Imperial College London

MSc EXAMINATION April 2015

This paper is also taken for the relevant Examination for the Associateship

COSMOLOGY

For Students in Quantum Fields and Fundamental Forces

Monday 27th April 2015: 14:00 to 17:00

Answer **THREE** *out of the following four questions.*

If answers to all four questions are handed in **only** *the first three will be marked.*

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the 3 answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in 3 answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

Conventions:

We use conventions as in lectures. In particular we take $(-,+,+,+)$ signature and choose units so that $\hbar = 1$ and $c = 1$.

You may find the following useful:

The Christoffel symbol is defined as,

$$\Gamma^{\mu}{}_{\alpha\beta} \equiv \frac{1}{2} g^{\mu\nu} \left(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta} \right) \ .$$

The covariant derivative of a vector field is,

$$\nabla_\mu v^\nu \equiv \partial_\mu v^\nu + \Gamma^{\nu}{}_{\mu\alpha} v^\alpha \ .$$

For the flat FRW metric $ds^2 = -dt^2 + a^2(t)\delta_{ij} dx^i dx^j$, the non-zero Christoffel components are,

$$\Gamma^{t}{}_{ij} = a\dot{a}\,\delta_{ij} \ , \qquad \Gamma^{i}{}_{tj} = \frac{\dot{a}}{a}\delta_{ij}$$

and the Friedmann equation is,

$$H^2 = \frac{8\pi G}{3}\rho$$

with stress energy conservation yielding,

$$\dot{\rho} + 3H\left(\rho + P\right) = 0 \, .$$

Go to the next page for questions

1. (i) Consider a homogeneous isotropic ideal gas of particles in Minkowski spacetime. Starting from the thermal Bose/Fermi distributions in phase space, derive the non-relativistic limit of the real space number density to show,

$$n = g\left(\frac{mkT}{2\pi}\right)^{\frac{3}{2}} e^{\frac{\mu-m}{kT}}$$

for T, m, μ, g the temperature, particle mass, chemical potential and internal degrees of freedom. You may use $\int_0^\infty dx\, x^2 e^{-x^2} = \sqrt{\pi}/4$.

[5 marks]

(ii) Consider nucleosynthesis and the weak processes $n+\bar{e} \to p+\bar{\nu}$, $n+\nu \to p+e$, $n \to p+e+\bar{\nu}$ that convert neutrons to protons, both of which can be treated as non-relativistic. Derive the equilibrium Saha equation and solve it for the neutron fraction.

[5 marks]

(iii) Use a dimensional argument to estimate Γ, the scattering rate per neutron, for the two body processes $n+\bar{e} \to p+\bar{\nu}$ and $n+\nu \to p+e$ in terms of the Fermi constant $G_W \simeq 10^{-5}/(GeV)^2$ and temperature. Give an approximate expression for the freeze out temperature for these processes, T_{freeze}, as a function of the effective number of relativistic degrees of freedom g_{eff} driving the expansion at the time of freeze out. (Recall for radiation $\rho = \frac{1}{2} g_{eff} a_R T^4$ where $a_R = \pi^2 k^4/15$.)

[3 marks]

(iv) In the approximation that these two body weak processes turn off instantaneously, explain how today's number density of Hydrogen, $n_{H,0}$, and Helium, $n_{He,0}$, in primordial gas clouds relate to the neutron fraction at T_{freeze}, and $n_{B,0}$, the baryon number density today.

Briefly explain why we can ignore the one body decay $n \to p+e+\bar{\nu}$.

[4 marks]

(v) Consider cosmology with N additional massless left-handed neutrinos. Assume the only change to nucleosynthesis is in T_{freeze} due to the dependence on g_{eff} you calculated above, and approximate,

$$g_{eff} \simeq 2 + \frac{7}{8} \times (3+N) \times 2 \times \left(\frac{4}{11}\right)^{\frac{4}{3}}$$

during nucleosynthesis. The mass fraction of today's primordial Helium is,

$$Y = \frac{4 n_{He,0}}{n_{H,0} + 4 n_{He,0}}$$

and is observed to be $Y \simeq 0.248 \pm 0.003$. Estimate $\frac{1}{Y}\frac{dY}{dN}$ and evaluate it for $N = 0$. Briefly comment on the significance of this value in relation to the observations of Y.

