Particle Symmetries

- Particle physics
- Lie gps , Lie Algebras
- Rep^n Theory

Books:

512.2/54 • Fulton and Harris - v. mathematical
- Georgi } more physics
- Jones }
- Cahn
- Ramond

1. Particles, The Standard Model

- In the SM we have:

  - quarks } 3 generations
  - leptons }
  - gauge bosons
  - Higgs boson
  - Forces strong nuclear, EM, Weak

    → gravity is outside the SM

- Gauge principle : $\underbrace{SU(3)}_{\text{colour}} \times \underbrace{SU(2) \times U(1)}_{EW}$

  8 gluons $\quad W^{\pm}, Z^{0}, \gamma$

- What is SU(3) ?

  → Special unitary gp. of 3x3 matrices

- $SU(N)$: $N \times N$ complex matrices (unitary) of det 1.

- $U(1)$: unitary group of dim 1, $e^{i\alpha}$ $\alpha \in [0, 2\pi)$.

- We characterise these particles with:

Gauge symmetries
- spin : matter spin $1/2$; spin 0 Higgs; Spin 1 gauge b.
- mass
- charge $\rightarrow$ irreducible repⁿ.

- Beyond the SM : graviton spin 2
  gravitino spin $3/2$.

- Spin - irreducible repⁿ under $SU(2)$.

- Mass - gluon : 0
  W, Z : 100 GeV
  γ : 0

  Spontaneous Symmetry breaking

  quarks : $\underbrace{u, d}_{1\,\text{MeV}}$, c, s, t, b
  1 GeV 500 MeV 200 GeV 10 GeV

	generations			
$\frac{2}{3}e$	u	c	t	spin under $SU(2)$
$-\frac{1}{3}e$	d	s	b	

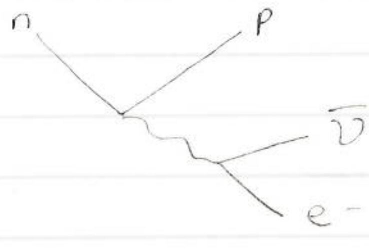
- Quarks come in doublets and have colour charge.
  → 3 colour states
  → antiquarks.

- They also possess charge.

Global Symmetries

- Spacetime symmetries. eg. Lorentz, discrete sym.
- Lepton no. eg. e^-

β-decay



- lepton no.
- charge
- baryon no.

are <u>all</u> preserved.

- Baryon no. : for each quark, the baryon no. is $1/3$.
- Isospin : $\binom{u}{d}$ transform in spin $\frac{1}{2}$. We say $\binom{u}{d}^{+1}_{-1}$.
- Flavour symmetry : 3 light quarks => interactions governed by ≈ SU(3) symmetry.

Spacetime Symmetries

- Spacetime translations
  → momentum and energy are conserved.
  → Nöther's Theorem
- Rotations
  → ang. mom. conserv.
- Boosts

Lorentz transform, 6 total

- Poincaré - 10 symmetries total.

- These are all global symmetries when there's no gravity, but gauge sym. when gravity is turned on.

  → Gravity is the gauge principle of ST sym.

Discrete Symmetries ...

- Parity P - space reversal $\underline{x} \to -\underline{x}$
  → broken by Weak interactions.

- Charge conjugation C, matter → antimatter
  → K-physics breaks CP sym. (also breaks T).

- Time Reversal T, $t \to -t$
  → CPT theorem - CPT is conserved for all processes.

CP is only lightly broken - need more violation to explain matter - antimatter asymmetry.

2. Groups

Defn: A gp. is a set G of elements $g \in G$ s.t. four pts are satisfied, w/ a binary form $G \times G \to G$. $(\cdot, \cdot)$

1. Closure $g_1, g_2 \in G$, then $g_1 \cdot g_2 \in G$.

2. Associativity $g_1, g_2, g_3 \in G$ then,

$(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$

3. Identity $e \in G$, $g \in G$, then $e \cdot g = g \cdot e = g$.

4. Inverse $\forall g \in G \ \exists ! \ g^{-1}$ s.t. $g \cdot g^{-1} = g^{-1} \cdot g = e$.
↳ unique

Examples

- Let n be an integer, and $n > 0$, the $\exists \ GL(n, \mathbb{C})$
  → gp of linear transf. in $\mathbb{C}^n$.

- $GL(n, \mathbb{C})$ is the set of $n \times n$ complex matrices w/ $\det \neq 0$.

- $g \in GL(n, \mathbb{C})$ is then $g_{n \times n}$, $\det g \neq 0$.

- Also have $GL(n, \mathbb{R})$.
  → real matrices

- No. of parameters (or the dimension) is n^2.

- $SL(n, \mathbb{C})$ is a gp. of $GL(n, \mathbb{C})$ matrices w/ $\det = +1$.
  → "S" = "special".
  → no. of parameters $= n^2 - 1$. (for $\mathbb{C}$ or $\mathbb{R}$).
  → $SL(n, \mathbb{C}) \subset GL(n, \mathbb{C})$.

E.g. $GL(2, \mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, a, b, c, d \in \mathbb{C}, \ ad - bc \neq 0 \right\}$

$$SL(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ s.t. } ad - bc = 1 \right\}.$$

2.1 Rotation Groups

- Consider a vec. space $\mathbb{R}^n$, $\overbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}^{= \underline{v}} \in \mathbb{R}^n$ and $x_i \in \mathbb{R}$.

- Define length, $L^2 = \sum_{i=1}^{n} x_i^2 = v^T v$.

- Consider the set of transformations, which preserve the length, $M \in GL(n, \mathbb{R})$.

ie. $v' = Mv$

preserve length

Length of v' : $L^2 = v^T M^T M v = v^T v$

$$\Rightarrow M^T M = \mathbb{1}_{n\times n} \quad \text{orthogonal matrix}$$

Start w/ n^2. Then remove n due to $M^TM = \mathbb{1}_{n\times n}$ $\to$ have n constraints

Orthogonal matrices preserve the length of vector.

$\longrightarrow$ group $O(n)$.

$\longrightarrow$ $\dim O(n) = \dfrac{n(n-1)}{2} = n^2 - \dfrac{n(n+1)}{2}$

E.g 3×3 $\quad$ 6 entries $\quad \dfrac{n(n+1)}{2}$ for $M^TM = \mathbb{1}$

- $\det(M^T M) = \det M^T \det M = (\det M)^2 = 1$

$\Rightarrow \det M = \pm 1.$

- Define $SO(n)$ as the set of $O(n)$ matrices w/ det $+1$.

E.g. $SO(2) \rightarrow$ rotations in a plane
$\rightarrow$ one dim, θ

$$\Rightarrow \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \theta \in [0, 2\pi).$$

$SO(3) \rightarrow 3$ angles as $\frac{3(3-1)}{2} = 3$

- Take a complex vector space $\mathbb{C}^n$, $v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$,

$x_i \in \mathbb{C}$.

$$L^2 = \sum_{i=1}^{n} |x_i|^2 = v^\dagger v.$$

What is the set of ~~complex~~ length preserving transformations in $\mathbb{C}^n$? Given by,

$$v' = Mv, \quad M \in GL(n, \mathbb{C}).$$

$$\begin{aligned} L^2 &= v'^\dagger v' \\ &= v^\dagger M^\dagger M v = v^\dagger v \end{aligned}$$

$\Rightarrow M^\dagger M = \mathbb{1}_{n\times n}$ M unitary

the product is

$\Rightarrow (M^\dagger M)^\dagger = M^\dagger M \Rightarrow$ Hermitian

$\dim_{\mathbb{R}} U(n) = n^2$

E.g. $U(2)$ is the set of 2×2 unitary matrices:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{C}.$$

or $\begin{pmatrix} a+id & b+ic \\ b-ic & a-id \end{pmatrix}$ $\rightarrow$ most general way to write this. Check!

Let h be Hermitian and 2×2:

$$h = \begin{pmatrix} a & b-ic \\ b+ic & d \end{pmatrix} \quad a, b, c, d \in \mathbb{R}.$$

- $SU(n)$ is the set of $U(n)$ matrices w/ det $+1$.

