{1}{UV}dU\,dV + r^2 d\Omega$$

Also have $UV = -e^{r_*/2M} = -e^{r/2M}(r/2M - 1)$

$$= -\frac{r}{2M}e^{r/2M}(1-2M/r)$$

~~⇒~~

$$\Rightarrow \boxed{ds^2 = -32M^3\frac{e^{-r/2M}}{r}dU\,dV + r^2 d\Omega}$$

with $r = r(U,V)$.

in region I coord transf^n $(t, r, \theta, \phi) \to (U, V, \theta, \phi)$

$$k = \partial_t = \frac{\partial U}{\partial t}\partial_U + \frac{\partial V}{\partial t}\partial_V + \underbrace{\cancel{\frac{\partial \theta}{\partial t}}}_{0}\partial_\theta + \underbrace{\cancel{\frac{\partial \phi}{\partial t}}}_{0}\partial_\phi$$

to work out $\frac{\partial U}{\partial t}$ & $\frac{\partial V}{\partial t}$ note that we are taking partial derivatives keeping r fixed

When r is fixed we have $U = \frac{c}{V}$ for constant c.

We also have $\frac{V}{U} = -e^{t/2M}$

$$\Rightarrow \quad V^2 = -c e^{t/2M}$$
$$U^2 = -c e^{-t/2M}$$

$$\Rightarrow \quad 2V \partial_t V = -c e^{t/2M} \frac{1}{2M} = + V^2/2M$$

$$\Rightarrow \quad \partial_t V = V/4M$$

Also $\quad 2U \partial_t U = +c e^{-t/2M}\left(\frac{+1}{2M}\right) = -U^2 \frac{1}{2M}$

$$\Rightarrow \quad \partial_t U = -U/4M$$

Hence

$$k = \partial_t = \frac{1}{4M}\left(V \partial_V - U \partial_U\right) \quad //$$

In these coords $k^\mu = \frac{1}{4M}(-U, V, 0, 0)$

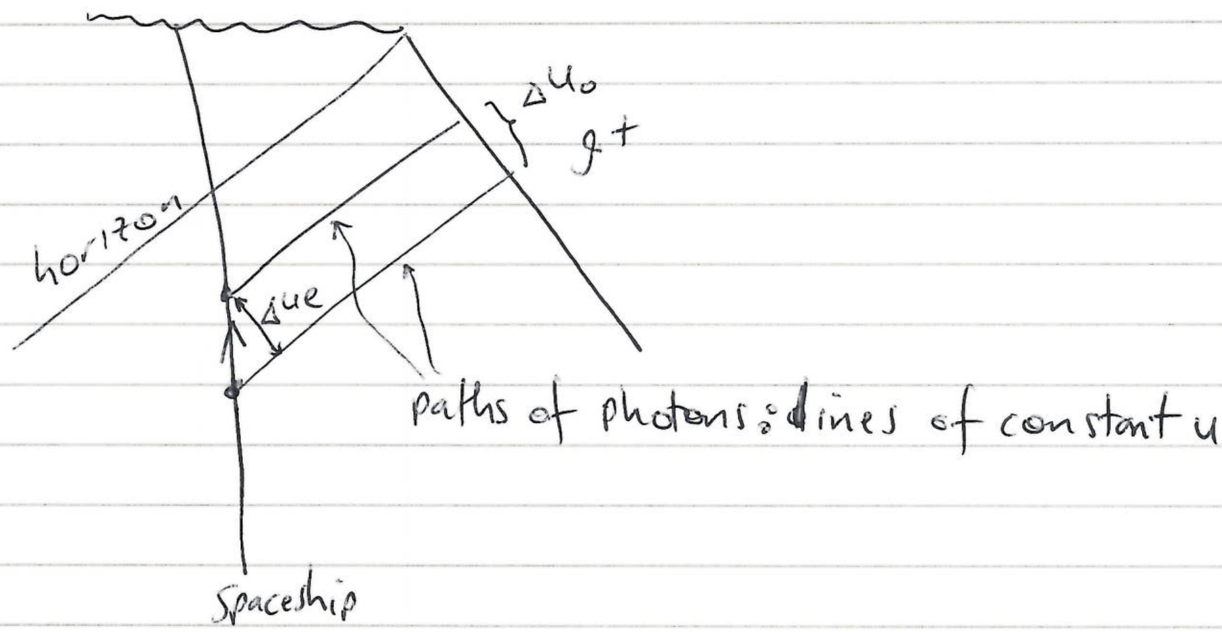
$$\|k\|^2 = k^\mu k^\nu g_{\mu\nu} = 2 k^U k^V g_{UV}$$
$$= 2\frac{1}{(4M)^2}(-U)(V)\left(-\frac{16M^3}{r}\right)e^{-r/2M} = -\left(1 - \frac{2M}{r}\right)$$

In schwartschild coord's.

$$k^\mu = (1, 0, 0, 0)$$

$$\|k\|^2 = g_{\mu\nu} k^\mu k^\nu = g_{tt} = -(1 - 2M/r) \checkmark$$

2.



Δu_e = difference in the u coordinate of 2 events where photons emitted

Δu_0 = " " " received

But $\Delta u_e = \Delta u_0$ because photons travel along lines of constant u.

$$ds^2 = -\left(1-\frac{2M}{r}\right)du^2 - 2\,du\,dr + r^2 d\Omega$$

For observer at $r=\infty$, have constant r, θ, ϕ

$$\Rightarrow ds^2 = -d\tau_0^2 = -du^2$$

$$\Rightarrow \Delta\tau_0 = \Delta u_0$$

The frequency of the photons at the emitter is ν_e

" " " " " " observer is ν_0

Redshift $\frac{\nu_e}{\nu_o} = \frac{(\Delta\tau_e)^{-1}}{(\Delta\tau_o)^{-1}} = \frac{\Delta\tau_o}{\Delta\tau_e} = \frac{\Delta u_e}{\Delta\tau_e} = \left.\frac{du}{d\tau}\right|_e$

We want to calculate $\left.\frac{du}{d\tau}\right|_e$ as spaceship approaches $r=2M$. We will write $\delta = r-2M$ and neglect $o(\delta^2)$ terms. Note $r = 2M\left(1+\frac{\delta}{2M}\right)$

Now for spaceship

$$-1 = -\left(1-\frac{2M}{r}\right)\left(\frac{du}{d\tau}\right)^2 - 2\frac{du}{d\tau}\frac{dr}{d\tau} \qquad (*)$$

So we need $\frac{dr}{d\tau}$

Recall from lectures.

