Imperial College London

MSc EXAMINATION May 2014

This paper is also taken for the relevant Examination for the Associateship

COSMOLOGY

For Students in Quantum Fields and Fundamental Forces

Friday 9^{th} May 2014: 14:00 to 17:00

Answer **THREE** *out of the following four questions.*

If answers to all four questions are handed in **only** *the first three will be marked.*

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the 3 answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in 3 answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

Conventions:

We use conventions as in lectures. In particular we take $(-, +, +, +)$ signature and choose units so that $\hbar = 1$ and $c = 1$.

You may find the following useful:

The Christoffel symbol is defined as,

$$\Gamma^{\mu}{}_{\alpha\beta} \equiv \frac{1}{2} g^{\mu\nu} \left(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta} \right) \ .$$

The covariant derivative of a vector field is,

$$\nabla_\mu v^\nu \equiv \partial_\mu v^\nu + \Gamma^{\nu}{}_{\mu\alpha} v^\alpha \ .$$

For the flat FRW metric $ds^2 = -dt^2 + a^2(t)\delta_{ij} dx^i dx^j$, the non-zero Christoffel components are,

$$\Gamma^{t}{}_{ij} = a\dot{a}\,\delta_{ij} \ , \qquad \Gamma^{i}{}_{tj} = \Gamma^{i}{}_{jt} = \frac{\dot{a}}{a}\delta_{ij}$$

and the Friedmann equation is $H^2 = \frac{8\pi G}{3}\rho$.

Go to the next page for questions

1. (i) Write the stress tensor for a perfect fluid in a general spacetime in terms of energy density ρ, pressure P and local 4-velocity u^μ. Write the components of the stress tensor for a homogeneous isotropic fluid in a flat FRW spacetime. Explicitly derive the fluid equation of motion,

$$\dot{\rho} + 3H\left(\rho + P\right) = 0$$

by computing the components of the stress tensor conservation equation.

[5 marks]

(ii) Consider a homogeneous isotropic fluid in flat FRW spacetime with equation of state $P = w\rho$ for constant w. Solve the Friedmann and fluid equations to find the scale factor $a(t)$ as a function of time t. *Derive* the values of w for which this cosmology is accelerating.

[4 marks]

(iii) Compute the observer particle (Hubble) horizon size, d_H, for this solution. What is the condition on w so that the particle horizon is finite?

[4 marks]

(iv) Consider a flat FRW universe filled with N homogeneous isotropic fluid components, which do not interact with each other, with densities ρ_i and equation of state parameters w_i with $i = 1, \ldots, N$. At a time $t = t_0$ a comoving observer sees a comoving source at a redshift Z. Define the *luminosity distance*, d_L, of the source as seen by the observer. Show that this can be written as the integral,

$$d_L = \frac{(1+Z)}{H(t_0)} \int_{\frac{1}{(1+Z)}}^{1} \frac{dx}{x^2 \sqrt{\sum_{i=1}^{N} \Omega_i x^{-3(1+w_i)}}}$$

where as usual $\Omega_i = \rho_i / \rho_{critical}$ at the time $t = t_0$.

[7 marks]

[Total 20 marks]

2. (i) Consider a homogeneous isotropic ideal gas. Recall that,

$$\rho = \int_0^\infty dp\, 4\pi p^2 n(p) E \ , \quad P = \int_0^\infty dp\, 4\pi p^2 n(p) \frac{p^2}{3E} \ .$$

State the density distribution function $n(p)$ for a Bose or Fermi gas made up of particles with mass m, chemical potential μ, and g degrees of freedom.

[4 marks]

(ii) Show that in the highly relativistic limit (where we may ignore mass and chemical potential) the energy density for a boson and fermion are,

$$\rho_{boson} = \frac{1}{2} g a_B T^4 \ , \qquad \rho_{fermion} = \frac{7}{16} g a_B T^4$$

with the pressure in both cases given by $P = \rho/3$, where $a_B = \pi^2 k^4/15$ is the radiation constant (in units $c = \hbar = 1$). You may use the integral,

$$\int_0^\infty dx \frac{x^3}{e^x \pm 1} = \frac{15 \mp 1}{240}\pi^4 \ .$$

[5 marks]

(iii) Use the first law of thermodynamics to derive that the entropy density s for a highly relativistic species with no chemical potential is,

$$s \ = \ \frac{4\rho}{3T} \ .$$

[3 marks]

(iv) Explain why the temperatures of the cosmic photon background T_γ and neutrino background T_ν are related as

$$\frac{T_\gamma}{T_\nu} \simeq \left(\frac{11}{4}\right)^{\frac{1}{3}} \ .$$

In particular you should derive the numerical factor in the above expression. (You may assume that when a relativistic species decouples it maintains a Bose/Fermi distribution with effective temperature $T_{eff} \propto 1/a$).