Suppose $M \in U(n)$, then $M^{+}M = \mathbb{1}$

$\Rightarrow \det M = e^{i\alpha}, \quad \alpha \in [0, 2\pi)$

as $\det M^{+} \det M = (\det M)^{*} (\det M)$

$$= |\det M|^2 = 1$$

$\Rightarrow \det M$ is a phase.

- Det. is discrete set for $O(n)$, but a whole parameter space for $U(n)$.

as $\alpha = 0$

$\Rightarrow$ $\dim SU(n) = n^2 - 1$

- $S_p(n)$ - symplectic matrices.

Set $\eta = \begin{pmatrix} 0_{n\times n} & \mathbb{1}_{n\times n} \\ -\mathbb{1}_{n\times n} & 0_{n\times n} \end{pmatrix}$ $\Rightarrow$ $2n \times 2n$ matrix

Length $L^2 = V^T \cdot \eta \cdot V$, $V = \begin{pmatrix} x_1 \\ \vdots \\ x_{2n} \end{pmatrix}$, $x_i \in \mathbb{R}$.

Linear transf. $M \in GL(2n, \mathbb{R})$

$\longrightarrow$ $M^T \eta M = \eta$ symplectic form

$S_p(n, \mathbb{R})$ is the set of matrices in $GL(2n, \mathbb{R})$ which satisfies the above condition.

E.g. Phase space, symplectic forms, Poisson brackets, generalised coord.

N.B. $M^T \Omega M = \Omega$, $\Omega = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$

$(M^T \Omega M)^T = M^T \Omega^T M = \Omega^T$

$\Rightarrow$ $\Omega^T = -\Omega$

$\Rightarrow$ antisymmetric form

no. of relations

$\Rightarrow$ ~~[illegible]~~ $= \dfrac{2n(2n-1)}{2}$.

- In $O(n)$ preserves the sym. form ; $Sp(n)$ preserves the antisym. form.

$\dim Sp(n) = n(2n+1)$ (as $(2n)^2 -$ no. of relations $= \dim$).

- The gps. discussed are called classical gps..

- Extension of rotation gps. :

$$\eta = \begin{pmatrix} \overbrace{1 \; \cdots \; 1}^{p} & \\ & \overbrace{-1 \; \cdots \; -1}^{q} \end{pmatrix}$$ p ones and q minus ones.

$\Rightarrow$ $p+q$ rows and columns.

Then have $O^T \eta O = \eta$ and $O(p,q)$ is the gp. matrix that preserves the length.

If $V \in \mathbb{R}^{p+q}$, $V = \begin{pmatrix} x_1 \\ \vdots \\ x_p \\ y_1 \\ \vdots \\ y_q \end{pmatrix}$

$$\Rightarrow L^2 = V^T \eta V = \sum_{i=1}^{p} x_i^2 - \sum_{j=1}^{q} y_i^2$$

This type of form is used in Special Rel.
$\Rightarrow$ $SO(3,1)$ and some discrete extensions of it.

- The Lorentz gp. is an extension of the rotation group.

Properties of the Classical Groups

1. $GL(n, \mathbb{C})$ - $n \times n$ matrices, $\det \neq 0$.

Dimension / no. of parameters $= n^2$.

2. $SL(n, \mathbb{C}) \subset GL(n, \mathbb{C})$ w/ $\det = +1$

Dimension $= \underbrace{n^2 - 1}$

why (-1)?

We have only one constraint on the matrices: that $\det = 1$. The det condition is just one constraining relation $\Rightarrow n^2 - 1$.

3. $O(n)$ - $n \times n$ matrices, $\det = \pm 1$, length preserving.

Dimension $= \frac{n(n-1)}{2}$ why?

For $M \in O(n)$ we have the orthogonality condition:

$$M^T M = \mathbb{1}_{n \times n}.$$

So for $M^T M$ we have n^2 entries, which are n^2 equ^{n}s as each element of $M^T M$ must be equal to an entry in $\mathbb{1}_{n \times n}$.

However, only $\frac{n(n+1)}{2}$ of the equ^{n}s are independ.

(as some of the entries in M^TM are repeated).

$$\Rightarrow \text{dim. } O(n) = \underbrace{n^2}_{\text{total no. of entries}} - \underbrace{\frac{n(n+1)}{2}}_{\text{no. of independ. equ}^n\text{s}}$$

$$= \frac{n(n-1)}{2}.$$

E.g. Let $M \in O(3)$ $\Rightarrow M = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_2 & c_2 & c_3 \end{pmatrix}$

$$\Rightarrow M^T = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

$$\mathbb{1} = MM^T = \begin{pmatrix} a_1^2+b_1^2+c_1^2 & a_1a_2+b_1b_2+c_1c_2 & a_3a_1+b_3b_1+c_3c_1 \\ a_1a_2+b_1b_2+c_1c_2 & a_2^2+b_2^2+c_3^2 & a_2a_3+b_2b_3+c_2c_3 \\ a_3a_1+b_3b_1+c_3c_1 & a_3a_2+b_3b_2+c_3c_2 & a_3^2+b_3^2+c_3^2 \end{pmatrix}$$

$\rightarrow$ n^2 equ^{n}s $= 9$ equ^{n}s

$\rightarrow$ $\frac{n(n+1)}{2}$ independ. $= \frac{3(3+1)}{2} = 6$ (i.e. 3 repeated equ^{n}s)

$\rightarrow$ dim $O(3) = 3$

For $n = 4$, we have 16 equ^{n}s, but only $\frac{4 \cdot 5}{2} = 10$ are independ.

$\rightarrow$ dim. $O(4) = 6$.

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A special subgroup of $O(n)$ is $SO(n)$. The $SO(n)$ matrices are $n \times n$ matrices with det $= +1$.

The dimension of $SO(n)$ is the same as for $O(n)$. This is because the same constraints apply to both $O(n)$ and $SO(n)$ ie. there are no additional constraints.

4. $U(n)$ – $n \times n$ matrices w/ complex entries, $|\det M|^2 = 1$, length preserving in $\mathbb{C}^n$.

As $|\det M|^2 = 1 \Rightarrow \det M = e^{i\alpha}$ for $\alpha \in [0, 2\pi)$.

ie. the det. of $U(n)$ matrices are phases.

$M \in U(n)$ must also satisfy:

$$M^{\dagger} M = \mathbb{1}_{n \times n}$$

$$\Rightarrow \dim_{\mathbb{R}} U(n) = n^2$$

Why? E.g. $U(2)$:

$$M \in U(2) \Rightarrow M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; \; M^{\dagger} = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$$

$$M^{\dagger} M = \begin{pmatrix} |a|^2 + |c|^2 & a^* b + c^* d \\ b^* a + d^* c & |b|^2 + |c|^2 \end{pmatrix} = \mathbb{1}_{2 \times 2}$$

4 equ^{n}s $= (2)^2 = (n)^2$.

Note that the product $(M^{+}M)$ is Hermitian:

$$(M^{+}M)^{+} = M^{+}M = \mathbb{1}$$

A special case of $U(n)$ is the subgroup $SU(n)$, where the $SU(n)$ matrices have $\det = +1$.

$$\det M = e^{i\alpha} = 1 \quad \Rightarrow \quad \alpha = 0 \quad \text{for } M \in SU(n).$$

This is one constraint, hence the dimension of $SU(n)$ is:

$$\dim. SU(n) = n^2 - 1.$$

5. Symplectic group, $Sp(n)$ - $2n \times 2n$ matrices of the form:

$$\eta = \begin{pmatrix} O_{n\times n} & \mathbb{1}_{n\times n} \\ -\mathbb{1}_{n\times n} & O_{n\times n} \end{pmatrix}.$$

η has symplectic form,

$$M^T \eta M = \eta \quad \text{for } M \in GL(2n, \mathbb{R})$$

The dimesion of $Sp(n) = n(2n+1)$.

