Problem Sheet 5

1

Evaluate the RHS

Define $\sqrt{} \equiv \sqrt{M^4 - J^2 - Q^2M^2}$

$$\frac{KA}{4\pi} = \frac{1}{M}\frac{\sqrt{}}{(2M^2 - Q^2 + 2\sqrt{})}\cdot(2M^2 - Q^2 + 2\sqrt{})$$

$$= \frac{1}{M(2M^2 - Q^2 + 2\sqrt{})}\left(2M^2\sqrt{} - Q^2\sqrt{} + 2(M^4 - J^2 - Q^2M^2)\right)$$

$$2\Omega_H J = \frac{2J^2}{M(2M^2 - Q^2 + 2\sqrt{})}$$

$$\Phi_H Q = \frac{1}{M}\frac{(M^2 + \sqrt{})Q^2}{(2M^2 - Q^2 + 2\sqrt{})}$$

$$\Rightarrow \frac{KA}{4\pi} + 2\Omega_H J + \Phi_H Q$$

$$= \frac{1}{M(2M^2 - Q^2 + 2\sqrt{})}\Big[2M^2\sqrt{} - \cancel{Q^2\sqrt{}} + 2M^4 - \cancel{2J^2} - 2Q^2M^2$$
$$+ \cancel{2J^2} + Q^2M^2 + \cancel{Q^2\sqrt{}}\Big]$$

$$= \frac{1}{M(2M^2 - Q^2 + 2\sqrt{})}\left[M^2\right]\left[2M^2 - Q^2 + \sqrt{}\right]$$

$$= M \quad //$$

2

K has units of acceleration $[K] = LT^{-2}$

$[A] = L^2$

$[K\,dA] = L^3 T^{-2}$ $\qquad$ $[G] = L^3/MT^2$

$\Rightarrow \quad [G^{-1} K\,dA] = M$

$[\Omega_H]$ = angular velocity = T^{-1}

$[J]$ = angular momentum = $L\,MLT^{-1}$

$[\Omega_H\,dJ] = ML^2T^{-2}$ = units of energy

Convert from ~~[illegible]~~ mass to energy via c^2.

$$\Rightarrow \quad dE \equiv c^2 dM = \frac{c^2}{8\pi G} K\,dA + \Omega_H\,dJ.$$

notice just G, & c appearing
(not $\hbar$) ~~[illegible]~~ – we are
just doing GR.

3 Initially Schwartschild with $\begin{cases} \text{Mass} = M_1 \\ \text{Area} = A_1 = 16\pi M_1^2 \end{cases}$

Final state 2 approx schwartschild black holes with Masses M_2 & M_3

Area $16\pi M_2^2$ & $16\pi M_3^2$

the approximation assumes very small gravitational binding energy

Energy conservation $M_1 \geq M_2 + M_3$

$$\Rightarrow M_1^2 \geq M_2^2 + M_3^2 + 2M_2M_3$$

Area theorem $16\pi M_1^2 \leq 16\pi (M_2^2 + M_3^2)$

$$\Rightarrow M_1^2 \leq M_2^2 + M_3^2$$

which can't happen.

Reverse process

Initially 2 non-rotating black holes which we assume are well approximated by schwarzschild with Masses M_2 & M_3
& Areas $16\pi M_2^2$ & $16\pi M_3^2$

They coalesce to single black hole with Mass M_1
Area $16\pi M_1^2$

~~By conservation of energy the smaller M_1 the larger the amount of~~
Energy radiated $E = M_2 + M_3 - M_1$ which can be used to do work. But we can't make M_1 arbitrarily small because of the area theorem.

$$\Rightarrow \quad M_1^2 \geq M_2^2 + M_3^2$$

Smallest M_1 can be is $M_1^{min} = \sqrt{M_2^2 + M_3^2}$

$\Rightarrow$ Maximum energy extractable is
$$M_2 + M_3 - \sqrt{M_2^2 + M_3^2}$$

Efficiency of the process

$$\eta = \frac{M_2 + M_3 - M_1}{M_2 + M_3} \leq 1 - \frac{\sqrt{M_2^2 + M_3^2}}{M_2 + M_3}$$

$$= 1 - \frac{\sqrt{1 + (M_3/M_2)^2}}{1 + (M_3/M_2)}$$

$$\leq 1 - \frac{1}{\sqrt{2}} \approx 29\%$$

& Saturated when $M_2 = M_3$.