[3 marks]

[Total 20 marks]

2. (i) Consider flat FRW spacetime $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a(t)^2\delta_{ij}dx^i dx^j$, and a particle with 4-momentum p^μ. Show for a free particle,

$$\frac{dp^i}{dt} = -\frac{1}{p^0}\Gamma^i{}_{\alpha\beta}p^\alpha p^\beta \,.$$

[3 marks]

(ii) Now defining $p = \sqrt{g_{ij}p^i p^j}$, use the above result to show that for a free particle in flat FRW,

$$\frac{dp}{dt} = -\frac{\dot{a}}{a}p \,.$$

[5 marks]

(iii) Assume $n = n(t, p)$ is the homogeneous isotropic phase space distribution for an ideal gas of particles. Derive the Boltzmann equation with collision term $C(t, p)$ using the result above to show,

$$\left[\left.\frac{\partial}{\partial t}\right|_p - p\,\frac{\dot{a}}{a}\left.\frac{\partial}{\partial p}\right|_t \right] n(t, p) = C(t, p) \,.$$

[4 marks]

(iv) Use this Boltzmann equation equation to derive an evolution equation for the real space number density $n(t)$,

$$\frac{1}{a^3(t)}\frac{d}{dt}\left(a^3(t)n(t)\right) = \int_0^\infty dp\, 4\pi p^2 C(t, p) \,.$$

[3 marks]

(v) Consider highly relativistic particles with phase space distribution $n(t, p)$, which decay into thermal products at temperature T. Take the decay part of the collision term to be,

$$C_{decay}(t, p) = -\Gamma\, n(t, p)$$

with Γ a constant, and further assume detailed balance. Assume that in thermal equilibrium the number density of the particles goes as,

$$n_{eq}(T) = q\,(kT)^3$$

for a constant q. Find the general solution for the real space number density $n(t)$ assuming temperature goes inversely with scale factor, so $T \sim 1/a$.

[5 marks]

[Total 20 marks]

3. (i) Consider a flat FRW universe. At time t_0, the scale factor and Hubble function are a_0 and H_0. Consider it filled with various matter components with equations of state $p_i = w_i \rho_i$ for constants w_i, and energy density fractions Ω_i at time t_0, and assume these do not interact with each other. A null ray starts at redshift Z_1 and finishes at Z_2 travelling a comoving radius R, where $R^2 = \Delta x^i \Delta x^i$. Show that,

$$R = \frac{1}{a_0 H_0} \int_{\frac{1}{1+Z_1}}^{\frac{1}{1+Z_2}} \frac{dy}{y^2 \sqrt{\sum_i \Omega_i y^{-3(1+w_i)}}} .$$

[5 marks]

(ii) Model our universe by flat FRW with radiation fluid and dust matter, with fractions Ω_r, Ω_m respectively (i.e. ignore dark energy). Assuming that radiation matter equality was at a redshift $Z_{eq} \gg 1$, what is Ω_r?

Using the above integral show that the comoving radius R moved by a photon between the big bang and last scattering at redshift $Z_{lss} \gg 1$ is approximately,

$$R \simeq \frac{2}{a_0 H_0} \left(\sqrt{\frac{1}{Z_{lss}} + \frac{1}{Z_{eq}}} - \sqrt{\frac{1}{Z_{eq}}} \right) .$$

[4 marks]

(iii) Likewise compute the comoving radius moved by a photon between last scattering and today, $Z = 0$.

Hence compute the angle (in degrees) a causal region at last scattering subtends in the sky today in this model, for the values $Z_{eq} = 3500$ and $Z_{lss} = 1100$.