Why? E.g. let $n = 2 \Rightarrow 4 \times 4$ matrix

$$\eta = \begin{pmatrix} O_{2\times 2} & \mathbb{1}_{2\times 2} \\ -\mathbb{1}_{2\times 2} & O_{2\times 2} \end{pmatrix}.$$

Let $M \in GL(4, \mathbb{R})$ and

$$M = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix}$$

Then, $$M^T \eta M = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{pmatrix} \overbrace{\begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \\ -a_1 & -a_2 & -a_3 & -a_4 \\ -b_1 & -b_2 & -b_4 & -b_4 \end{pmatrix}}^{\eta M}$$

$$\equiv \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

There are $(2n)^2 = 16$ equ^{n}s. However, the diagonal $\mathrm{diag}(M^T \eta M) = 0$, so there's no need to impose equ^{n}s on the diagonal.

$\Rightarrow (2n)^2 - 2n$ equ^{n}s.

However, the off-diagonal equ^{n}s only differ by a factor of (-1)

$\Rightarrow$ there's a doubling of the remaining equ^{n}s

$\therefore$ the no. of ~~ol~~ independ. equ^{n}s $= \dfrac{2n(2n-1)}{2}$

So, the dimension $= (2n)^2 - \dfrac{2n}{2}(2n-1)$

$= n(2n+1)$.

- The dim. of $O(p,q) = \frac{(p+q)(p+q-1)}{2}$

$\rightarrow$ ie. the same form as $O(n)$.

The dim. of the Lorentz gp $O(3,1) = 6$
$\rightarrow$ 3 rotations and 3 boosts.

2.2 Representations of Gps.

- A repⁿ. ρ of gp G is a homo. from $G \rightarrow GL(n, \mathbb{C})$, which preserves the gp multiplication law.

ie. $g_1, g_2 \in G$, then $\rho(g_1) \cdot \rho(g_2) = \rho(g_1 g_2)$.

- ρ assigns an $n \times n$ matrix to every element in G.

- n is called the dimension of the repⁿ.

E.g. The fundamental repⁿ of the $O(3)$ 3-vec. $\underline{v}$.

$\underline{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, rotation matrix is 3×3 (rotates the vector).

Spin-$\frac{1}{2}$ particles appear is 2 dim. repⁿ of $SU(2)$.

The spacetime vectors are 4d. vectors $(\underline{x}, t)$, acted by 4×4 matrices of $O(3,1)$.

- $\exists$ ∞ many repⁿs.

- The gps. discussed are continous gps.

2.3 Continuous Groups

- Need the concept of a generator.

- Ham. generates t-translations ; mom. generates spatial translations.

- Consider a wavefunc^n $\psi(x)$, which we shift by "a":

$$\psi(x+a) = \psi(x) + a\psi'(x) + \frac{a^2}{2}\left(\frac{\partial}{\partial x}\right)^2 \psi(x) + \dots$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\left(a\frac{\partial}{\partial x}\right)^n \psi(x)$$

$$\triangleq \exp\left(a\frac{\partial}{\partial x}\right)\psi(x)$$

Let $p = -i\hbar\frac{\partial}{\partial x} \Rightarrow \triangleq \exp\left(\frac{iap}{\hbar}\right)\psi(x)$

$\Rightarrow$ momentum is a generator of spatial shifts!

- Similarly $H = i\hbar\frac{\partial}{\partial t}$; $e^{-iHt_0/\hbar}\psi(x,t) = \psi(x,t+t_0)$.

- Every conserved quantity has an associated generator.

- Take $SO(2)$ - look for matrix generators of $SO(2)$.

Parameterise matrix M

$$M = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \qquad \text{Rotation of angle } \alpha.$$

look at behaviour at α v. small

$$\Rightarrow M = \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix} + O(\alpha^2)$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \alpha \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{X} + O(\alpha^2)$$

$X \rightarrow$ generator of $SO(2)$ rotations

Then compute $e^{\alpha X}$

$$e^{\alpha X} = \sum_{n=0}^{\infty} \frac{(\alpha X)^n}{n!}$$

$X^2 = -\mathbb{1}$

$$= \underbrace{\sum_{n=0}^{\infty} \frac{(\alpha X)^{2n}}{(2n)!}}_{\text{even powers}} + \underbrace{\sum_{n=0}^{\infty} \frac{(\alpha X)^{2n+1}}{(2n+1)!}}_{\text{odd powers}}$$

$$= \sum_{n=0}^{\infty} (\alpha)^{2n} \frac{(-\mathbb{1})^n}{(2n)!} + X \sum_{n=0}^{\infty} \frac{(\alpha)^{2n+1} (-1)^n}{(2n+1)!}$$

$$= \cos\alpha \cdot \mathbb{1} + X \sin\alpha = M.$$

- X is the generator of rotations in 2 space dim.

3 space-dim. (SO(3)).

- Have 3 angles:

$$\begin{pmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ Rotates around the z-axis.

$$= \mathbb{1}_3 + \theta_3 \underbrace{\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\text{generator of rotations around } x_3 \to X_3} + O(\theta_3^2)$$

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

$$\exp\{\theta_1 X_1 + \theta_2 X_2 + \theta_3 X_3\} = ?$$

Need $[X_i, X_j]$ to figure out exp.

We find: $[X_1, X_2] = X_3$
$[X_2, X_3] = X_1$
$[X_3, X_1] = X_2$

- As differential op. we can say,

$\underline{L} = \underline{x} \times \underline{p}$, where

$$L_x = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$L_y = -i\hbar \left(z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right)$$

$$L_z = -i\hbar \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right)$$

ie. the ang. mom. op in QM.

$\Rightarrow \quad [L_i, L_j] = i\hbar \, \varepsilon_{ijk} L_k.$

L_i act on vector spaces.

- For the matrix analysis we have done,

$$[X_i, X_j] = \varepsilon_{ijk} X_k$$

for X_i 3x3 matrices for the 3d rep^n of $SO(3)$.

- We $\therefore$ have an algebra defined by $[\cdot, \cdot]$.
  $\longrightarrow$ Lie Algebra

2.4 The Lie Algebra

Def^n: A Lie Algebra is a set of element $\mathcal{A}$, which forms a vector space, with a binary operation.

$$\mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}.$$

ie. $[\circ, \circ] \in \mathcal{A}.$

- They satisfy:

The Lie Algebra

(a) Closure : $A, B \in \mathcal{R}$, $[A, B] \in \mathcal{R}$

(b) Bilinearity : $\alpha, \beta \in \mathbb{C}$ then $[\alpha A + \beta B, C] \stackrel{\Delta}{=}$

$\stackrel{\Delta}{=} \alpha [A, C] + \beta [B, C]$ w/ $A, B, C \in \mathcal{R}$.

(c) $[A, B] = -[B, A]$

(d) Jacobi Identity ; $A, B, C \in \mathcal{R}$

$$\big[[A,B], C\big] + \big[[B,C], A\big] + \big[[C,A], B\big] = 0.$$

- Algebra generates the rotations around the $\mathbb{1}$.
  $\longrightarrow$ gives local properties.

A gp. contains all the rotations - even large ones.

2.5. The Lie Algebra of SU(2)

- Lie Algebra of SU(2) is the same as that of SO(3).

- $[L_i, L_j] = \varepsilon_{ijk} L_k$

$\longrightarrow$ construct rep^{n}s that obey this algebra.

- Consider a set of lowering/raising op :

$$J_\pm = \frac{i}{\sqrt{2}}\left(L_1 \pm i L_2\right) \quad , \quad J_3 = i L_3$$

$$\left.\begin{aligned} [J_3, J_\pm] &= \pm J_\pm \\ [J_+, J_-] &= J_3 \end{aligned}\right\}$$

$\mathbb{Z}_2$ symmetry:

$J_+ \longleftrightarrow J_-$ ie $+ \leftrightarrow -$ symmetry

This symmetry is called Weyl symmetry.

- Construct a set of states in finite dim. vector space.

$$\left.\begin{aligned} &J \underbrace{|\phi\rangle}_{\text{state}} \\ &J^2 |\phi\rangle \end{aligned}\right\} \text{ act op. on states}$$

Every state is characterised by two quantum no. $|\underbrace{\alpha}_{\varepsilon\text{-value of } J^2}, \underbrace{m}_{\varepsilon\text{-value of } J_3}\rangle$

Can do this as $[J^2, J_3] = 0$.

