Black Holes - Jerome Gauntlett

Notes on differential forms

Let M be an n-dimensional manifold. A p-form is a tensor of type $(0,p)$ that is totally anti-symmetric:

$$A_{\mu_1\dots\mu_p} = A_{[\mu_1\dots\mu_p]} \tag{1}$$

A 0-form is simply a function and a 1-form is a co-vector. We must have $p \leq n$.

Wedge product: Consider a p-form, A, and a q-form, B. We define the $(p+q)$ form $A \wedge B$ via

$$(A\wedge B)_{\mu_1\dots\mu_p\nu_1\dots\nu_q} = \frac{(p+q)!}{p!q!} A_{[\mu_1\dots\mu_p}B_{\nu_1\dots\nu_q]} \tag{2}$$

We immediately have

$$A\wedge B = (-1)^{pq} B\wedge A \tag{3}$$

and a corollary is $A \wedge A = 0$ if p is odd.

In a given set of coordinates we can consider a basis for one-forms dx^μ. Using the wedge product we can obtain a basis for p-forms and we can write

$$A = \frac{1}{p!} A_{\mu_1\dots\mu_p} dx^{\mu_1}\wedge\dots\wedge dx^{\mu_p} \tag{4}$$

Exterior derivative: This is a derivative operation that takes a p-form, A, to a $p+1$ form, dA whose components are

$$(dA)_{\nu_1\dots\nu_{p+1}} = (p+1)\partial_{[\nu_1}A_{\nu_2\dots\nu_{p+1}]} \tag{5}$$

The factor $(p+1)$ corresponds to the fact that we can think of d as the operation $dx^\rho \wedge \partial_\rho$ in the sense:

$$\begin{aligned} dA &= \frac{1}{p!}\partial_\rho A_{\mu_1\dots\mu_p} dx^\rho\wedge dx^{\mu_1}\wedge\dots\wedge dx^{\mu_p} \\ &= \frac{1}{p!}\partial_{[\rho} A_{\mu_1\dots\mu_p]} dx^\rho\wedge dx^{\mu_1}\wedge\dots\wedge dx^{\mu_p} \\ &= \frac{1}{(p+1)!}\left((p+1)\partial_{[\nu_1}A_{\nu_2\dots\mu_{p+1}]}\right) dx^{\nu_1}\wedge dx^{\nu_2}\wedge\dots\wedge dx^{\nu_{p+1}} \end{aligned} \tag{6}$$

If A is a p-form and B is a q-form we have the Leibniz rule:

$$d(A\wedge B) = (dA)\wedge B + (-1)^p A\wedge(dB) \tag{7}$$

We also have the important property that

$$d^2 = 0 \tag{8}$$

We say a form A is "closed" if $dA = 0$. We say a form is "exact" if $A = dB$. Clearly an exact form is closed from (8), but the converse is NOT true in general (and leads to study of cohomology).

We have not yet assumed that we have a metric defined on M. Let us now do so, with components $g_{\mu\nu}$. This gives a unique Levi-Civita covariant derivative ∇. It is useful to note that we can write

$$(dA)_{\nu_1\dots\nu_{p+1}} = (p+1)\nabla_{[\nu_1} A_{\nu_2\dots\nu_{p+1}]} \tag{9}$$

Volume n-form: The metric allows us to define a volume n-form, ϵ, via

$$\begin{aligned}\epsilon &= \sqrt{|g|}dx^1 \wedge \cdots \wedge dx^n \\ &= \frac{1}{n!}\sqrt{|g|}\epsilon(\mu_1,\dots,\mu_n)dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_n}\,,\end{aligned} \tag{10}$$

and thus

$$\epsilon_{\mu_1\dots\mu_n} = \sqrt{|g|}\epsilon(\mu_1,\dots,\mu_n)\,, \tag{11}$$

where $\epsilon(\mu_1,\dots,\mu_n)$ is the object (not the components of a tensor!) which equals $+1$ if $(\mu_1,\dots,\mu_n)$ is an even permutation of $(1,2,\dots,n)$, equals -1 if $(\mu_1,\dots,\mu_n)$ is an odd permutation of $(1,2,\dots,n)$ and equals zero if any index is repeated. Note that this definition is coordinate independent because the transformation of $\sqrt{|g|}$ and $dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_n}$ compensate each other.

We can raise indices using the metric to get

$$\begin{aligned}\epsilon^{\mu_1\dots\mu_n} &= g^{\mu_1\nu_1}\dots g^{\mu_n\nu_n}\epsilon_{\nu_1\dots\nu_n} \\ &= g^{\mu_1\nu_1}\dots g^{\mu_n\nu_n}\sqrt{|g|}\epsilon(\nu_1,\dots,\nu_n) \\ &= det(g)^{-1}\epsilon(\mu_1,\dots,\mu_n)\sqrt{|g|} \\ &= \pm\frac{1}{\sqrt{|g|}}\epsilon(\mu_1,\dots,\mu_n)\end{aligned} \tag{12}$$

where the upper plus sign arises when we have Riemannian geometry and the lower minus sign arises when we have Lorentzian geometry. A useful fact is that

$$\epsilon^{\mu_1\dots\mu_p\nu_{p+1}\dots\nu_n}\epsilon_{\rho_1\dots\rho_p\nu_{p+1}\dots\nu_n} = \pm p!(n-p)\delta^{\mu_1\dots\mu_p}_{\rho_1\dots\rho_p} \tag{13}$$

where $\delta^{\mu_1\dots\mu_p}_{\rho_1\dots\rho_p} \equiv \delta^{\mu_1}_{[\rho_1}\dots\delta^{\mu_p}_{\rho_p]}$. We also have $\nabla_\rho\epsilon_{\mu_1\dots\mu_n} = 0$.