[4 marks]

(iv) Now introduce an inflationary epoch as a period of exact de Sitter with N e-foldings of inflation, and $H =$constant. Assume an instantaneous transition to the radiation/matter era discussed above, at a redshift $Z_{rad} \gg Z_{eq}, Z_{lss}$. Show that the minimum number of e-folds, N_{min}, required to solve the horizon problem in this model is given approximately by,

$$e^{N_{min}} \simeq \frac{2 Z_{rad}}{\sqrt{Z_{eq}}} .$$

[6 marks]

(v) For a reheat temperature corresponding to $kT \simeq 10^{20} GeV$ and $Z_{eq} = 3500$, compute N_{min}.

[1 mark]

[Total 20 marks]

4. (i) Consider a real inflaton scalar field ϕ with potential $V(\phi)$ and governed by the equation of motion $\nabla^2\phi = V'(\phi)$. Take the spacetime to be flat FRW, but allow the inflaton to have small inhomogeneous fluctuations, $\delta\phi(t,x)$, about a homogeneous and isotropic solution $\phi_{cl}(t)$, so that $\phi(t,x) = \phi_{cl}(t) + \delta\phi(t,x)$. Explicitly compute that to linear order in $\delta\phi$,

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} - \frac{1}{a^2}\delta^{ij}\partial_i\partial_j\delta\phi + V''(\phi_{cl})\delta\phi = 0 \ .$$

[6 marks]

(ii) Consider a solution with comoving wavenumber k_i,

$$\delta\phi(t,x) = \delta\phi_{k_j}(t)e^{-ik_ix^i} + c.c.$$

Now approximate the inflating spacetime as de Sitter with H =const, and take $V'' \simeq 0$. Show then that the time dependence,

$$\delta\phi_{k_i}(t) = \frac{1}{\sqrt{2k(2\pi)^3}}e^{+\frac{ik}{a(t)H}}\left(\frac{1}{a(t)} + \frac{iH}{k}\right)$$

with $k = \sqrt{\delta^{ij}k_ik_j}$ provides a solution.

[4 marks]

(iii) We quantise the inflaton in this de Sitter using the above modes, writing,

$$\delta\hat{\phi}(t,x) = \int d^3k_i\left(a_{k_i}\delta\phi_{k_i}(t)e^{-ik_ix^i} + a^\dagger_{k_i}\delta\phi_{k_i}(t)^\star e^{+ik_ix^i}\right)$$

with creation and annihilation operators, a_{k_i} and $a^\dagger_{k_i}$, obeying the standard relations,

$$[a_{k_i}, a_{q_j}] = 0 \ , \quad [a_{k_i}, a^\dagger_{q_j}] = \delta^{(3)}(k_i - q_i) \ .$$

We write the equal time 2-point function of fluctuations in vacuum in terms of a dimensionless power spectrum $\Delta^2(t,k)$, so,

$$\langle 0|\delta\hat{\phi}(t,x)\delta\hat{\phi}(t,y)|0\rangle = \int\frac{d^3k_i}{k^3}\,\Delta^2(t,k)\,e^{-ik_i(x^i-y^i)} \ .$$

By explicit computation show that for super horizon modes $\Delta^2(t,k)$ is a constant determined only by H.

[5 marks]

(iv) Consider a potential $V = \lambda\,\phi^4$. Assume we require slow roll, with at least 60 e-folds of inflation, and that slow roll lasts until ϕ reaches the minimum of the potential. Consider the large scale CMB temperature fluctuations whose dimensionless power spectrum, $\Delta^2_{(T)}$, is estimated in terms of Δ^2 above as,

$$\Delta^2_{(T)} \sim \frac{H^2}{\dot{\phi}^2}\Delta^2 \ .$$

Given that this should be of order $\sqrt{\Delta^2_{(T)}} \sim 10^{-5}$, estimate the magnitude of the parameter λ in the potential.

[5 marks]

[Total 20 marks]

 End of examination paper