Note:

$$J^2 = J_+ J_- + J_- J_+ + J_3^2.$$

$$[J^2, J_\pm] = 0.$$

- Consider, $J_+ |\alpha, m\rangle$

Then, $J^2 J_+ |\alpha, m\rangle = J_+ J^2 |\alpha, m\rangle$

$$= \alpha J_+ |\alpha; m\rangle$$

$\Rightarrow J_+ |\alpha, m\rangle$ has α $\mathcal{E}$-value under J^2.

- Under J_3:

$$J_3 J_+ |\alpha, m\rangle = \Big(\underbrace{[J_3, J_+]}_{= J_+} + J_+ J_3\Big) |\alpha, m\rangle$$

$$= (m+1) J_+ |\alpha, m\rangle$$

$\Rightarrow J_+ |\alpha, m\rangle$ has $\mathcal{E}$-value $(m+1)$

$\rightarrow$ raises $\mathcal{E}$-value under J_3.

- $J_- |\alpha, m\rangle$ has $\mathcal{E}$-value $(m-1)$ under J_3.

$\rightarrow$ lowers $\mathcal{E}$-value under J_3

Has $\mathcal{E}$-value α under J^2.

- Finite dim. space $\Rightarrow$ can't raise/lower forever.

Use J_+/J_- op. to construct the irrep^n's and to completely reduce red. rep^n's.

$\downarrow$ useful later to find rep^n's of general Lie Algebras.

Have critical value $m = j$ for which

$$J_+ |\alpha, j\rangle = 0.$$

Also, critical $m = j'$ for which,

$$J_- |\alpha, j'\rangle = 0.$$

Jump by integers when raise/lower

$\Rightarrow j - j' = n$, n integer and non-negative.

j: highest spin cmpt.

j': lowest spin cmpt.

- Evaluate:

$$J^2|\alpha,j\rangle = (\underbrace{J_+J_-}_{[J_+,J_-]=J_3} + J_-J_+ + J_3^2)|\alpha,j\rangle$$

$J_+|\alpha,j\rangle = 0$

$$= (j^2+j)|\alpha,j\rangle$$

$$\Rightarrow \alpha = j^2 + j$$

$$J^2|\alpha,j'\rangle = (J_+J_- + \overbrace{J_-J_+}^{[J_-,J_+]=-J_3} + J_3^2)|\alpha,j'\rangle$$

$$= (j'^2 - j')|\alpha,j'\rangle$$

$J_-|\alpha,j'\rangle = 0$

$$\Rightarrow \alpha = j'^2 - j'$$

$\therefore$ have an equ^n between the 2 j's.

$$j(j+1) = j'(j'-1)$$

$$= -j'(1-j')$$

$\Rightarrow$ $j' = j+1$ or $j' = -j$

as defined $j - j' = n \geq 0$.

- The set of m values is going from $-j$ to j in jumps of 1.

- Have $2j+1$ states.

- $n = j - j' = 2j$.

n is twice the spin - The rep^{n}s of SU(2) (which are finite dimensional) are characterised by a +ve integer $n = 2j$
↳ actually non-negative

- n is called the highest weight of the rep^n. of dim $n+1$.

- $-n$ is the lowest weight of the rep^n, by Weyl symmetry.

- These are irreducible rep^{n}s, which are characterised by the highest weight n (non-negative integer)

For $n=0$: 1 state $\longrightarrow$ trivial rep^n
$\alpha = 0$.

$n=1$: 2dim rep^n $\longrightarrow$ spin-$\frac{1}{2}$; σ_i Pauli matrices

simplest, "defining" rep^n of SU(2).

$$\sigma_i = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

$n=2$: 3-dim rep^n $\longrightarrow$ vec. rep^n or the adjoint rep^n.

3 = dim. of the algebra

$n=3$: 4-dim rep^n $\longrightarrow$ spin-$\frac{3}{2}$, gravitino.

$n=4$: 5-dim. rep^n $\longrightarrow$ spin 2, graviton -
- traceless second rank sym. tensor.

- Consider a spin-$\frac{1}{2}$ state, V_α $\alpha = 1, 2$.

- We can construct a spin-1 one state by taking 2 such states $V_\alpha V_\beta = a_{\alpha\beta}$

$\longrightarrow$ 3 cmpts. $V_1^2, V_1 V_2, V_2^2$.

- Can construct spin $3/2$ states w/ 3 such states:

$$a_{\alpha\beta\gamma} = V_\alpha V_\beta V_\gamma$$

$\longrightarrow$ V_1^3, $V_1^2 V_2$, $V_1 V_2^2$, V_2^3 ie. 4 ~~states~~ compts.

$3/2$ $\quad$ $1/2$ $\quad$ $-1/2$ $\quad$ $-3/2$

highest weight $\longleftrightarrow$ lowest weight

- A lattice:
$\Rightarrow n = 5$ (as 6-dim rep^n) $\quad$ 5 $\quad$ 3 $\quad$ 1 $\quad$ −1 $\quad$ −3 $\quad$ −5
$\times \frac{1}{2}$ to get spin

−6 $\quad$ −4 $\quad$ −2 $\quad$ 0 $\quad$ 2 $\quad$ 4 $\quad$ 6

Spin 3, $n = 7$

~

- The construction of SU(2) ireps. generalises to other compact Lie gps.
  - $\longrightarrow$ the method used is called the highest weight construction.
  - $\longrightarrow$ the J_3 values are called weights.

2.6 Lie Algebras

su(2) $[X_i, X_j] = \varepsilon_{ijk} X_k$

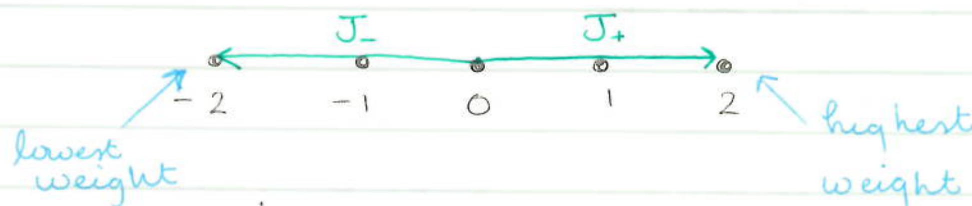
∞-many rep^{n}s of $SU(2)$ - irreducible.
$\rightarrow$ parameterised by a non-neg. integer n, where $n = 2j$ (twice spin).

For a fixed n there are $(n+1)$ different states, parameterised by $-j \leq m \leq j$ in jumps of one.

We call n the heighest weight, and $2m$ is called the weights, which is an integer.

Given n, we can construct all the weights.

E.g. $n = 2$



$\Rightarrow$ two lowering / raising op.

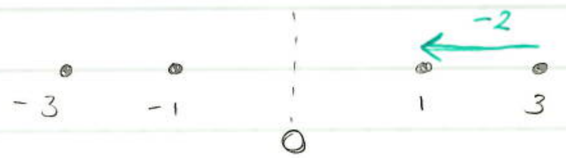
$\Rightarrow$ $2m = -2, 0, 2$ (as $n = 2 \Rightarrow j = 1$).

The set of all points generated by $\alpha = 2$, is the set of all even integers. $\longrightarrow$ root lattice.

J^+ - raising op. $\longrightarrow$ positive root $\underline{\alpha}$

E.g. If $n = 3 \Rightarrow j = 3/2 \Rightarrow$ the set of weights is
$-3, -1, 1, 3$

ie the weights are odd
$\Rightarrow$ 4-dim rep^n.



