SO(3) - Rotations

- Consider vectors $\underline{A}$ and $\underline{B}$
  → can define vector calculus.

Transform in the trivial rep^n.

Scalar product $\underline{A} \cdot \underline{B} = A_i B_j \delta_{ij}$ $i, j = 1, \ldots 3.$

$k = 1, \ldots 3.$

Generate another vector

Cross product $\underline{A} \times \underline{B} = A_i B_j \varepsilon_{ijk} = C_k$

Tensor product (2nd rank). $A_i B_j + A_j B_i - \underbrace{a}_{a = \frac{2}{3}} \delta_{ij} \underline{A} \cdot \underline{B}$ symmetric

$= D_{ij}$

$D_{ij} \delta_{ij} = 0$ ie traceless.

What is a?

Check : $A_i B_j \delta_{ij} + A_j B_i \delta_{ij} - a \overbrace{\delta_{ij} \delta_{ij}}^{3} A_i B_j$

$\Rightarrow (1 + 1 - 3a) \underline{A} \cdot \underline{B} = 0$

$\Rightarrow a = \frac{2}{3}$

- Translate this to the language of rep^{n}s.

- How does a vector transform under SO(3)?

Recall SU(2) has ∞ many rep^{n}s parameterised by $[n]$ $n = 0, 1 \ldots$ of dim $n+1$. n is the highest weight.

→ $\underline{A}$ and $\underline{B}$ are in the 3 dim rep^n of SO(3).

$\Rightarrow [2]$ of SU(2) or SO(3)

↳ $n = 2j$

The tensor product $[2][2]$:

in QM $\overbrace{1 \times 1}^{\text{tensor product}} = 2 + 1 + 0$ add^n of ang. mom.

Then, $[2][2] = \underset{\substack{\text{dim } 5}}{[4]} + \underset{\substack{\text{dim } 3 \\ \underline{A} \times \underline{B}}}{[2]} + \underset{\substack{\text{dim } 1 \\ \underline{A} \cdot \underline{B}}}{[0]}$

- The scalar product is sym. under exchange of $\underline{A}$ and $\underline{B}$.

- The cross product is antisym.

- The 2^{nd} rank traceless tensor product is sym.

- If we take $\mathrm{Sym}^2 [2] = [4] + [0]$.

  Antisym. two vectors $\underset{\text{"wedge" 2}}{\Lambda^2} [2] = [2]$

  Symmeterise and antisymmeterise vectors.

  ie. get another vec.

- When we take a collection of bosons, we symmeterise their wavefunc^n.

- We antisymmeterise for fermions

- Symmetric product

Suppose we have n variables x_i where $i = 1, \dots n$. (real or complex).

Construct a bilinear $x_i x_j = a_{ij}$

$\Rightarrow a_{ij}$ symmetric

$\therefore$ how many compts. does a_{ij} have?

N.B. n here is the dim. of the rep^n eg. [2] has $n=3$.

There are $\frac{n(n+1)}{2}$ components.

We count by setting $i \leq j$ ie. $i = 1, \dots j$
$j = 1, \dots n$

- Now let $b_{ijk} = x_i x_j x_k$

Order them again s.t. $i \leq j \leq k$

$$\sum_{k=1}^{n} \sum_{j=1}^{k} \sum_{i=1}^{j} 1 = \frac{(n+2)(n+1)n}{6}$$

$$= \binom{n+2}{3}$$

- Consider the product

$$\underbrace{\prod_{i=1}^{n} \frac{1}{1-x_i}}_{\text{Generator of symmetrisations. } \to \text{ Bose-Einstein statistics}} \overset{\text{taylor}}{=} 1 + \sum_{i=1}^{n} x_i + \sum_{i \leq j} x_i x_j$$
$$+ \sum_{i \leq j \leq k} x_i x_j x_k + \dots \sum_{i_1 \leq i_2 \leq \dots \leq i_m} x_{i_1} \dots x_{i_m}$$

Set $x_i = x$ then expand $\frac{1}{(1-x)^n}$ $n = \dim$ of rep^n

$$\Rightarrow \frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k. \qquad = \frac{(n+k-1)!}{(n-1)!\,k!}$$

- Hence, the k^{th} - sym. product of n elements has $\binom{n+k-1}{k}$ different compts.

Exercise: Show that the k^{th} antisym. product of n elements has $\binom{n}{k}$. $\frac{n!}{k!(n-k)!}$

V: 3 dim rep^n of $SU(3)$

$Sym^2\, 3 = 6$ ie. 6 independ. compts.