$$\left(\frac{dr}{d\tau}\right)^2 = (1-\varepsilon^2)\left(\frac{R_{max}}{r} - 1\right) \qquad R_{max} = \frac{2M}{1-\varepsilon^2}$$

$$= \cancel{(1-\varepsilon^2)}\frac{1}{r}\left(\frac{2M}{\cancel{(1-\varepsilon^2)}} - (1-\varepsilon^2)r\right)$$

$$\simeq \varepsilon^2\left(1 - \frac{\delta}{2M\varepsilon^2}\right)$$

$$\Rightarrow \frac{dr}{d\tau} \simeq \varepsilon\left(1 - \frac{\delta}{4M\varepsilon^2}\right)$$

Hence, from $(*)$

$$-\frac{\delta}{2M}\left(\frac{du}{d\tau}\right)^2 + 2\varepsilon\left(1-\frac{\delta}{4M\varepsilon^2}\right)\frac{du}{d\tau} + 1 \simeq 0$$

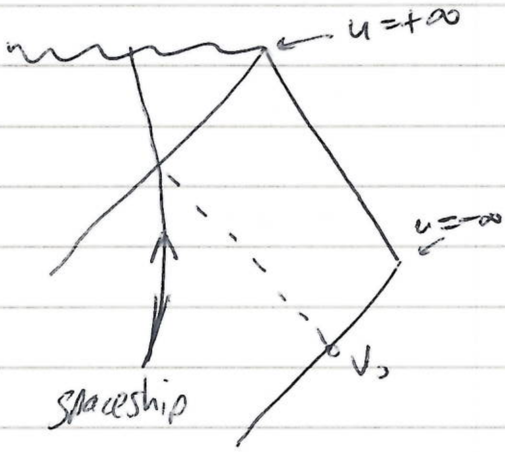
$$\Rightarrow \frac{du}{d\tau} = \frac{-2\varepsilon(1-\delta/4M\varepsilon^2) \pm \left[4\varepsilon^2\left(1-\frac{\delta}{4M\varepsilon^2}\right)^2 + 4\frac{\delta}{2M}\right]^{1/2}}{-2\left(\frac{\delta}{2M}\right)}$$

expand square root & neglect $o(\delta^2)$:

$$\Rightarrow \frac{du}{d\tau} \approx \frac{-2\mathcal{E}(1-\delta/4m\mathcal{E}^2) \pm (4\mathcal{E}^2)^{1/2}}{-2(\delta/2M)}$$

$\frac{du}{d\tau} > 0$ so take lower sign

$$\Rightarrow \left.\frac{du}{d\tau}\right|_e \simeq \frac{4M\mathcal{E}}{\delta}$$



The emitter crosses horizon $r=2M$ at some value of v, v_0 say.

As $r \to 2M$
$\Rightarrow v \to v_0$
$\Rightarrow t \to v_0 - r_*$
$\Rightarrow u \to v_0 - 2r_*$

$$\Rightarrow u \to v_0 - 2\left(r + 2M \ln\left(\frac{r-2M}{2M}\right)\right)$$

and $u \to +\infty$ as $\delta \to 0$ since $\ln(r-2M) = \ln\delta$

$$\Rightarrow u \approx v_0 - 2(2M) - 4M \ln(\delta/2M)$$
$$\simeq v_0 - 4M + 4M \ln 2M - 4M \ln\delta$$

or $u \approx \underbrace{\text{constant}}_{v_0} - 4M \ln\delta$

or $\delta^{-1} \simeq e^{(u-u_0)/4M}$

Thus $\left.\frac{du}{d\tau}\right|_e \approx 4M\mathcal{E}\, e^{(u-u_0)/4M}$ = redshift

Recall that u = proper time at infinity. Hence, redshift increases exponentially in proper time at ∞ and signals disappear in time of order $4M$. //

3. $X^0 = \cos\theta$
$X^I = \sin\theta\,\mu^I$

$(X^0)^2 + (X^I)^2 = \cos^2\theta + \sin^2\theta \underbrace{\mu^I\mu^I}_{1} = \cos^2\theta + \sin^2\theta = 1$

So parametrises S^N

choose

$0 \leq \theta \leq \pi$ so that we cover $-1 \leq X^0 \leq 1$

$ds^2 = dX^0 dX^0 + dX^I dX^I$

$dX^0 = -\sin\theta\, d\theta$
$dX^I = \cos\theta\, d\theta\, \mu^I + \sin\theta\, d\mu^I$

$$ds^2 = \sin^2\theta\, d\theta^2 + [\cos^2\theta\, d\theta^2 \mu^I\mu^I + \sin^2\theta\, d\mu^I d\mu^I + 2\cos\theta\sin\theta\, d\theta\, \mu^I d\mu^I]$$

$$= \sin^2\theta\, d\theta^2 + \cos^2\theta\, d\theta^2 + \sin^2\theta\,(d\Omega_{N-1}) + 2\cos\theta\sin\theta\, d\theta\, \mu^I d\mu^I$$

Note that $\mu^I\mu^I = 1$
$\Rightarrow 2\mu^I d\mu^I = 0$

Hence
$$ds^2 = d\theta^2 + \sin^2\theta\, d\Omega_{N-1}$$

4. As in lectures. Conformal compactification $(\hat{M}, \hat{g})$ has

$$d\hat{s}^2 = -4\,d\tilde{u}\,d\tilde{v} + \sin^2(\tilde{v}-\tilde{u})\,d\Omega$$
$$-\pi/2 \le \tilde{u} \le \tilde{v} \le \pi/2$$

$u = t - r \qquad u = \tan\tilde{u}$
$v = t + r \qquad v = \tan\tilde{v}$

i^+ eg r fixed & $t \to \infty \iff u, v \to \infty \iff \tilde{u} = \pi/2$, $\tilde{v} = \pi/2$

Note that ~~...~~ the radius of the S^2 is $\sin(\tilde{v}-\tilde{u}) = 0$. Hence is a point

i^- eg r fixed + $t \to \infty \iff u, v \to -\infty \iff \tilde{u} = -\pi/2$, $\tilde{v} = -\pi/2$

$\sin(\tilde{v}-\tilde{u}) = 0$ so this is also a point

i^0 eg t finite & $r \to \infty \iff u \to -\infty$, $v \to +\infty \iff \tilde{u} = -\pi/2$, $\tilde{v} = +\pi/2$

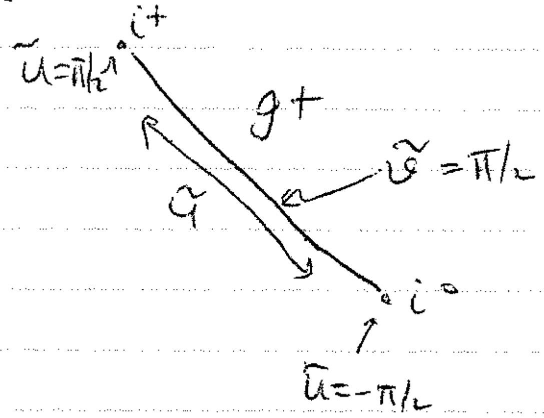
$\sin(\tilde{v}-\tilde{u}) = \sin(\pi) = 0$ and so again is a point.