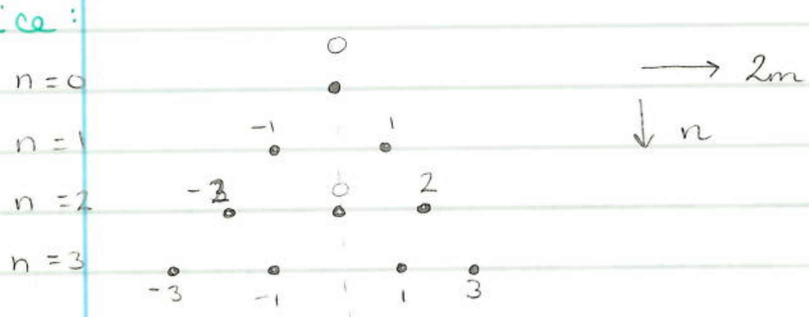
This corresponds to half-integer spin.

Odd weights $\Longleftrightarrow$ half integer spin
Even weights $\Longleftrightarrow$ integer spin

N.B. If $\underline{\alpha}$ +ve $\Rightarrow$ raising by units of 2
If $\underline{\alpha}$ −ve $\Rightarrow$ lower by units of 2

- Weyl symmetry : $\mathbb{Z}_2$
  - $\longrightarrow$ for every +ve weight $\exists$ an image which is a −ve weight.
  - $\longrightarrow$ the heighest weight has a corresponding lowest weight, which is the lower bound.

Lattice:



one weight
two weights
three weights
↓
etc.

The root-lattice is a sublattice of the weight lattice.

The root lattice does not contain all weights
(generated by the simple roots)
→ introduce the weight lattice, which has all integer points.
(generated by the fundamental weights.)
→ the root lattice is a sublattice of index 2.

- We denote the weight lattice by Λ, and the root lattice by ~~Λ~~ Λ_r

$$\Rightarrow \frac{\Lambda}{\Lambda_r} = \mathbb{Z}_2 .$$

For $SU(n)$, this generalises to $\Lambda / \Lambda_r = \mathbb{Z}_n$.

ie. root lattice - take integer multiples of 2
weight lattice - take integers multiples of 1.

SO(3) repⁿ and SU(2) repⁿ

- A rotation op. is given by, say around the 3rd - axis, $\exp(i J_3 \theta)$.

Consider : $e^{i J_3 \theta} |\alpha, m\rangle = e^{im\theta} |\alpha, m\rangle$.

where $\theta \in [0, 2\pi]$ ie. $\theta = \theta + 2\pi$.

$\therefore$ the no. $e^{im\theta} = e^{im(\theta + 2\pi)}$

$$\Rightarrow e^{im2\pi} = 1 \quad \Rightarrow m \text{ integer}$$

- SO(3) repⁿs have integer spin, while SU(2) repⁿs have both integer and half-integer spin.

- As a matrix, we can write,

$$J_3 = \begin{pmatrix} j & & & 0 \\ & j-1 & & \\ & & \ddots & \\ 0 & & & -j \end{pmatrix}, \quad \text{(ie. diagonalised)}.$$

heighest weight

by a choice of basis $|\alpha, m\rangle$.

- $e^{iJ_3\theta} = \mathrm{diag}\left(e^{ij\theta}, e^{i(j-1)\theta}, \ldots, e^{i(-j)\theta}\right).$

- A nice feature:

$$\mathrm{Tr}\; e^{iJ_3\theta} = \sum_{m=-j}^{j} e^{im\theta}.$$

Define $X = e^{i\theta/2}$.

Then, $\mathrm{Tr}\; e^{iJ_3\theta} = \sum_{m=-j}^{j} X^{2m} \equiv f_n(x)$. → character

- Suppose we rotate around another axis, say we pick J_2.

$\longrightarrow e^{i\beta J_2}$.

$O e^{iH}|x\rangle$

$= O e^{iH} \underbrace{O^{-1} O}_{\text{new states}} |x\rangle$ Take: $e^{i\beta J_2} e^{iJ_3\theta} e^{-i\beta J_2}$, but the Tr. of this is preserved ie remains as $\mathrm{Tr}\; e^{iJ_3\theta}$.

- Then, $f_n(x)$ is invariant under conjugation by a matrix $M \in GL(n, \mathbb{C})$.

- The character of an SU(2) rep^n of highest weight n is given by;

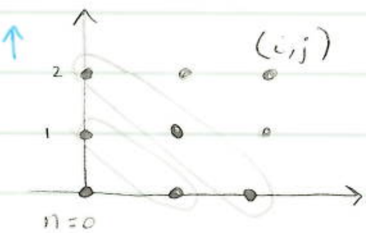
$$f_n(X) = \sum_{m=-n/2}^{n/2} X^{2m}$$

E.g.

$$\left.\begin{array}{ll} n=0 & f_0(X) = 1 \\ n=1 & f_1(X) = X + 1/X \\ n=2 & f_2(X) = X^2 + 1 + X^{-2} \end{array}\right\} \text{reproduces the lattice}$$

- Try to ~~(count)~~ generate <u>all</u> weights ~~for~~ all $SU(2)$ rep^{n}s.

heighest weight ↑



Known as the positive quadrant.

Pick two dummy variables, t_1, t_2, s.t. the power of t_1, t_2 is the point on the lattice.

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} t_1^{\,i} t_2^{\,j} = f(t_1, t_2)$$

$$= \frac{1}{(1-t_1)(1-t_2)}$$

- Set $t_1 = tX$ and $t_2 = \frac{t}{X}$. want to generate heighest weight

Then, $f(t_1, t_2) = \dfrac{1}{(1-tX)(1-t/X)}$

Taylor expand

$$= \sum_{n=0}^{\infty} f_n(X)\, t^n$$

$f(tx, t/x)$ is generating all characters of $SU(2)$.

- t_1 and t_2 are called fugacities.
  t is the fugacity of the h.w., n.
  X is the fugacity for the quantum no. under J_3.

Example

$$\frac{1}{(1-tx)(1-t/x)} = \underset{\substack{\uparrow \\ n=0}}{1} + \overbrace{\underset{\substack{\uparrow \\ n=1}}{(x + 1/x)}}^{f_1(x)} t + \overbrace{\underset{\substack{\uparrow \\ n=2}}{(x^2 + 1 + 1/x^2)}}^{f_2(x)} t^2 + O(t^3)$$

- Hence, we can find all the weights of $SU(2)$ by utilising the character, $f_n(x)$.

- $SU(2)$ Algebra : 3 generators X_i which satisfy :

$$[X_i, X_j] = \varepsilon_{ijk} X_k.$$

SU(3) Algebra

- 8 generators (as n^2-1)
  $\longrightarrow$ Gell-Mann matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \frac{1}{\sqrt{3}}$$

- Traceless generators as $\det(e^{i\lambda_i\theta}) = 1$.
  $\longrightarrow$ condition on SU(N)

Also $\underbrace{\lambda_a^\dagger = \lambda_a}_{}$ and $\underbrace{\mathrm{Tr}(\lambda_a \lambda_b) = 2\delta_{ab}}_{}$.

=> the SU(3) matrices are unitary

=> det = +1

where $a = 1, \dots 8$

We can define $T_a = -\frac{1}{2}\lambda_a$ and $T_a^\dagger = -T_a$

anti-hermitian

Then, $[T_a, T_b] = \underbrace{f_{abc}}_{} T^c$

structure constants for the algebra

- We have 3 different SU(2) subalgebras:

Need these to construct the rep^{n}s

$\lambda_1, \lambda_2, \lambda_3$

$\lambda_4, \lambda_5, \underbrace{\frac{\sqrt{3}}{2}\lambda_8 + \frac{1}{2}\lambda_3}_{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}}$

$\lambda_6, \lambda_7, \frac{\sqrt{3}}{2}\lambda_8 - \frac{1}{2}\lambda_3$

- 3 dim. rep^{n}s of the SU(3) algebra
  $\longrightarrow$ the defining rep^n.

- We have 2 diagonal matrices $- \lambda_3, \lambda_8$

$\Rightarrow [\lambda_3, \lambda_8] = 0$

$\Rightarrow$ all of the states can be parameterised by E-values of λ_3, λ_8.

- λ_3, λ_8 form a maximally commuting set of matrices $\therefore$ any state is characterised by their E-values.
  $\longrightarrow$ $\therefore$ will have 2 highest weights or a highest weight vector that is 2-dim.