5 LIGO EVENT

$M_1 = 36\, M_\odot$
$M_2 = 29\, M_\odot$

$M_3 = 62\, M_\odot$

$\frac{a_3}{M_3} = \left(\frac{J_3}{M_3^2}\right) = (0.68)$ ie. 68% of extremal value.

$$\text{Area} = 4\pi\left[2M^2 + 2\left[M^4 - J^2\right]^{1/2}\right]$$
$$= 4\pi\left[2m^2 + 2M^2\left[1 - \left(\frac{a}{M}\right)^2\right]^{1/2}\right]$$

is $\frac{A_1}{4\pi} + \frac{A_2}{4\pi} \leq \frac{A_3}{4\pi}$

the biggest the left hand side could be is if the initial black holes are both Schwarzschild each with $a_i = 0$. If we assume this, then

$$\frac{A_1}{4\pi} + \frac{A_2}{4\pi} = 4M_1^2 + 4M_2^2 = 8548\, M_\odot^2$$

$$\& \ \frac{A_3}{4\pi} = 4\pi\left[2M_3^2 + 2M_3^2\left[1 - (0.68)^2\right]^{1/2}\right]$$
$$= 13\,324\, M_\odot^2$$

& Area law satisfied

$3M_\odot$ released which gives energy $= 3M_\odot c^2$
$= 5.4 \times 10^{47}$ Joules

Coalescence time: $\sim 15\, ms^{-1}$

$\Rightarrow$ Peak power $\sim 3.6 \times 10^{49}$ Watts.

6

Initial state is Kerr black hole with Mass M & rotation parameter a with $a = J/M$ & $a > 0$
After absorbing a particle it settles down to another stationary (Kerr) black hole

First Law: $dM = \frac{\kappa}{8\pi} dA + \Omega_H dJ$

$$\kappa > 0 \quad , \quad \Omega_H = \frac{J}{2M} \frac{1}{M^2 + \sqrt{M^4 - J^2}} > 0$$

Second Law says

$$dA \geq 0$$

$$\frac{\kappa}{8\pi} dA = dM - \Omega_H dJ$$

In the Penrose process, the black hole loses mass
So $dM < 0 \Rightarrow \Omega_H dJ < 0$ for $dA \geq 0$
$\Rightarrow dJ < 0$

The angular momentum must decrease.

Now consider initial black hole with mass, M_0 & angular momentum J_0 to start.
Initially

$$A = 4\pi (r_+^2 + a_0^2) = 8\pi \left(M_0^2 + \sqrt{M_0^4 - J_0^2} \right)$$

The most efficient process has $dA = 0$ which means $dM = \Omega_H dJ$

The process will cease when $J = 0$
At this stage we will have schwartzschild black hole with mass M_1 & are the same as we

started:

$\cancel{A_0 =}$

$\cancel{A_1 = 16\pi M_1^2 = A_0 = 128\pi}$

$$A_0 = A_1$$

$$\Rightarrow \quad 8\pi\left(M_0^2 + \sqrt{M_0^4 - J_0^2}\right) = 16\pi M_1^2$$

$$\Rightarrow \quad M_1 = \frac{1}{\sqrt{2}}\left[M_0^2 + \sqrt{M_0^4 - J_0^2}\right]^{1/2}$$

the maximum possible energy that can be extracted is

$$M_0 - M_1 = M_0 - \frac{1}{\sqrt{2}}\left(M_0^2 + \sqrt{M_0^4 - J_0^2}\right)^{1/2}$$