$\mathscr{I}^-$ eg v finite + $u \to -\infty \iff \tilde{u} = -\pi/2$, $|\tilde{v}| < \pi/2$

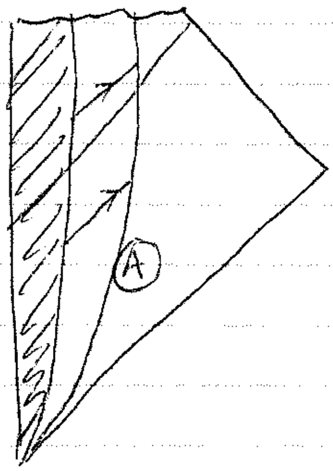
Notice $\sin(\hat{v}-\hat{u}) \neq 0$. Hence, $\mathscr{I}^+$ is then product of the open interval $|\tilde{v}| < \pi/2$ with S^2 or $\mathbb{R} \times S^2$

$\mathscr{I}^+$ similar: u finite + $v \to +\infty$: $|\tilde{u}| < \pi/2$, $|\tilde{v}| = \pi/2$

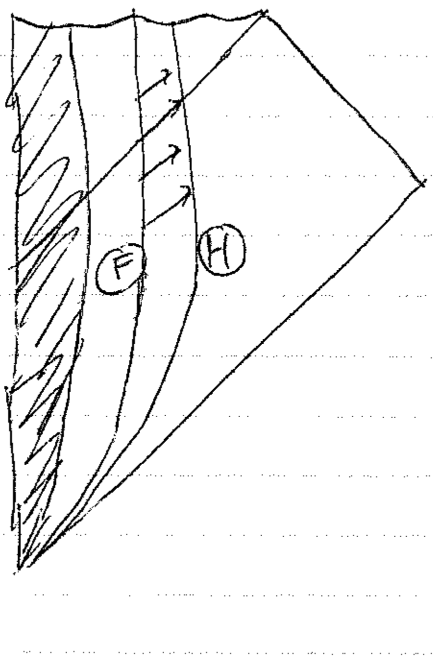
Notice, eg for g^+ that as $\tilde{v} \to \pm\pi/2$ the radius of the $S^2 \to 0$ and we reach the ~~pts~~ points i^+ & i^0, respectively.



5.



the arrowed lines represent radial geodesics from the star to the observer on worldline A. Yes, they still see the star.



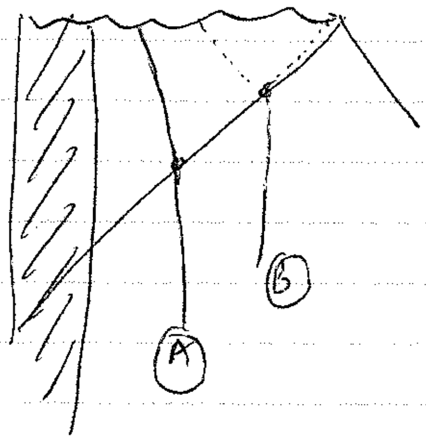
H represents worldline of head

F represents worldline of feet.

the arrowed lines are radial null geodesics from feet to the head. Yes they see them.

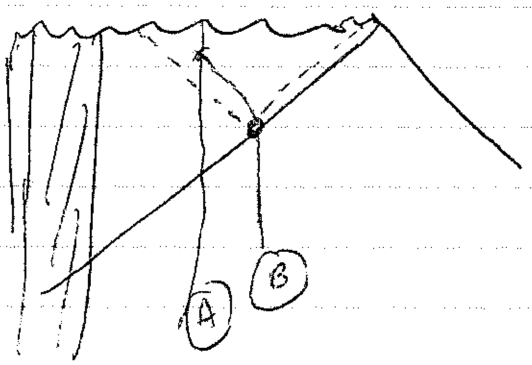
It depends.

~~Various possibilities~~



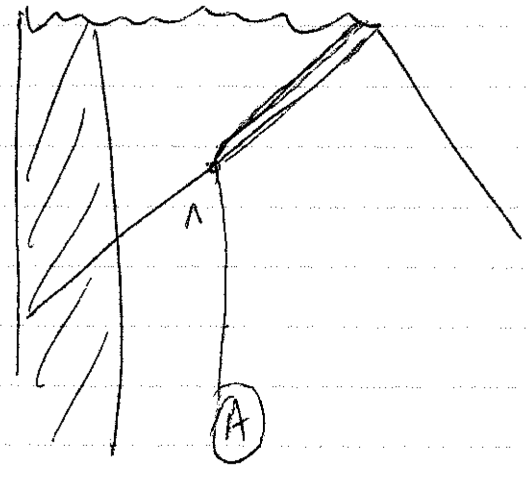
no matter what B does, they won't meet A

On the other hand, they can meet



(~~[illegible]~~ accelerated

Or



this worldline of A will maximize the possibility of meeting.

4

$$\frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g} = \frac{1}{\sqrt{-g}}\frac{1}{2\sqrt{-g}}\partial_\mu(-g)$$

$$= \frac{1}{2(-g)}(-g)g^{\rho\sigma}\partial_\mu g_{\rho\sigma}$$

$$= \frac{1}{2}g^{\rho\sigma}\partial_\mu g_{\rho\sigma}$$

On the other hand we have

$$\Gamma^\rho_{\rho\mu} = \frac{1}{2}g^{\rho\sigma}\left(g_{\sigma\rho,\mu} + \underbrace{g_{\sigma\mu,\rho} - g_{\rho\mu,\sigma}}_{\text{antisymmetric on } \rho \text{ and } \sigma}\right)$$

$$= \frac{1}{2}g^{\rho\sigma}\partial_\mu g_{\rho\sigma}$$

Hence

$$\Gamma^\rho_{\rho\mu} = \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g} \quad //$$

i) $\nabla_\mu V^\mu = \partial_\mu V^\mu + \Gamma^\mu_{\mu\sigma}V^\sigma$

$$= \partial_\sigma V^\sigma + \left(\frac{1}{\sqrt{-g}}\partial_\sigma\sqrt{-g}\right)V^\sigma$$

$$= \frac{1}{\sqrt{-g}}\partial_\sigma\left[\sqrt{-g}\,V^\sigma\right] \quad //$$