[8 marks]

[Total 20 marks]

3. (i) Consider a space-time with metric $ds^2 = \tilde{g}_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu = -dt^2 + g_{ij}(t,x) dx^i dx^j$ with $\tilde{x}^\mu = (t, x^i)$. Write down the Liouville equation for the phase space distribution function of free particles $n(t, x^i, p_j)$. Show that for free particles following trajectories $x^i(t)$ with 4-momenta $p^\mu = (p^0, p^i)$ then,

$$\frac{dx^i}{dt} = \frac{p^i}{p^0} \, .$$

[3 marks]

(ii) Use the fact that certain Christoffel components for this space-time are,

$$\tilde{\Gamma}^i{}_{tt} = 0 \, , \quad \tilde{\Gamma}^i{}_{tk} = \frac{1}{2} g^{ij} \partial_t g_{jk} \, , \quad \tilde{\Gamma}^i{}_{kl} = \Gamma^i{}_{kl}$$

where $\Gamma^i{}_{kl}$ are Christoffel components for the spatial metric g_{ij}, to show that for free particles,

$$\frac{dp_i}{dt} = \frac{1}{2} \frac{1}{p^0} p^j p^k \partial_i g_{jk} \, .$$

[7 marks]

(iii) Show that for a homogeneous isotropic gas in a flat FRW space-time, so $g_{ij} = a^2(t)\delta_{ij}$, the Boltzmann equation is,

$$\left(\left. \frac{\partial}{\partial t} \right|_p - Hp \left. \frac{\partial}{\partial p} \right|_t \right) n(t,p) = C$$

where $p = \sqrt{g^{ij} p_i p_j}$ and C is the collision term representing particle interactions.

[4 marks]

(iv) Recall the particle number density for flat FRW is $N(t) = \int_0^\infty dp \, 4\pi p^2 n(t,p)$. Show that $N(t)$ obeys,

$$\frac{1}{a^3} \frac{d\left(a^3 N\right)}{dt} = \int_0^\infty dp \, 4\pi p^2 C$$

[3 marks]

(v) Consider photons which are kept in thermal equilibrium in the early universe by interactions with matter. Show that if these interactions quickly turn off as the universe expands and cools, the free streaming photons still maintain a Bose distribution but with an effective temperature that redshifts as $T_{eff} \propto 1/a$.

[3 marks]

[Total 20 marks]

4. (i) Consider a real scalar field ϕ with potential $V(\phi)$ and governed by the equation of motion $\nabla^2\phi = V'(\phi)$. For the homogeneous isotropic flat FRW solution the scalar field and Friedmann equations are,

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \, , \qquad H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V\right) \, .$$

Consider ϕ to be the inflaton. Recall the *slow-roll* condition is $\dot{\phi}^2 \ll V(\phi)$. Show by differentiating w.r.t. time that this implies $|\ddot{\phi}| \ll |V'(\phi)|$. State the scalar and Friedmann equations in the slow roll approximation, and derive the condition on the potential,

$$|V'| \ll \sqrt{G} V \, .$$

[3 marks]

(ii) We now consider the inflaton scalar to be inhomogeneous but keep the metric as flat FRW. By writing $\phi(t,x) = \phi_{cl}(t) + \delta\phi(t,x)$ so that $\phi_{cl}(t)$ is the homogeneous slow roll solution, and $|\delta\phi| \ll 1$ is a small inhomogeneity, show that to linear order in $\delta\phi$,

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} - \frac{1}{a^2}\delta^{ij}\partial_i\partial_j\delta\phi + V''(\phi_{cl})\delta\phi = 0 \, .$$

[6 marks]

(iii) We may write a solution $\delta\phi(t,x) = \delta\phi_{k_j}(t)e^{ik_ix^i}$ in terms of a constant co-moving wavenumber k_i. What equation must $\delta\phi_{k_i}(t)$ obey? Show that for sufficiently early times during inflation, we may write an approximate solution for a given k_i in WKB form as,

$$\delta\phi_{k_i}(t) \simeq c_{k_i} f(t) e^{-ik\int\frac{dt}{a(t)}}$$

with $k = \sqrt{\delta^{ij}k_ik_j}$ and c_{k_i} a constant, where you should deduce the leading order behaviour of $f(t)$.

[5 marks]

(iv) Write down a quantised inflaton field operator $\hat{\phi}(t,x)$ in terms of creation and annihilation operators $a^\dagger_{k_i}$ and a_{k_i} that obey standard commutation relations, and the solutions $\delta\phi(t,x) = \delta\phi_{k_i}(t)e^{ik_ix^i}$. The conjugate momentum for the inflaton in flat FRW is $\pi(t,x) = a^3(t)\dot{\phi}(t,x)$. Determine the constants of integration c_{k_i} that appear in the early time approximate solutions for the $\delta\phi_{k_i}(t)$ above in order to obtain the conventional quantisation (the Bunch-Davies vacuum) for the inflaton. You may find it useful to recall that $\int d^3k_j e^{ik_ix^i} = (2\pi)^3\delta^3(x^k)$.

[6 marks]

[Total 20 marks]

 End of examination paper