Efficiency

$$\eta = \frac{M_0 - M_1}{M_0} = 1 - \frac{1}{\sqrt{2}}\left(1 + \sqrt{1 - \left(\frac{J_0^2}{M_0^4}\right)}\right)^{1/2}$$

the maximum efficiency is achieved if the initial black hole is extremal with $J_0^{\cancel{2}} = M_0^2$ & then

$$\eta_{max} = 1 - 1/\sqrt{2} \approx 29\%.$$

7

a) i) $\xi \wedge d(A\cdot B) = 0 \iff \xi_{[\mu} \nabla_{\nu]}(A\cdot B) = 0$

$\iff \xi_{[\mu} \nabla_{\nu]}\left(A_{\rho_1 \cdots \rho_p} B^{\rho_1 \cdots \rho_p}\right) = 0$

$\iff \left(\xi_{[\mu} \nabla_{\nu]} A_{\rho_1 \cdots \rho_p}\right) B^{\rho_1 \cdots \rho_p} + A_{\rho_1 \cdots \rho_p}\, \xi_{[\mu}\nabla_{\nu]} B^{\rho_1 \cdots \rho_p} = 0$

but on N $\quad A_{\rho_1 \cdots \rho_p} = 0$

$\Rightarrow \left(\xi_{[\mu} \nabla_{\nu]} A_{\rho_1 \cdots \rho_p}\right) B^{\rho_1 \cdots \rho_p} = 0$ on N

this is true for any $B^{\rho_1 \cdots \rho_p}$

$\Rightarrow \xi_{[\mu} \nabla_{\nu]} A_{\rho_1 \cdots \rho_p} = 0$ on N

ii) $A_\mu = \xi^\nu \nabla_\nu \xi_\mu - \kappa \xi_\mu$

Note $A_\mu = 0$ on N by definition of κ

$\xi_{[\sigma} \nabla_{\rho]} A_\mu = 0$ on N

$\iff \left(\xi_{[\sigma} \nabla_{\rho]} \xi^\nu\right)\left(\nabla_\nu \xi_\mu\right) + \xi^\nu \xi_{[\sigma} \nabla_{\rho]} \nabla_\nu \xi_\mu$

$= \left(\xi_{[\sigma} \nabla_{\rho]} \kappa\right) \xi_\mu + \kappa\, \xi_{[\sigma} \nabla_{\rho]} \xi_\mu$ on N

$\iff \left(\xi_{[\sigma} \nabla_{\rho]} \xi^\nu\right)\left(\nabla_\nu \xi_\mu\right) + \xi^\nu \cancel{\kappa}\, \xi_{[\sigma} R^{\delta}{}_{\rho]\nu\mu} \xi_\delta$

$= \xi_\mu \left(\xi_{[\sigma} \nabla_{\rho]} \kappa\right) + \kappa\, \xi_{[\sigma} \nabla_{\rho]} \xi_\mu$ on N

b) Frobenius $\Rightarrow \xi_{[\rho}\nabla_{\mu}\xi_{\nu]} = 0$ on N

$\Rightarrow \xi_\rho \nabla_\mu \xi_\nu + \xi_\mu \nabla_\nu \xi_\rho + \xi_\nu \nabla_\rho \xi_\mu = 0$ "

since $\nabla_\mu \xi_\nu = \nabla_{[\mu}\xi_{\nu]}$

$\Rightarrow \xi_\rho \nabla_\mu \xi_\nu + \xi_\mu \nabla_\nu \xi_\rho - \xi_\nu \nabla_\mu \xi_\rho = 0$ "

$\Rightarrow \xi_\rho \nabla_\mu \xi_\nu + 2\xi_{[\mu}\nabla_{\nu]}\xi_\rho = 0$ on N

Hence

$(\xi_{[\sigma}\nabla_{\rho]}\xi^\nu)\nabla_\nu \xi_\mu = -\frac{1}{2}(\xi^\nu \nabla_\sigma \xi_\rho)\nabla_\nu \xi_\mu$ on N