Cartan Subalgebra

diagonal generators

- The set of maximally commuting generators.

- The rank is the no. of such generators.

No. of roots = dim − rank $\Rightarrow$ No. of roots for $SU(3) = 8 - 2 = 6$.

$\rightarrow$ rank of $SU(3) = 2$. $\Rightarrow$ rep^n are 2 dim.
$\rightarrow$ rank of $SU(n) = n-1$.

- Use new basis as more convenient for computations.

Diagonal elements are E-values of the 3dim rep^n

Pick $h_1 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $h_2 = \begin{pmatrix} \sqrt{3}/2 & 0 & 0 \\ 0 & \sqrt{3}/2 & 0 \\ 0 & 0 & -\sqrt{3} \end{pmatrix}$

$\rightarrow$ diagonal, $[h_1, h_2] = 0$.
$\rightarrow$ all the rep^{n}s will be E-values of h_1, h_2.

- Raising / lowering op:

The weights are ... $(1/2, \sqrt{3}/6)$, $(-1/2, \sqrt{3}/6)$ and $(0, -\sqrt{3}/3)$. This forms the weight linear combination of generators eg. like $J_\pm$.

$$e_+^1 = \begin{pmatrix} 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad e_-^1 = \begin{pmatrix} 0 & 0 & 0 \\ 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$e_+^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \qquad e_-^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 \end{pmatrix} = (e_+^2)^T$$

~

The roots form a regular hex.

$$e_+^3 = \begin{pmatrix} 0 & 0 & 1/\sqrt{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad e_-^3 = (e_+^3)^T$$

The roots are the differences of weights as the corresponding generators must take us from one weight to another.

$\rightarrow$ traceless, not hermitian.

- Replace $h_i \rightarrow H_i$ as rep^{n}s we construct will be $n \times n$, not 3×3.

Explicitly compute f_{abc}.

$E_\pm^1$ not in max. commuting algebra as it only commutes w/ H_2 and not also H_1.

$$[H_1, H_2] = 0$$

commutation relation w/ Cartan subalg.

$$[H_1, E_\pm^1] = \pm E_\pm^1 \qquad [H_2, E_\pm^1] = 0.$$

$$[H_1, E_\pm^2] = \mp \frac{1}{2} E_\pm^2 \qquad [H_2, E_\pm^2] = \pm \frac{\sqrt{3}}{2} E_\pm^2$$

SU(2) subalgebra

$$[H_1, E_\pm^3] = \pm \frac{1}{2} E_\pm^3 \qquad [H_2, E_\pm^3] = \pm \frac{\sqrt{3}}{2} E_\pm^3.$$

"like" terms

Like $[J_+, J_-] = J_3$

$$[E_+^1, E_-^1] = H_1 \qquad [E_+^2, E_-^2] = \frac{\sqrt{3}}{2} H_2 - \frac{1}{2} H_1$$

$$[E_+^3, E_-^3] = \frac{\sqrt{3}}{2} H_2 + \frac{1}{2} H_1$$

mixed terms

$$[E_+^1, E_+^2] = \frac{1}{\sqrt{2}} E_+^3 \qquad [E_-^1, E_-^2] = \frac{-1}{\sqrt{2}} E_-^3$$

$$[E_+^1, E_-^3] = -\frac{1}{2} E_-^2 \qquad [E_-^1, E_+^3] = \frac{1}{\sqrt{2}} E_+^2$$

$$[E_+^2, E_-^3] = \frac{1}{\sqrt{2}} E_-^1 \qquad [E_-^2, E_+^3] = \frac{-1}{\sqrt{2}} E_+^1.$$

- Each commutation has a meaning that we need to figure out.

- Denote a state by $|\underbrace{\lambda_1, \lambda_2}_{\text{highest weight}}, \underbrace{p, q}_{\text{E-values of } H_1, H_2}\rangle$

So, $$H_1 |\lambda_1, \lambda_2, p, q\rangle = p|\lambda_1 \lambda_2 p q\rangle$$

$$H_2 |\lambda_1 \lambda_2 p q\rangle = q|\lambda_1 \lambda_2 p q\rangle.$$

What are the allowed values of p, q and how many states?

$$E_\pm^1 |\lambda_1 \lambda_2 p q\rangle \qquad \text{raises } p \text{ by } \pm 1$$

$$= C_{\lambda_1, \lambda_2, p, q} |\lambda_1 \lambda_2 (p \pm 1) q\rangle.$$

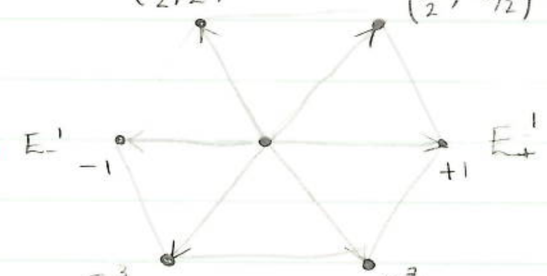
Use $H_1 H_2 E_\pm^2 |\ \rangle$ to find E-values

$$\hookrightarrow E_\pm^2 |\lambda_1 \lambda_2 p q\rangle \qquad \text{raises } p \text{ by } \mp\tfrac{1}{2} \text{ and } q \text{ by } \pm\tfrac{\sqrt{3}}{2}.$$

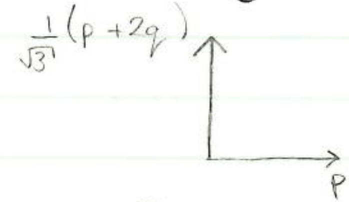
$$= C'_{\lambda_1 \lambda_2 p q} |\lambda_1 \lambda_2, p \mp \tfrac{1}{2}, q \pm \tfrac{\sqrt{3}}{2}\rangle$$

$E^3_\pm$ changes (p,q) to $(p,q) \pm \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

Equilateral Δs.



Have a regular hexagon.



plotted on this axis

The +ve roots correspond to raising op.

3 positive roots
3 negative roots } 3 SU(2) subalgebras

The highest weight has the property that we cannot raise it.

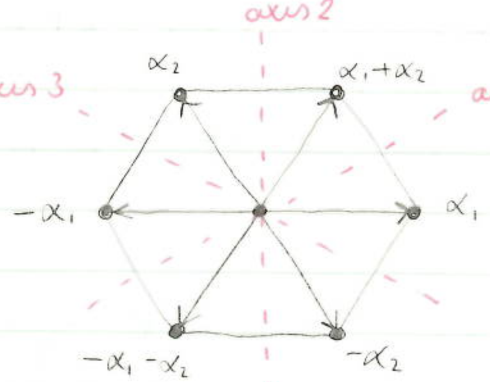
$\Rightarrow$ α_1, α_2, $\alpha_1 + \alpha_2$

$-\alpha_1$, $-\alpha_2$, $-\alpha_1 - \alpha_2$ by Weyl sym.

$\sim$

Rotations of $2\pi/3$ + reflections in y-axis leave the set of weights invariant

α_1, α_2 are the root vectors



- - - - Weyl reflection

} action by +ve roots

SU(2) associated w/ each root direction

- H_1, H_2 have zero weight (ie. middle/centre of hexagon).

- $\{\alpha\}$ are the vectors we add as we raise and lower.

- Have 3 $su(2)$ subalgebras
  - → one from $\alpha_1 / -\alpha_1$
  - → generated by $\pm\alpha_2$
  - → and the final by $\pm(\alpha_1+\alpha_2)$.

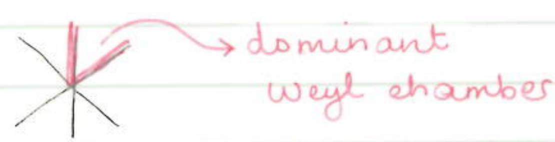
The subalgebras do not commute amongst themselves.

- Each $su(2)$ has a Weyl reflection
  - → can start with one pt (eg α_1) and can get all the rest from Weyl reflection
  - → eg. can get $(-\alpha_1 - \alpha_2)$ from $-\alpha_2$.