ii) $\nabla^2\phi \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu\phi$

$$= \nabla_\mu(g^{\mu\nu}\nabla_\nu\phi)$$

$$= \nabla_\mu(\nabla^\mu\phi)$$

$$= \frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}\,\nabla^\mu\phi\right]$$

$$= \frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}\,g^{\mu\nu}\partial_\nu\phi\right] \quad //$$

iii) $\nabla_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} + \Gamma^\mu_{\mu\sigma} F^{\sigma\nu} + \underbrace{\Gamma^\nu_{\mu\sigma}}_{\text{symmetric}} \underbrace{F^{\mu\sigma}}_{\text{antisymmetric}}$

$$= \partial_\mu F^{\mu\nu} + \Gamma^\mu_{\mu\sigma} F^{\sigma\nu}$$

$$= \partial_\sigma F^{\sigma\nu} + \frac{1}{\sqrt{-g}} \partial_\sigma \sqrt{-g}\, F^{\sigma\nu}$$

$$= \frac{1}{\sqrt{-g}} \partial_\sigma \left[\sqrt{-g}\, F^{\sigma\nu} \right]$$

Note it was crucial that $F_{\mu\nu} = -F_{\nu\mu}$ to get this cancellation & hence result.

iv) $A_t = Q/r$ & $A_\phi = -P\cos\theta$

$\Rightarrow F_{tr} = \partial_t A_r - \partial_r A_t = Q/r^2$

$F_{\theta\phi} = \partial_\theta A_\phi - \partial_\phi A_\theta = P\sin\theta$

<u>Metric</u>
$$\begin{cases} g_{tt} = -f & f = 1 - \frac{2m}{r} \\ g_{rr} = f^{-1} & \\ g_{\theta\theta} = r^2 & \det g = -r^4 \sin^2\theta \\ g_{\phi\phi} = r^2 \sin^2\theta & \Rightarrow \sqrt{-g} = r^2 \sin\theta \end{cases}$$

$$F^{\mu\nu} = g^{\mu\rho} g^{\nu\sigma} F_{\rho\sigma}$$

$$F^{tr} = g^{tt} g^{rr} F_{tr} = -F_{tr} = -Q/r^2$$

$$F^{\theta\phi} = g^{\theta\theta} g^{\phi\phi} F_{\theta\phi} = \frac{P\sin\theta}{r^4 \sin^2\theta} = \frac{P}{r^4 \sin\theta}$$

$$\sqrt{-g}\, F^{tr} = \cancel{r^2} \sin\theta \left(-\frac{Q}{\cancel{r^2}} \right) = -Q\sin\theta$$

$$\sqrt{-g}\, F^{\theta\phi} = r^2 \cancel{\sin\theta} \frac{P}{r^4 \cancel{\sin\theta}} = \frac{P}{r^2}$$

Max well's eq'ns $\nabla_\mu F^{\mu\nu} = 0$

$$\Leftrightarrow \quad \partial_\mu \left(\sqrt{-g}\, F^{\mu\nu} \right) = 0$$

$\underline{\nu = t}$ $$\partial_\mu \left(\sqrt{-g}\, F^{\mu t} \right) = \partial_r \left(\sqrt{-g}\, F^{rt} \right)$$
$$= \partial_r \left(\mp Q \sin\theta \right) = 0$$

$\underline{\nu = r}$ $$\partial_\mu \left(\sqrt{-g}\, F^{\mu r} \right) = \partial_t \left(\sqrt{-g}\, F^{tr} \right) = 0$$

$\underline{\nu = \theta}$ $$\partial_\mu \left(\sqrt{-g}\, F^{\mu \theta} \right) = \partial_\phi \left(\sqrt{-g}\, F^{\phi\theta} \right)$$
$$= \partial_\phi \left(-P/r^2 \right) = 0$$

$\underline{\nu = \phi}$ $$\partial_\mu \left(\sqrt{-g}\, F^{\mu\phi} \right) = \partial_\theta \left(\sqrt{-g}\, F^{\theta\phi} \right)$$
$$= \partial_\theta \left(P/r^2 \right) = 0 \qquad //$$

Max well's eq'ns $\nabla_\mu F^{\mu\nu} = 0$

$$\Leftrightarrow \partial_\mu \left(\sqrt{-g}\, F^{\mu\nu} \right) = 0$$

<u>$\nu = t$</u> $\quad \partial_\mu \left(\sqrt{-g}\, F^{\mu t} \right) = \partial_r \left(\sqrt{-g}\, F^{rt} \right)$

$$= \partial_r \left(+ Q \sin\theta \right) = 0$$

<u>$\nu = r$</u> $\quad \partial_\mu \left(\sqrt{-g}\, F^{\mu r} \right) = \partial_t \left(\sqrt{-g}\, F^{tr} \right) = 0$

<u>$\nu = \theta$</u> $\quad \partial_\mu \left(\sqrt{-g}\, F^{\mu \theta} \right) = \partial_\phi \left(\sqrt{-g}\, F^{\phi\theta} \right)$

$$= \partial_\phi \left(- P / r^2 \right) = 0$$

<u>$\nu = \phi$</u> $\quad \partial_\mu \left(\sqrt{-g}\, F^{\mu \phi} \right) = \partial_\theta \left(\sqrt{-g}\, F^{\theta\phi} \right)$

$$= \partial_\theta \left(P / r^2 \right) = 0 \qquad //$$

(v)

$$\nabla_\mu S^\mu{}_\nu = \partial_\mu S^\mu{}_\nu + \Gamma^\mu_{\mu\rho} S^\rho{}_\nu - \Gamma^\rho_{\mu\nu} S^\mu{}_\rho$$

$$= \partial_\rho S^\rho{}_\nu + \frac{1}{\sqrt{-g}}\partial_\rho \sqrt{-g}\, S^\rho{}_\nu - \Gamma^\rho_{\mu\nu} S^\mu{}_\rho$$

$$= \frac{1}{\sqrt{-g}}\partial_\rho\left(\sqrt{-g}\, S^\rho{}_\nu\right) - \Gamma^\rho_{\mu\nu} S^\mu{}_\rho$$

Now

$$\Gamma^\rho_{\mu\nu} S^\mu{}_\rho = \frac{1}{2} g^{\rho\lambda}\left(g_{\lambda\mu,\nu} + g_{\lambda\nu,\mu} - g_{\mu\nu,\lambda}\right) S^\mu{}_\rho$$

$$= \frac{1}{2}\left(g_{\lambda\mu,\nu} + \underbrace{g_{\lambda\nu,\mu} - g_{\mu\nu,\lambda}}_{\text{antisymmetric on } \mu \,\&\, \lambda}\right)\underbrace{S^{\mu\lambda}}_{\text{symmetric on } \mu \,\&\, \lambda}$$

$$= \frac{1}{2} g_{\lambda\mu,\nu} S^{\mu\lambda}$$

$$\Rightarrow \nabla_\mu S^\mu{}_\nu = \frac{1}{\sqrt{-g}}\partial_\rho\left(\sqrt{-g}\, S^\rho{}_\nu\right) - \frac{1}{2}\partial_\nu\left(g_{\lambda\mu}\right) S^{\lambda\mu} \quad //$$