$= -\frac{1}{2}(\nabla_\sigma \xi_\rho)(\xi^\nu \nabla_\nu \xi_\mu)$ "

$= -\frac{\kappa}{2}(\nabla_\sigma \xi_\rho)\xi_\mu$ "

$= -\frac{\kappa}{2}(-2\xi_{[\sigma}\nabla_{\rho]}\xi_\mu)$ "

$= \kappa\, \xi_{[\sigma}\nabla_{\rho]}\xi_\mu$ on N

and equation (3) follows immediately from (1)

c) Have shown $\xi_\rho \nabla_\mu \xi_\nu + 2\xi_{[\mu}\nabla_{\nu]}\xi_\rho = 0$ on N

$\Rightarrow \underbrace{(\xi_{[\sigma}\nabla_{\delta]}\xi_\rho)}_{1}(\nabla_\mu \xi_\nu) + \xi_\rho(\xi_{[\sigma}\nabla_{\delta]}\nabla_\mu \xi_\nu)$

$+ 2\underbrace{(\xi_{[\sigma}\nabla_{\delta]}\xi_{[\mu})}_{2}\nabla_{\nu]}\xi_\rho + 2\xi_{[\mu}\xi_{|[\sigma}\nabla_{\delta]|}\nabla_{\nu]}\xi_\rho = 0$

bars on 2 indices.

on N

term ①: $-\frac{1}{2}(\xi_\rho \nabla_\sigma \xi_\delta)(\nabla_\mu \xi_\nu)$ using equation (2)

term ②:

$$(\xi_{[\sigma}\nabla_{\delta]}\xi_\mu)\nabla_\nu\xi_\rho - \xi_{[\sigma}\nabla_{\delta]}\xi_\nu\nabla_\mu\xi_\rho$$

$$= -\frac{1}{2}\xi_\mu(\nabla_\sigma\xi_\delta)\nabla_\nu\xi_\rho + \frac{1}{2}\xi_\nu(\nabla_\sigma\xi_\delta)\nabla_\mu\xi_\rho$$

$$= (\nabla_\sigma\xi_\delta)\left(-\frac{1}{2}\right)(\xi_\mu\nabla_\nu\xi_\rho - \xi_\nu\nabla_\mu\xi_\rho)$$

$$= -(\nabla_\sigma\xi_\delta)\,\xi_{[\mu}\nabla_{\nu]}\xi_\rho$$

$$= \frac{1}{2}(\nabla_\sigma\xi_\delta)\,\xi_\rho\nabla_\mu\xi_\nu$$

where we used equation (2) 3 times.

this cancels with term ① & so we are left with

$$\xi_\rho\,\xi_{[\sigma}\nabla_{\delta]}\nabla_\mu\xi_\nu = -2\left(\xi_{[\sigma}\nabla_{\delta]}\nabla_{[\nu}\xi_{|\rho|}\right)\xi_{\mu]}$$

where we just moved the position of ξ_μ in the expression.

$$\Rightarrow \xi_\rho\,\xi_{[\sigma}R^{\gamma}{}_{\delta]\mu\nu}\,\xi_\gamma = -2\,\xi_\sigma R^{\gamma}{}_{\delta\nu\rho}\,\xi_\gamma\,\xi_\mu$$

$$= +2\,\xi_\sigma R^{\gamma}{}_{\delta\rho\nu}\,\xi_\mu\,\xi_\gamma$$

$$= 2\,\xi_{[\sigma}R^{\gamma}{}_{\delta]\rho[\nu}\,\xi_{\mu]}\,\xi_\gamma$$

d)