- The hexagon has a reflection symmetry
  - → Weyl gp. is S_3 ie. the permutation group in 3 elements

N.B. Not including rotation symmetry. Reflections (sucessive reflections) do not commute amongst themselves.

- Using Weyl symmetry bypasses the lengthy $[\ ,\ ]$ calculations we had to do.

- We have 6 wedges 

  - → if we know rep^n in one wedge, we know rep^n in all wedges due to Weyl sym.

- Wedges are called Weyl chambers.

- What's special about the dominant Weyl chamber?

- Require highest weight that vanishes on action by the +ve roots.

- For a given highest weight λ_1, λ_2, we demand :

$$E_+^i \, |\lambda_1 \, \lambda_2 , \, p = \lambda_1 , \, q = \lambda_2 \rangle = 0$$

- The dominant chamber contains all the possible highest weights.

- Suppose we are not in the dom. chamb.

Acting by α_2 moves us to dom chamb

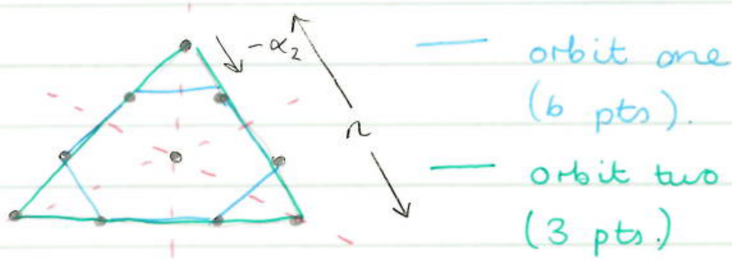
all the highest weights live here
Applying +ve roots here keeps us in this chamber.

- 4 different regions of a chamber.
  (i) pts. inside the chamber
  (ii) on the boundary 2 lines
  1 pt. (origin).

- Pick the pt. at the origin
  $\Rightarrow (\lambda_1, \lambda_2 = 0)$ trivial rep^n of dim 1.

Pick highest weight to be along the vertical axis
$\longrightarrow$ apply Weyl reflection

Should look like



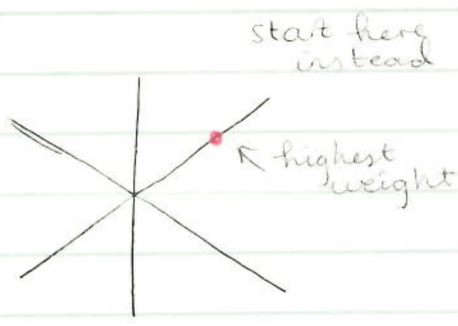
This is a 10 d rep^n

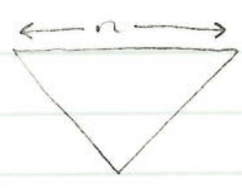
- We have a set of triangles with edge of length n and the no. of pts. is $\frac{(n+1)(n+2)}{2}$

E.g. in above example, $n = 3$.

- A class of rep^{n}s of SU(3) is parameterised of dim $\frac{(n+1)(n+2)}{2}$

- $n = 0$ gives the trivial rep^n.



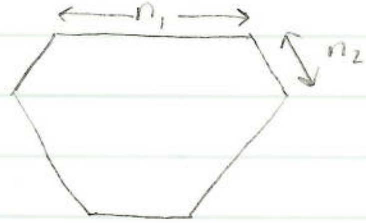


No. of pts $= \frac{(n+1)(n+2)}{2}$

- Pick pt. inside wedge eg. $\alpha_1 + \alpha_2$.
  $\rightarrow$ 8-dim rep^n $\rightarrow$ 6 pts in orbit, 2 in the centre of zero weight.

- For highest weights inside the wedge has the shape of the polygon is a hexagon.

Can have:



No. of pts is $\dfrac{(n_1+1)(n_2+1)(n_1+n_2+2)}{2}$.

- Triangle is a special case where $n_2 = 0$.

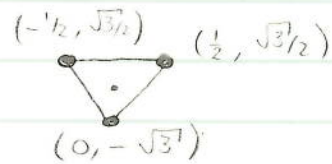
- All irrep^{n}s of $SU(3)$ are parameterised by 2 highest weights n_1, n_2 of dimension:

$$\frac{(n_1+1)(n_2+1)(n_1+n_2+2)}{2}$$

Examples

- $[n_1, n_2] = 0 \Rightarrow \in$ family of $\triangle$s.

has length 1 and three pts.



$\Rightarrow$ 3 dim rep^n.

The pts. are exactly the values for (p, q).

- $[0, 1]$

Also 3 dim.

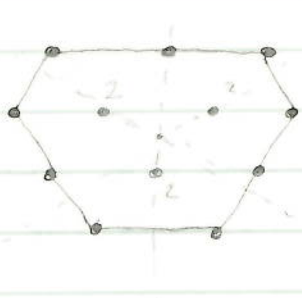
- $[2, 0]$

6pts.

Multiplicity of each weight is the same on each layer. For $n_1 > n_2$, there are n_2+1 hexagonal layers and then the layers become triangular. For hexagonal layers, the multiplicity $\uparrow$ by 1 moving inwards; $\triangle$ layers all have multiplicity (n_2+1).

- $[0, 2]$

- $[2, 1]$



$\dim = \frac{3 \cdot 2 \cdot 5}{2} = 15$

ie need 15 pts on diagram

Multiplicity of 2.

No. of pts $= 3\times 3 + 3 \times 2 = 15$
$\Rightarrow$ 15 dim. rep^n.

$\sim$

- $SU(3)$ has dim $8 = \underset{\text{rank}}{2} + \underset{\text{+ve roots}}{3} + \underset{\text{-ve roots}}{3}$.

$\uparrow$ are 2-dim vectors

$\rightarrow$ No. of simple roots is equal to the rank.

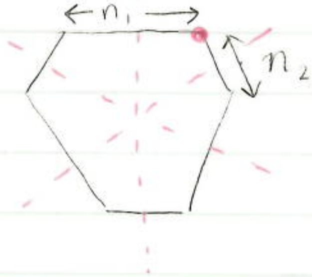
If a weight is annihilated by the generators of all the simple roots, it is the highest weight of the rep^n.

- We have 2 simple roots α_1, α_2.
  $\rightarrow$ all other roots are a linear combination of the simple roots.

- An irrep^n of $SU(3)$ is parameterised by 2 non-ve integers, n_1 and n_2 of dim.

$$\frac{(n_1+1)(n_2+1)(n_1+n_2+2)}{2}.$$

These rep^{n}s are characterised by hexagons.



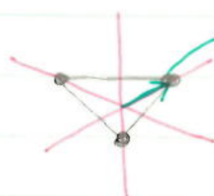
• highest weight → always in the dominant Weyl chamber.

- - - - Weyl symmetry

- n_1, n_2 parameterise the size of the hexagon, and define the dim of the rep^n.

- Consider rep^n $[1, 0]$

Fundamental rep^n

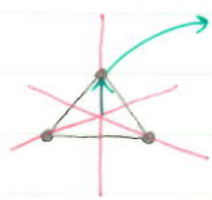


→ highest weight → at the boundary of the chamber

Define vector ↗ λ_1 (a 2 vector).

Rep^n $[0, 1]$

Anti-fundamental rep^n

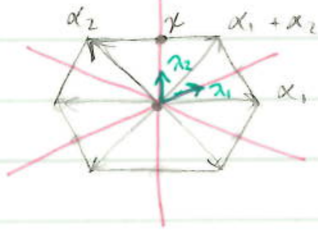


→ highest weight. Again on the boundary

Call ↑ λ_2.

$\therefore$ λ_1 is the highest weight of $[1, 0]$ and λ_2 the highest weight of $[0, 1]$.

N.B. λ_i are "smaller" vectors than α_i.



$$\lambda_1 = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2$$
$$\lambda_2 = \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2$$
} fundamental weights

$\alpha_1 = 2\lambda_1 - \lambda_2$
$\alpha_2 = -\lambda_1 + 2\lambda_2$

So, define $A = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

transit from $\Lambda_w \leftrightarrow \Lambda_r$ w/ matrix A.

and $\lambda_i = A_{ij}\alpha_j$ the λ_i are the weight vec.