7. Schwarzschild $ds^2 = -\left(1-\frac{2M}{r}\right)dv^2 + 2dv\,dr + r^2 d\Omega$

$\tilde t = v - r$

$d\tilde t = dv - dr \qquad dv = (d\tilde t + dr)$

$$ds^2 = -\left(1-\frac{2m}{r}\right)\left(d\tilde t^2 + dr^2 + 2d\tilde t\,dr\right) + 2d\tilde t\,dr + 2dr^2 + r^2 d\Omega$$

$$= -\left(d\tilde t^2 + dr^2 + \cancel{2d\tilde t\,dr}\right) + \cancel{2d\tilde t\,dr} + 2dr^2 + \frac{2M}{r}\left(d\tilde t^2 + dr^2 + 2d\tilde t\,dr\right) + r^2 d\Omega$$

$$\boxed{ds^2 = -d\tilde t^2 + dr^2 + r^2 d\Omega + \frac{2M}{r}\left(d\tilde t + dr\right)^2}$$

We now introduce cartesian coords $\begin{cases} x + iy = re^{i\phi}\sin\theta \\ z = r\cos\theta \end{cases}$

$\cancel{dx + idy = (dr + ird\phi)e^{i\phi}}$

$\cancel{dx^2 + dy^2 = dr^2 +}$

$$dx + idy = (dr\sin\theta + r\cos\theta\,d\theta + r\sin\theta\,i\,d\phi)e^{i\phi}$$

$$dz = dr\cos\theta - r\sin\theta\,d\theta$$

$$dx^2+dy^2 = dr^2\sin^2\theta + r^2\cos^2\theta\, d\theta^2 + \cancel{2r\cos\theta\sin\theta\, dr\, d\theta}$$
$$+ r^2\sin^2\theta\, d\phi^2$$
$$+ dr^2\cos^2\theta + r^2\sin^2\theta\, d\theta^2 - \cancel{2r\sin\theta\cos\theta\, dr\, d\theta}$$
$$= dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi)^2$$
$$= dr^2 + r^2\, d\Omega \checkmark$$

$$ds^2 = -d\bar{t}^2 + dx^2+dy^2+dz^2 + \frac{2M}{r}(d\bar{t}+dr)^2$$

$r = r(x,y,z)$ via $r^2 = x^2+y^2+z^2$

$$\Rightarrow \quad dr = \frac{x}{r}dx + \frac{y}{r}dy + \frac{z}{r}dz$$

~~$ds^2 = \eta\, dx\, dx + fk$~~

In these coord's

$$g_{\mu\nu} = \eta_{\mu\nu} + f K_\mu K_\nu$$

where $K_\mu = \left(1, \frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right)$ & $f = \frac{2M}{r}$

First check conditions with respect to $\eta_{\mu\nu}$:

1) Notice $K^2 = -1 + \frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2} = 0$ Null

2) $K^\rho \nabla_\rho K_\mu = K^\rho \partial_\rho K_\mu$
$$= \partial_t K_\mu + \frac{x}{r}\partial_x K_\mu + \frac{y}{r}\partial_y K_\mu + \frac{z}{r}\partial_z K_\mu$$
$$\swarrow$$
$z \quad 0$

$\underline{\mu = t}$: $K^\rho D_\rho K_t = 0$

$K_\mu = 1$

$\underline{\mu = x}$: $K_x = \frac{x}{r}$

$$K^\rho D_\rho K_x = \frac{x}{r}\partial_x\left(\frac{x}{r}\right) + \frac{y}{r}\partial_y\left(\frac{x}{r}\right) + \frac{z}{r}\partial_z\left(\frac{x}{r}\right)$$

$$= \frac{x}{r}\left(\frac{1}{r} - \frac{x}{r^2}\frac{x}{r}\right) + \frac{yx}{r}\left(\frac{-1}{r^2}\right)\frac{y}{r} + \frac{zx}{r}\left(\frac{-1}{r^2}\right)\frac{z}{r}$$

$$= \frac{x}{r^2} - \frac{x^3}{r^4} - \frac{xy^2}{r^4} - \frac{xz^2}{r^4}$$

$$= \frac{x}{r^2} - x\frac{r^2}{r^4}$$

$$= 0$$

Similarly for $\mu = y$ & z by symmetry

Finally, check conditions with respect to $g_{\mu\nu}$:

Firstly $g^{\mu\nu} = \eta^{\mu\nu} - fK^\mu K^\nu$

where $K^\mu = \eta^{\mu\nu} K_\nu$

check $g^{\mu\nu} g_{\nu\sigma} = (\eta^{\mu\nu} - fK^\mu K^\nu)(\eta_{\nu\sigma} + fK_\nu K_\sigma)$

$$= \delta^\mu_\sigma + fK^\mu K_\sigma - fK^\mu K_\sigma + 0$$

$$= \delta^\mu{}_\sigma \quad \checkmark$$

thus $K^\mu = \eta^{\mu\nu} K_\nu = g^{\mu\nu} K_\nu$ also.

$K^2 = g^{\mu\nu} K_\mu K_\nu$

$$= (\eta^{\mu\nu} - fK^\mu K^\nu)(K_\mu)(K_\nu)$$

$$= \eta^{\mu\nu} K_\mu K_\nu$$

$$= 0 \quad \checkmark$$

$$K^\rho D_\rho K_\mu = K^\rho(\partial_\rho K_\mu - \Gamma^\sigma_{\rho\mu} K_\sigma)$$

we have shown $K^\rho \partial_\rho K_\mu = 0$ already.
So, just have to show

$$K^\rho K_\sigma \Gamma^\sigma_{\rho\mu} = 0$$

$$\Leftrightarrow \quad K^\rho K_\sigma \tfrac{1}{2} g^{\sigma\delta}(g_{\delta\rho,\mu} + g_{\delta\mu,\rho} - g_{\rho\mu,\delta}) = 0$$

$$\Leftrightarrow \quad \underbrace{K^\rho K^\delta}_{\text{symmetric on } \rho \,\&\, \delta}(g_{\delta\rho,\mu} + \underbrace{g_{\delta\mu,\rho} - g_{\rho\mu,\delta}}_{\text{antisymmetric on } \delta \,\&\, \rho}) = 0$$

$$\Leftrightarrow \quad K^\rho K^\delta \frac{\partial}{\partial x^\mu}(g_{\delta\rho}) = 0$$

$$\Leftrightarrow \quad K^\rho K^\delta \frac{\partial}{\partial x^\mu}(\eta_{\delta\rho} + f K_\delta K_\rho) = 0$$

$$\Leftrightarrow \quad K^\rho K^\delta \frac{\partial}{\partial x^\mu}(f K_\delta K_\rho) = 0$$

& this is true since $K^2 = 0$ //.

