IMPERIAL COLLEGE

MSc EXAMINATION

Black Holes

Time Allowed:	3 hours
Date:	16th May 2008
Time:	2pm–5pm
Instructions:	**Answer Question 1 (40%) and TWO out of Questions 2, 3 and 4 (30% each).**
Marks:	The marks shown are indicative of those the examiners intend to assign.

DO NOT TURN TO THE FIRST PAGE OF THE QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY THE INVIGILATOR

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Question 1.

(a) (*10 marks*) Write down the Schwarzschild metric for a black hole of mass M in Schwarzschild coordinates (t, r, θ, ϕ). Derive the ingoing and outgoing null Eddington-Finkelstein (EF) coordinates, v and u respectively and calculate the Schwarzschild metric in terms of (v, r, θ, ϕ) and (u, r, θ, ϕ).

What is a coordinate singularity? Prove that $r = 2M$ is a coordinate singularity. How do we know that $r = 0$ is not a coordinate singularity?

(b) (*15 marks*) Draw the Carter-Penrose diagram for the spacetime of a spherically symmetric star that collapses to form a black hole. Label *all* its features and boundaries and explain the *physical* significance of all these features and boundaries. Draw an ingoing and an outgoing null geodesic.

(c) (*7 marks*) On the Carter-Penrose diagram from part (b) (or on a separate copy diagram) draw the worldline of an observer who remains at a fixed radial coordinate far from the black hole and the worldline of a spaceship that falls towards and into the black hole. Use the diagram to explain qualitatively what the observer far from the black holes sees as the spaceship falls towards and into the hole.

Explain qualitatively why it is impossible to avoid the singularity once one has fallen across the horizon.

(d) (*8 marks*) Draw the Carter-Penrose diagram of the maximally extended charged Reissner-Nordstrom (RN) black hole. Label all its features and boundaries.

Use your diagram to explain why radiation becomes infinitely blue-shifted at the inner horizon of the RN spacetime. Could the maximally extended RN solution describe a physical black hole? Explain your answer.

Question 2.

Consider the Kerr metric for a rotating black hole of mass M and angular momentum $J = Ma$,

$$ds^2 = -\frac{(\Delta - a^2 \sin^2 \theta)}{\Sigma} dt^2 - 2a \sin^2 \theta \frac{(r^2 + a^2 - \Delta)}{\Sigma} dt d\phi$$
$$+ \frac{((r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta)}{\Sigma} \sin^2 \theta d\phi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2$$

where

$$\Sigma = r^2 + a^2 \cos^2 \theta$$
$$\Delta = r^2 - 2Mr + a^2$$

and $M > a$.

(a) (*8 marks*) Where is the event horizon? Show that the Killing vector $k = \frac{\partial}{\partial t}$ is timelike at large r. Show also that k becomes spacelike in a region, called the ergoregion, outside the horizon. Derive the equation for the boundary of the ergoregion and sketch the ergoregion and show its relation to the event horizon.

(b) (*5 marks*)

Consider a spaceship, not necessarily freely falling, orbiting the black hole at a fixed radius in the equatorial plane ($\theta = \frac{\pi}{2}$). Let $\Omega = \frac{d\phi}{dt}$ be its "angular velocity with respect to an observer at infinity." Show that when the orbit lies inside the ergoregion but outside the horizon, Ω cannot be zero.

(c) (*12 marks*) Calculate the area of the event horizon.

Show that, in a process in which the mass and angular momentum both change, the Second Law of Black Hole Mechanics allows the mass of the black hole to decrease if $J \neq 0$. What is the limiting minimum mass that the resulting black hole can have? Deduce the limiting maximum amount of energy that can be extracted from the black hole.

(d) (*5 marks*) Describe the Penrose process for extraction of energy from a rotating black hole.

Question 3.

(a) (*8 marks*) Describe how a free massive real scalar field is quantised in a stationary spacetime with a Cauchy surface. Define the vacuum state.

(b) (*8 marks*) Suppose a spacetime (M, g) has the form of a sandwich: there are two non-intersecting Cauchy surfaces, Σ_1 and Σ_2 such that the spacetime is stationary to the past of Σ_1 and to the future of Σ_2, and in between it is time dependent.

Describe how this may lead to particle production. Derive the formula, in terms of Bogoliubov coefficients, for the expectation value of the number of particles in a certain mode as measured by an observer in the far future, in the vacuum state as defined by an observer in the far past?

Why is this sandwich spacetime example relevant to the derivation of Hawking radiation in the spacetime of gravitational collapse to a black hole?

(c) (*10 marks*) The Hawking temperature of a Reissner-Nordstrom black hole of mass M and charge Q is

$$T_H = \frac{1}{4\pi r_+^2}(r_+ - r_-) \ , \tag{1}$$

where $r_\pm = M \pm \sqrt{M^2 - Q^2}$.

Consider such a black hole in thermal equilibrium with an infinite reservoir of radiation at temperature T_H. Explain why the black hole is unstable if $Q = 0$.

The specific heat of a charged black hole of mass M and *fixed* charge Q is

$$C = T_H \frac{\partial S_{BH}}{\partial T_H} |_Q$$

Assuming that the entropy of a black hole is given by $S_{BH} = A/4$ where A is the area of the event horizon, show that the specific heat of a Reissner-Nordstrom black hole is

$$C = \frac{2S_{BH}\sqrt{M^2 - Q^2}}{M - 2\sqrt{M^2 - Q^2}} \ .$$

(d) (*4 marks*) Find the range of values of M for fixed charge Q for which the black hole is in equilibrium with an infinite reservoir of radiation at its Hawking temperature.

Question 4.

The metric for a spherically symmetric black hole with mass parameter μ in 5 dimensions is the 5-dimensional Schwarzschild metric:

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega_3^2$$

where

$$f(r) = 1 - \frac{\mu}{r^2} ,$$

and $d\Omega_3^2$ is the round metric on the 3-sphere

$$d\Omega_3^2 = d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2) .$$

For this spacetime:

(a) (*5 marks*) Find the ingoing and outgoing Eddington-Finkelstein coordinates, v and u, respectively. Show that $r = \mu^{\frac{1}{2}}$ is a coordinate singularity.

(b) (*10 marks*) An observer falls freely from rest at radial coordinate $r = R_{max}$, radially into the black hole. Calculate the proper time it takes for that observer to reach the singularity at $r = 0$.

(c) (*10 marks*)

Prove that $r = \mu^{\frac{1}{2}}$ is a null hypersurface and a Killing horizon.

(d) (*5 marks*)

Calculate the surface gravity, κ, of the black hole horizon. What is the Hawking temperature of the black hole?

You may assume that all formulae for 4-dimensional black holes hold for 5-dimensional black holes.