Hodge dual: With a metric and hence a volume form, given a p-form A, we can define an $(n-p)$-form $*A$, the Hodge dual, via[1]

$$*A_{\mu_1\dots\mu_{n-p}} = \frac{1}{p!}\epsilon_{\mu_1\dots\mu_{n-p}\nu_1\dots\nu_p}A^{\nu_1\dots\nu_p} \tag{14}$$

One can show that

$$*(*A) = \pm(-1)^{p(n-p)}A \tag{15}$$

and also

$$(*d*A)_{\mu_1\dots\mu_{p-1}} = \pm(-1)^{p(n-p)}\nabla^\nu A_{\mu_1\dots\mu_{p-1}\nu} \tag{16}$$

[1]Some other people define $A_{\mu_1\dots\mu_{n-p}} = \frac{1}{p!}\epsilon_{\nu_1\dots\nu_p\mu_1\dots\mu_{n-p}}A^{\nu_1\dots\nu_p}$ which leads to some different signs in places.

If we denote $\mathbb{1}$ as the trivial 0-form (function) which is just 1 everywhere we have $*\mathbb{1} = \epsilon$.

Integration: We first define the integral of an n-form A over an n-dimensional manifold M. We can write $A = \frac{1}{n!}A_{\mu_1\dots\mu_n}dx^{\mu_1}\wedge\dots\wedge dx^{\mu_n} = A_{1\dots n}dx^1\wedge\dots\wedge dx^n$ and we define

$$\int_M A \equiv \int dx^1\dots dx^n A_{1\dots n} = \int d^n x A_{1\dots n} \tag{17}$$

where the right hand side is usual integration. One can show that this is a coordinate independent definition. Note that this definition did not require a metric.

Suppose now we have a metric and hence a volume form ϵ. We can then define the volume of M to be $Vol(M) = \int_M \epsilon$ (which might be infinite). We can also use ϵ to define the integral of a function f on M:

$$\int_M f \equiv \int_M f\epsilon = \int d^n x\sqrt{|g|}f \tag{18}$$

and the latter expression should be familiar.

Stokes Theorem: Let M be an oriented n-dimensional manifold with boundary ∂M then for an $(n-1)$-dimensional form A we have

$$\int_M dA = \int_{\partial M} A \tag{19}$$

Notice that this theorem does not require a metric.

Gauss Law or Divergence Theorem: Let M be an n-dimensional manifold with boundary ∂M, metric $g_{\mu\nu}$ and volume form ϵ. Let V be a one-form, then $A = *V$ is an $(n-1)$-form and Stokes Theorem says

$$\int_M dA = \int_{\partial M} A = \int_{\partial M} *V \tag{20}$$

We now want to reexpress the left and right hand sides. From (16) we have

$$\begin{aligned} *d*V &= \pm(-1)^{n-1}(\nabla^\mu V_\mu)\mathbb{1} \\ \Rightarrow \pm d*V &= \pm(-1)^{n-1}(\nabla^\mu V_\mu)*(\mathbb{1}) \\ \Rightarrow d*V &= (-1)^{n-1}(\nabla^\mu V_\mu)\epsilon \end{aligned} \tag{21}$$

where $\mathbb{1}$ is the trivial 0-form (function) which is 1 everywhere and as noted above $*\mathbb{1} = \epsilon$. The left hand side of (20) is thus

$$\int_M dA = (-1)^{n-1}\int_M d^n x\sqrt{|g|}\nabla_\mu V^\mu \tag{22}$$

We now consider the right hand side of (20). We first assume that ∂M is specified by an outward pointing normal vector n^μ, with $n^2 = n^\mu n^\nu g_{\mu\nu} = \mp 1$ depending on whether the normal is time-like or spacelike. The induced metric on ∂M is then given by $h_{\mu\nu} = g_{\mu\nu} \pm n_\mu n_\nu$ (note that $h_{\mu\nu}n^\nu = 0$). This induced metric h can be used to define a volume $(n-1)$-form, $\bar{\epsilon}$ on ∂M. We then calculate

$$*V = \frac{1}{(n-1)!}\epsilon_{\mu_1\dots\mu_{n-1}}{}^{\nu}V_\nu dx^{\mu_1}\wedge\dots\wedge dx^{\mu_{n-1}}$$

$$
\begin{aligned}
&= (-1)^{n-1}\frac{1}{(n-1)!}\epsilon^{\nu}{}_{\mu_1\dots\mu_{n-1}}V_\nu dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_{n-1}} \\
&= (-1)^{n-1}\frac{1}{(n-1)!}(n^\nu V_\nu)\bar{\epsilon}_{a_1\dots a_{n-1}}dx^{a_1}\wedge\cdots\wedge dx^{a_{n-1}} \\
&= (-1)^{n-1}(n^\nu V_\nu)\bar{\epsilon}
\end{aligned} \tag{23}
$$

where x^a are coordinates on ∂M. To get from the second to the third line is a bit fiddly: one can use coordinates so that the boundary is defined by $x^1 = 0$, the normal n as a one-form (i.e $n_\mu = g_{\mu\nu}n^\nu$) is proportional to dx^1 and the coordinates on ∂M are $x^a = (x^2, \dots, x^n)$. We can now use this in the right hand side of (20) and combining with (22) we finally have the result

$$
\int_M d^n x\sqrt{|g|}\nabla_\mu V^\mu = \int_{\partial M} d^{n-1}x\sqrt{|h|}n^\mu V_\mu \tag{24}
$$

which you have been using in some form in your studies for a while.