(i)

all permu'ns

$$(A\wedge B)_{\mu_1\cdots\mu_p\nu_1\cdots\nu_q} = \frac{(p+q)!}{p!\,q!} A_{[\mu_1\cdots\mu_p} B_{\nu_1\cdots\nu_q]}$$

$$= \frac{1}{p!\,q!}\left\{ A_{\mu_1\cdots\mu_p} B_{\nu_1\cdots\nu_q} + \cdots\cdots \right.$$

$$= \frac{1}{p!\,q!}\left\{ B_{\nu_1\cdots\nu_q} A_{\mu_1\cdots\mu_p} + \cdots\cdots \right.$$

$$= \frac{(p+q)!}{p!\,q!} B_{[\nu_1\cdots\nu_q} A_{\mu_1\cdots\mu_p]}$$

$$= (B\wedge A)_{\nu_1\cdots\nu_q\mu_1\cdots\mu_p}$$

$$= (B\wedge A)_{\mu_1\cdots\mu_p\nu_1\cdots\nu_q}\,(-1)^{qp}$$

we move e.g. ν_1 past p of the μ_i + then do that for the rest of the $\nu_2\cdots\nu_q$

$$\Rightarrow A\wedge B = (-1)^{p.q} B\wedge A \quad //$$

(ii)

$$d(dA)_{\nu_1\cdots\nu_{p+2}} = (p+2)\,\partial_{[\nu_1}\, dA_{\nu_2\cdots\nu_{p+2}]}$$

$$= (p+2)(p+1)\,\partial_{[\nu_1}\partial_{\nu_2} A_{\nu_3\cdots\nu_{p+2}]}$$

$$= 0$$

since $\partial_{\nu_1}\partial_{\nu_2}$ is symmetric on interchange of ν_1 + ν_2

Note in the first step we know dA is a $(p+1)$-form and we take the d of that object.

(iii)

$$(A\wedge B)_{\mu_1\cdots\mu_p\nu_1\cdots\nu_q} = \frac{(p+q)!}{p!\,q!}\; A_{[\mu_1\cdots\mu_p} B_{\nu_1\cdots\nu_q]}$$

$$d(A\wedge B)_{\rho\mu_1\cdots\mu_p\nu_1\cdots\nu_q} = (p+q+1)\,\partial_{[\rho}(A\wedge B)_{\mu_1\cdots\mu_p\nu_1\cdots\nu_q]}$$

$$= (p+q+1)\,\partial_{[\rho}\left(\frac{(p+q)!}{p!\,q!}\; A_{\mu_1\cdots\mu_p} B_{\nu_1\cdots\nu_q]}\right)$$

$$= (p+q+1)\frac{(p+q)!}{p!\,q!}\Big(\partial_{[\rho} A_{\mu_1\cdots\mu_p} B_{\nu_1\cdots\nu_q]} + A_{[\mu_1\cdots\mu_p}\partial_{\rho} B_{\nu_1\cdots\nu_q]}\Big)$$

$$= \frac{(p+q+1)!}{p!\,q!}\left(\frac{(p+1)!\,q!}{(p+1+q)!}\frac{1}{(p+1)}(dA\wedge B)_{\rho\mu_1\cdots\mu_p\nu_1\cdots\nu_q} + \frac{p!\,(q+1)!}{(p+q+1)!}\frac{1}{q+1}(A\wedge dB)_{\mu_1\cdots\mu_p\rho\nu_1\cdots\nu_q}\right)$$

Where we've call $(dA)_{\rho\mu_1\cdots\mu_p} = (p+1)\,\partial_{[\rho}A_{\mu_1\cdots\mu_p]}$

$$(dB)_{\rho\nu_1\cdots\nu_q} = (q+1)\,\partial_{[\rho}B_{\nu_1\cdots\nu_q]}$$

$$= (dA\wedge B)_{\rho\mu_1\cdots\mu_p\nu_1\cdots\nu_q} + \underbrace{(A\wedge dB)_{\mu_1\cdots\mu_p\rho\nu_1\cdots\nu_q}}_{(-1)^p(A\wedge dB)_{\rho\mu_1\cdots\mu_p\nu_1\cdots\nu_q}}$$

$$\Rightarrow\ dA\wedge B = dA\wedge B + (-1)^p A\wedge dB \quad //$$

(iv)

$(**A)_{\mu_1 \cdots \mu_p}$

Now $(*A)$ is a $(n-p)$ form, so take $*$ of that:

$$*(*A)_{\mu_1 \cdots \mu_p} = \frac{1}{(n-p)!} \varepsilon_{\mu_1 \cdots \mu_p}{}^{\mu_{p+1} \cdots \mu_n} (*A)_{\mu_{p+1} \cdots \mu_n}$$

$$= \frac{1}{(n-p)!} \varepsilon_{\mu_1 \cdots \mu_p}{}^{\mu_{p+1} \cdots \mu_n} \frac{1}{p!} \varepsilon_{\mu_{p+1} \cdots \mu_n}{}^{\nu_1 \cdots \nu_p} A_{\nu_1 \cdots \nu_p}$$

$$= \frac{1}{(n-p)!} \frac{1}{p!} \varepsilon_{\mu_1 \cdots \mu_p}{}^{\mu_{p+1} \cdots \mu_n} (-1)^{p \cdot n-p} \varepsilon^{\nu_1 \cdots \nu_p}{}_{\mu_{p+1} \cdots \mu_n} A_{\nu_1 \cdots \nu_p}$$

"Moved p indicies past n-p"

$$= \frac{1}{(n-p)!} \frac{1}{p!} (-1)^{p \cdot n-p} \cdot (\pm)\, p!\, (n-p)!\, \delta^{\nu_1 \cdots \nu_p}_{\mu_1 \cdots \mu_p} A_{\nu_1 \cdots \nu_p}$$

$$= \pm (-1)^{p(n-p)} A_{\mu_1 \cdots \mu_p}$$

$$\Rightarrow \; **A = \pm (-1)^{p(n-p)} A \quad //$$

Again Work "from the left" i.e. take $*$ of the $(n-p+1)$-form $(d*A)$:

(V) $$(*d*A)_{\mu_1\cdots\mu_{p-1}}$$

$$= \frac{1}{(n-p+1)!}\,\varepsilon_{\mu_1\cdots\mu_{p-1}}{}^{\rho\nu_1\cdots\nu_{n-p}}\,(d*A)_{\rho\nu_1\cdots\nu_{n-p}}$$

$$= \frac{1}{(n-p+1)!}\,\varepsilon_{\mu_1\cdots\mu_{p-1}}{}^{\rho\nu_1\cdots\nu_{n-p}}\,(n-p+1)\,\nabla_{[\rho} *A_{\nu_1\cdots\nu_{n-p}]}$$

$$= \frac{1}{(n-p)!}\,\varepsilon_{\mu_1\cdots\mu_{p-1}}{}^{\rho\nu_1\cdots\nu_{n-p}}\,\nabla_\rho *A_{\nu_1\cdots\nu_{n-p}}$$

$$= \frac{1}{(n-p)!}\,\varepsilon_{\mu_1\cdots\mu_{p-1}}{}^{\rho\nu_1\cdots\nu_{n-p}}\,\nabla_\rho\left(\frac{1}{p!}\varepsilon_{\nu_1\cdots\nu_{n-p}}{}^{\sigma_1\cdots\sigma_p}A_{\sigma_1\cdots\sigma_p}\right)$$

$$= \frac{1}{(n-p)!}\,\frac{1}{p!}\,\varepsilon_{\mu_1\cdots\mu_{p-1}}{}^{\rho\nu_1\cdots\nu_{n-p}}\,\varepsilon_{\nu_1\cdots\nu_{n-p}}{}^{\sigma_1\cdots\sigma_p}\,\nabla_\rho A_{\sigma_1\cdots\sigma_p}$$

$$= \frac{1}{(n-p)!}\,\frac{1}{p!}\,(-1)^{(n-p)\cdot p}\,\underbrace{\varepsilon^{\nu_1\cdots\nu_{n-p}}{}_{\mu_1\cdots\mu_{p-1}\rho}\,\varepsilon_{\nu_1\cdots\nu_{n-p}}{}^{\sigma_1\cdots\sigma_p}}_{\pm\, p!\,(n-p)!\,\delta^{\sigma_1\cdots\sigma_{p-1}\sigma_p}_{\mu_1\cdots\mu_{p-1}\rho}}\,\nabla^\rho A_{\sigma_1\cdots\sigma_p}$$

$$= \pm\,(-1)^{(n-p)p}\,\nabla^\rho A_{\mu_1\mu_2\cdots\mu_{p-1}\rho}$$

9.
(i) $g_{\mu\nu} g^{\nu\sigma} = \delta^{\sigma}_{\mu} \Rightarrow \delta g_{\mu\nu} g^{\nu\sigma} + g_{\mu\nu} \delta g^{\nu\sigma} = 0$

$$\Rightarrow \delta g_{\mu\nu} g^{\nu\sigma} = - g_{\mu\nu} \delta g^{\nu\sigma}$$

$$\Rightarrow \delta g_{\mu\rho} = - g_{\mu\nu} g_{\rho\sigma} \delta g^{\nu\sigma} \quad //$$

(ii)

$$\Gamma^{\mu}{}_{\nu\rho} = \tfrac{1}{2} g^{\mu\sigma} \left(g_{\sigma\nu,\rho} + g_{\sigma\rho,\nu} - g_{\nu\rho,\sigma} \right)$$

$$\delta \Gamma^{\mu}{}_{\nu\rho} = \frac{1}{2} \delta g^{\mu\sigma} \left(g_{\sigma\nu,\rho} + g_{\sigma\rho,\nu} - g_{\nu\rho,\sigma} \right)$$

$$+ \underbrace{\frac{1}{2} g^{\mu\sigma} \left(\delta g_{\sigma\nu,\rho} + \delta g_{\sigma\rho,\nu} - \delta g_{\nu\rho,\sigma} \right)}$$

$$\frac{1}{2} g^{\mu\sigma} \left(\delta g_{\sigma\nu;\rho} + \delta g_{\sigma\rho;\nu} - \delta g_{\nu\rho;\sigma} \right)$$

$$+ \frac{1}{2} g^{\mu\sigma} \Big(\cancel{\Gamma_{\rho\sigma}^{\delta} \delta g_{\delta\nu}} + \cancel{\Gamma_{\nu\sigma}^{\delta} \delta g_{\delta\rho}} - \cancel{\Gamma_{\sigma}{}^{\delta}{}_{\nu} \delta g_{\delta\rho}}$$
$$+ \Gamma_{\rho\nu}^{\delta} \delta g_{\sigma\delta} + \Gamma_{\nu\rho}^{\delta} \delta g_{\sigma\delta} - \cancel{\Gamma_{\sigma\rho}^{\delta} \delta g_{\nu\delta}} \Big)$$

$$= \frac{1}{2} g^{\mu\sigma} \left(\delta g_{\sigma\nu;\rho} + \delta g_{\sigma\rho;\nu} - \delta g_{\nu\rho;\sigma} \right)$$

$$+ \cancel{\frac{1}{2} \delta g^{\mu\sigma} \left(g_{\sigma\nu,\rho} + g_{\sigma\rho,\nu} - g_{\nu\rho,\sigma} \right)}$$

$$+ \underbrace{g^{\mu\sigma} \delta g_{\sigma\delta} \Gamma^{\delta}_{\rho\nu}}$$

$$\cancel{- \delta g^{\mu\sigma} g_{\sigma\delta} \frac{1}{2} g^{\delta\gamma} \left(g_{\gamma\rho,\nu} + g_{\gamma\nu,\rho} - g_{\rho\nu,\gamma} \right)}$$

$$\Longrightarrow \delta\Gamma^{\mu}{}_{\nu\rho} = \tfrac{1}{2} g^{\mu\sigma}\left(\delta g_{\sigma\nu;\rho} + \delta g_{\sigma\rho;\nu} - \delta g_{\nu\rho;\sigma}\right) \quad //$$