Any vector in the dom. Weyl chamber is just a linear combination of λ_1, λ_2.

The highest weight is $\lambda = n_1\lambda_1 + n_2\lambda_2$

$\Rightarrow$ λ_1, λ_2 are called fundamental weights.

- The weight lattice $\Lambda_w = \mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2$
  integer

- The root lattice $\Lambda_r = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$.

Cyclic gp. of order n.

$\mathbb{Z}_n = \dfrac{\mathbb{Z}}{n\mathbb{Z}} = \{0, 1, \ldots, n-1\}$. w/ addition modulo n.

$\rightarrow$ compare Λ_w and Λ_r : $\dfrac{\Lambda_w}{\Lambda_r} = \mathbb{Z}_3$.

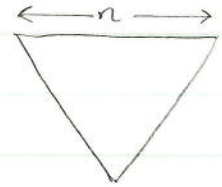
1 pt. in Λ_r $\Rightarrow$ 3 pts. in Λ_w and vice versa.

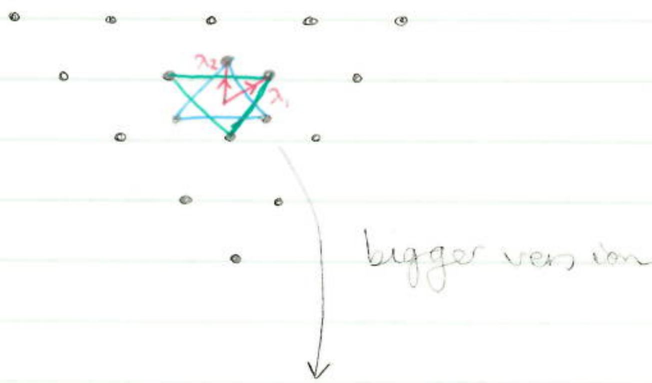
E.g. $\omega^n = 1$ $\Rightarrow \omega = e^{2\pi i k/n}$ where $k = 0, 1, \ldots (n-1)$

i.e. Reflecting pts. at λ_2 gives us a total of 3 pts.
But reflecting from pt. directly above λ_2 (χ in the diagram above) gives 6 pts.

Examples of Cyclic Group

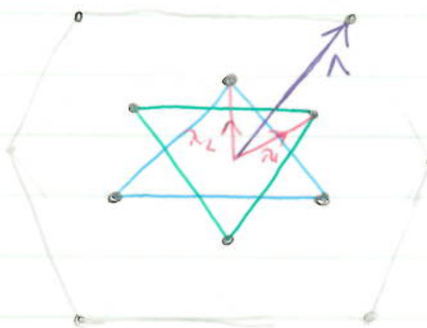
Rep^{n}s $[n, 0]$





∇ is 3 dim rep^n

∇ is the other 3 dim rep^n.



$\Lambda = \lambda_1 + \lambda_2 = \alpha_1 + \alpha_2$

Just get our hexagon again

The set of rep^{n}s given by $\lambda = n\lambda_1$

- Consider a 3-vec. $V_i \in \mathbb{C}$, $i = 1, 2, 3$.

Let $a_{ij} = V_i V_j$; how many components for a?

$\rightarrow$ 3×3 symmetric $\Rightarrow$ a_{ij} has 6 compts.

Let $a_{ijk} = v_i v_j v_k$
$\longrightarrow$ 27 compts., but only 10 independ.

Let $a_{i_1 \ldots i_n} = v_{i_1} \ldots v_{i_n} \Rightarrow \frac{(n+1)(n+2)}{2}$ independ compts.

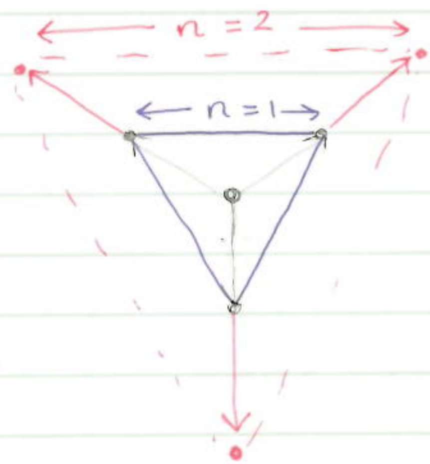
This is because a is an n^{th}-rank symmetric tensor.

We compute the n^{th} sym. product $\mathrm{Sym}^n[1,0] = [n,0]$
$\longrightarrow$ n^{th} sym. product of the 3 dim rep^n.

ie. to find dimension of $\overset{\leftarrow n \rightarrow}{\nabla}$ we count the no. of pts. in the triangle.

- Consider the 3 weights of fundamental rep^n of SU(3). Take all possible multiplications

Constructed a Δ of $n=2$ from $n=1$.

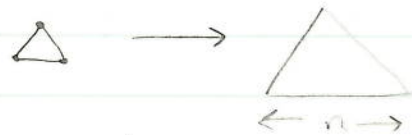


where ↗ and ↗ are vectors.

Note: Vectors ↗ are of the same length as Δ are equilateral. Same w/ ↑.

Just used λ_1 here and then applied successively to get the n-length triangle

- The $\mathrm{Sym}^n [0,1] = [0,n]$.



- Introduce lower and upper indices:

w^i, $i = 1, 2, 3$ antifundamental index

$w_i \Rightarrow$ fundamental index.

- Now consider:

Tensor product

$$a_i^{\,j} = \underset{[1,0]}{v_i}\ \underset{[0,1]}{w^j}$$

different from a_{ij}, a_{ijk} from before.

→ Now we have 9 compts.
→ Draw the lattice for this

Just add the E-values of the rep^{n}s being tensor producted, then decompose.

ie vec add^n.

from multiplying antipodal pts

$$\text{ie. } \underbrace{\overset{3}{[1,0]}\ \overset{3}{[0,1]}}_{\text{tensor product}} = \overset{8}{[1,1]} + \overset{1}{[0,0]}$$

trivial rep^n

3 antiquarks
3 quarks

8 gluons
(mesons).

dim 8

dim 1

SU(3) Algebra :

- 8 generators that transform in the adj. repⁿ
- 2 Cartan gen.
- 3 +ve roots $\longrightarrow$ 2 simple roots, 2 fundamental weights

  3 -ve roots.

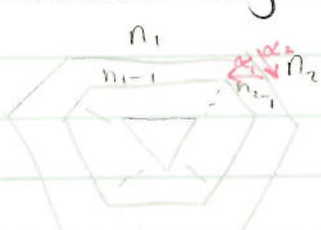
- The SU(3) repⁿ is given by two integer no. $[n_1, n_2]$

The dim. of $[n_1, n_2] = \dfrac{(n_1+1)(n_2+1)(n_1+n_2+2)}{2}$

- The highest weight λ of the repⁿ $[n_1, n_2]$ is given by :

$$\lambda = n_1 \lambda_1 + n_2 \lambda_2$$

- Have diminishing hexagons layer by layer :



Eventually there will be a layer that is a $\triangle$

Outer layer has multiplicity 1, next layer has 2 etc. until we hit the $\triangle$, then the multiplicity remains constant.

- Start with λ (highest weight) then act with vectors inwards to get to the centre.
  → also use Weyl sym.

Symmetric Products

- $\mathrm{Sym}^n [1,0] = [n,0]$

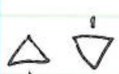
Highest weight λ_1 in $[1,0] \Rightarrow n\lambda_1$ in $[n,0]$.

- $\mathrm{Sym}^n [0,1] = [0,n]$.

- Sym^n  = 

Sym^n  = 

Tensor Product

 =  + ·

dim $3 \quad 3 = 8 + 1$

$[1,0]$ - fundamental rep^n dim 3

$[0,1]$ - antifundamental rep^n dim $\underline{3}$ or $\overline{3}$.

$\Rightarrow \quad 3 \times \underline{3} = 8 + 1.$

↳ adjoint rep^n