LHS

$$\frac{1}{2}\left(\xi_\rho \xi_\sigma R^\gamma{}_{\delta\mu\nu}\xi_\gamma - \xi_\rho \xi_\delta R^\gamma{}_{\sigma\mu\nu}\xi_\gamma\right)$$

contract on ρ & δ

$$\frac{1}{2}\left(\underbrace{\xi^\rho \xi_\sigma \xi^\gamma}_{\substack{\text{symmetric}\\ \text{on } \rho \,\&\, \gamma}} \underbrace{R_{\gamma\rho}}_{\substack{\text{a-symmetric}\\ \text{on}\\ \rho \,\&\, \gamma}}{}_{\mu\nu} - \underbrace{\xi^\rho \xi_\rho}_{\substack{\downarrow\\ 0}} R^\gamma{}_{\sigma\mu\nu}\xi_\gamma\right)$$

since null $\underline{\text{on } N}$

$= 0$. on N

RHS

$$\frac{1}{2}\left(\xi_\sigma R^\gamma{}_{\delta\rho[\nu}\xi_{\mu]} - \xi_\delta R^\gamma{}_{\sigma\rho[\nu}\xi_{\mu]}\right)\xi_\gamma$$

contract on ρ & δ

$$\Rightarrow -\xi_\gamma \xi_\sigma R^\gamma{}_{[\nu}\xi_{\mu]} - \underbrace{\xi^\rho R^\gamma{}_{\sigma\rho[\nu}\xi_{\mu]}\xi_\gamma}$$

$$\underbrace{-\frac{1}{2}\xi^\rho \xi^\gamma\left[R_{\gamma\sigma\rho\nu}\xi_\mu - R_{\gamma\sigma\rho\mu}\xi_\nu\right]}$$

using $R_{[\mu\nu\rho]\sigma}=0$ $\downarrow$

$$\underbrace{-\frac{1}{2}\xi^\rho \xi^\gamma\left[-\left(R_{\rho\gamma\sigma\nu}+R_{\sigma\rho\gamma\nu}\right)\xi_\mu + \left(R_{\sigma\rho\gamma\mu}+R_{\rho\gamma\sigma\mu}\right)\xi_\nu\right]}$$

$$\underbrace{+\frac{1}{2}\xi^\rho \xi^\gamma R_{\sigma\rho\gamma\nu}\xi_\mu - \frac{1}{2}\xi^\rho \xi^\gamma R_{\sigma\rho\gamma\mu}\xi_\nu}$$

$$\frac{1}{2}\xi^\rho \xi^\gamma R_{\sigma\rho\gamma[\nu}\xi_{\mu]}$$

Since LHS $= 0 \Rightarrow$ RHS $= 0$ on N

$$\Rightarrow 0 = -\xi_{[\mu} R_{\nu]}{}^{\gamma} \xi_{\gamma} \xi_{\sigma} + \xi_{[\mu} R^{\gamma}{}_{\nu]\sigma\rho} \xi^{\rho} \xi_{\gamma}$$

$$\Rightarrow -\xi_{[\mu} R_{\nu]}{}^{\gamma} \xi_{\gamma} \xi_{\sigma} = \xi_{[\mu} R^{\gamma}{}_{\nu]\rho\sigma} \xi^{\rho} \xi_{\gamma} \quad \underline{\text{on } N}$$

Relabel indices in this
$\mu \to \sigma$
$\nu \to \rho$
$\gamma \to \delta$
$\rho \to \nu$
$\sigma \to \mu$

$$\Rightarrow -\xi_{[\sigma} R_{\rho]}{}^{\delta} \xi_{\delta} \xi_{\mu} = \xi_{[\sigma} R^{\delta}{}_{\rho]\nu\mu} \xi^{\nu} \xi_{\delta}$$

sub into (3)

$$\Rightarrow -\xi_{[\sigma} R_{\rho]}{}^{\delta} \xi_{\delta} \xi_{\mu} = \xi_{\mu} \xi_{[\sigma} \nabla_{\rho]} k$$

& since $\xi_{\mu} \neq 0$

$$\Rightarrow -\xi_{[\sigma} R_{\rho]}{}^{\delta} \xi_{\delta} = \xi_{[\sigma} \nabla_{\rho]} k \quad \underline{\text{On } N}$$