(iii)

$$\delta R = \delta\left(R_{\mu\nu} g^{\mu\nu}\right) = \left(\delta R_{\mu\nu}\right) g^{\mu\nu} + R_{\mu\nu}\,\delta g^{\mu\nu}$$

$$R_{\nu\sigma} = R^{\mu}{}_{\nu\mu\sigma}$$

$$= \partial_\mu \Gamma^{\mu}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}_{\nu\mu} + \Gamma^{\mu}_{\mu\delta}\Gamma^{\delta}_{\nu\sigma} - \Gamma^{\mu}_{\sigma\delta}\Gamma^{\delta}_{\nu\mu}$$

$$\Longrightarrow \delta R_{\nu\sigma} = \underbrace{\partial_\mu\left(\delta\Gamma^{\mu}_{\nu\sigma}\right) - \partial_\sigma\left(\delta\Gamma^{\mu}_{\nu\mu}\right)}$$

$$\nabla_\mu\left(\delta\Gamma^{\mu}_{\nu\sigma}\right) - \nabla_\sigma \delta\Gamma^{\mu}_{\nu\mu}$$

$$-\Big(\Gamma^{\mu}_{\mu\delta}\,\delta\Gamma^{\delta}_{\nu\sigma} - \cancel{\Gamma^{\delta}_{\mu\nu}\,\delta\Gamma^{\mu}_{\delta\sigma}} - \cancel{\Gamma^{\delta}_{\mu\sigma}\,\delta\Gamma^{\mu}_{\nu\delta}}$$

$$- \cancel{\Gamma^{\mu}_{\sigma\delta}\,\delta\Gamma^{\delta}_{\nu\mu}} + \Gamma^{\delta}_{\sigma\nu}\,\delta\Gamma^{\mu}_{\delta\mu} + \cancel{\Gamma^{\delta}_{\sigma\mu}\,\delta\Gamma^{\mu}_{\nu\delta}}\Big)$$

$$+\delta\Gamma^{\mu}_{\mu\delta}\,\Gamma^{\delta}_{\nu\sigma} + \Gamma^{\mu}_{\mu\delta}\,\delta\Gamma^{\delta}_{\nu\sigma} - \cancel{\delta\Gamma^{\mu}_{\sigma\delta}\,\Gamma^{\delta}_{\nu\mu}} - \cancel{\Gamma^{\mu}_{\sigma\delta}\,\delta\Gamma^{\delta}_{\nu\mu}}$$

$$\Longrightarrow \delta R_{\nu\sigma} = \nabla_\mu\left(\delta\Gamma^{\mu}_{\nu\sigma}\right) - \nabla_\sigma\left(\delta\Gamma^{\mu}_{\nu\mu}\right)$$

$$\Rightarrow \quad g^{\nu\sigma}\,\delta R_{\nu\sigma} = g^{\nu\sigma}\,\nabla_\mu\left(\delta\Gamma^\mu_{\nu\sigma}\right) - g^{\nu\sigma}\,\nabla_\sigma\left(\delta\Gamma^\mu_{\nu\mu}\right)$$

$$= \nabla_\mu\left(g^{\nu\sigma}\,\delta\Gamma^\mu_{\nu\sigma}\right) - \nabla^\nu\left(\delta\Gamma^\mu_{\nu\mu}\right)$$

$$= \nabla_\mu\left[g^{\nu\sigma}\,\frac{1}{2}\,g^{\mu\delta}\left(2\,\delta g_{\delta\nu;\sigma} - \delta g_{\nu\sigma;\delta}\right)\right]$$

$$-\nabla^\nu\left[\frac{1}{2}\,g^{\mu\delta}\left(\delta g_{\delta\cancel{\nu};\mu} + \delta g_{\delta\mu;\nu} - \delta g_{\cancel{\nu}\mu;\delta}\right)\right]$$

since 2 terms antisym. in μ & δ

$$= \nabla^\delta\,\nabla^\nu\,\delta g_{\delta\nu} - \frac{1}{2}\,\nabla^\delta\,\nabla_\delta\,\delta g_{\nu\sigma}\,g^{\nu\sigma}$$

$$-\frac{1}{2}\,\nabla^\nu\,\nabla_\nu\,\delta g_{\delta\mu}\,g^{\mu\delta}$$

$$= \nabla^\delta\Big[\underbrace{\left(\nabla^\nu\,\delta g_{\delta\nu}\right) - \left(\nabla_\delta\,\delta g_{\nu\sigma}\right)g^{\nu\sigma}}_{\equiv V_\delta}\Big]$$

$$= \nabla^\delta V_\delta$$

(iv)

$$\delta\sqrt{-g} = \frac{1}{2}\frac{1}{\sqrt{-g}}\,(-g)\,g^{\mu\nu}\,\delta g_{\mu\nu} = \frac{1}{2}\sqrt{-g}\,g^{\mu\nu}\,\delta g_{\mu\nu}$$

$$= -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\,\delta g^{\mu\nu}$$

(v)

$$\delta(\sqrt{-g}\,R) = (\delta\sqrt{-g})R + \sqrt{-g}\,\delta R$$

$$= -\tfrac{1}{2}\sqrt{-g}\,g_{\mu\nu}\,\delta g^{\mu\nu} R + \sqrt{-g}\left[R_{\mu\nu}\,\delta g^{\mu\nu} + \nabla^{\delta} V_{\delta}\right]$$

$$= \sqrt{-g}\left(R_{\mu\nu} - \tfrac{1}{2} R\, g_{\mu\nu}\right)\delta g^{\mu\nu} + \sqrt{-g}\,\nabla^{\mu} V_{\mu}$$

$$\delta\left(\sqrt{-g}\,F_{\mu\nu}F^{\mu\nu}\right) = (\delta\sqrt{-g})F_{\mu\nu}F^{\mu\nu} + \sqrt{-g}\,\delta\left(F^{\mu\nu}F_{\mu\nu}\right)$$

$$= -\tfrac{1}{2}\sqrt{-g}\,g_{\mu\nu}\,\delta g^{\mu\nu} F^2$$

$$+ \underbrace{\underbrace{\sqrt{-g}\,\delta\left[F_{\rho\sigma}\, g^{\mu\rho} g^{\sigma\nu} F_{\mu\nu}\right]}_{\sqrt{-g}\,2\,\delta g^{\mu\rho}\left(F_{\rho\sigma}\, g^{\sigma\nu} F_{\mu\nu}\right)}}_{\sqrt{-g}\,2\,\delta g^{\mu\nu}\left(F_{\nu\sigma} F_{\mu}{}^{\sigma}\right)}$$

$$= \sqrt{-g}\left[2F_{\mu\sigma}F_{\nu}{}^{\sigma} - \tfrac{1}{2}F^2 g_{\mu\nu}\right]\delta g^{\mu\nu}$$

$$\mathcal{L} = \sqrt{-g}\left[R - F_{\mu\nu}F^{\mu\nu}\right]$$

$$\delta\mathcal{L} = \sqrt{-g}\left[R_{\mu\nu} - \tfrac{1}{2} R\, g_{\mu\nu} - 2\left(F_{\mu\sigma}F_{\nu}{}^{\sigma} - \tfrac{1}{4}F^2 g_{\mu\nu}\right)\right]\delta g^{\mu\nu}$$

$$+ \sqrt{-g}\,\nabla^{\mu} V_{\mu} \quad //$$