For V_i 3 dim rep^n of $SO(3)$, $[2]$ of $SO(3)$

$$\underset{\substack{\downarrow \\ 6 \text{ compts.}}}{Sym^2[2]} = \underset{5}{[4]} \underset{+}{+} \underset{1}{[0]}$$

<u>Note</u>: $a_{ij} = \underbrace{V_i V_j + V_j V_i}_{\text{Need this to get 6 compts}} - \frac{2}{3}\delta_{ij} V_i V_j$

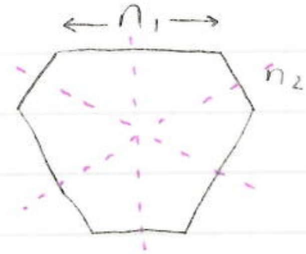
→ instructed to do this by the decomp. of the $Sym^2[2]$ product.

→ construct $SO(3)$ rep^{n}s from vectors.

<u>Recall</u> an irrep. of $SU(2)$ is denoted by the highest weight n $[n]$, $n = 0, 1, \ldots$ non-negative, dim $n+1$.

<u>Recall</u>: irreps of $SU(3)$ are denoted by two highest weights n_1, n_2 $[n_1, n_2]$ of dim. $\frac{(n_1+1)(n_2+1)(n_1+n_2+2)}{2}$.

Graphical repⁿ:



Hexagon invariant under Weyl gp. S_3.

* Symmetric Product / Antisymmetric Product + Tensor Product.

Sym. — bosons
Antisym — fermions
Tensor — many particle $\mathcal{H}$ space.

<u>Examples</u>: $SU(2)$.

$[1][1] = [2] \oplus [0]$
$2 \times 2 \quad 3 + 1$

$[1][2] = [3] + [1]$
$2 \times 3 \quad 4 + 2$

$[1][3] = [4] + [2]$
$2 \times 4 \quad 5 + 3$

$[1][n] = [n+1][n-1]$
$\underbrace{2 \times (n+1)} \quad (n+2) + n$

In terms of weights :

$[1][1]$  

$= 2, 0, 0, -2$

Get same result as before.

$= [2] + [0]$

$[1][2] =$ 

$= 3, 1, -1, 1, -1, -3$

$= [3] + [1]$ as before

$[1][n] =$ 

$= [n+1] + [n-1]$

Symmetric Product

$$\mathrm{Sym}^n[1] = \underset{n+1}{[n]}$$

What is the dim. on the LHS ?

Consider $\underbrace{Sym^2[1]}_{3} = [2]$ 3 - ie. 3 states

→ how many objects left when symmetrising them twice

E.g. no. 1 and 2 - symmetrise twice
11, 22, 21 (no 12 as 21 = 12).

ie. $Sym^2\{1,2\} = \{11, 22, 12\}$

<u>Recall:</u> $$\prod_{i=1}^{n} \frac{1}{1-x_i} = 1 + \sum_{i}^{n} x_i + \sum_{i\leq j=1}^{n} x_i x_j + \sum_{i\leq j\leq k}^{n} x_i x_j x_k + \dots$$

Set $x_i = x$ to get dim. of sym. product

$$\Rightarrow \frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k$$

n objects, symmetrise k times

$$= 1 + nx + \left(\frac{n+1}{2}\right) n x^2 + \frac{(n+2)(n+1)}{3!} x^3 + \dots$$

→ no. of n objects symmetrised k times is $\binom{n+k-1}{k}$

dim of rep^n [1]

For $Sym^n[1]$, we have $n=2$, $k=n$.
$$\Rightarrow \binom{n+1}{n} = \frac{(n+1)!}{n!} = n+1$$

$\mathrm{Sym}^2[2] = [4] + [0]$

$6 \qquad\qquad 5 + 1$

$\left.\begin{array}{l} n = 3 \\ k = 2 \end{array}\right\}$

Using the graphical rep^n:

$\mathrm{Sym}^2 \Big($  $\Big) =$ 

$4, 2, 0, 0, -2, -4$

→ 6 weights (not nine like for tensor product)

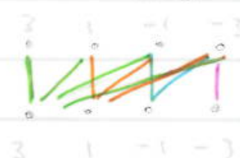
→ comes from order of $i \leq j \leq k \ldots$

$=$ 

$\mathrm{Sym}^2 \begin{pmatrix} +3 & 1 & -1 & -3 \\ \circ & \circ & \circ & \circ \end{pmatrix}$

$= \overbrace{6, 4, 2, 0}^{3}, \overbrace{2, 0, -2}^{1}, \overbrace{-2, -4}^{-1}, \overbrace{-6}^{-3} = [6] + [2]$

ie.



$\underbrace{\mathrm{Sym}^2 \underbrace{[3]}_{4}}_{10} = [6] + [2]$

$7 + 3$

$k = 2, \ n = 4$

General formulas :

$\mathrm{Sym}^2[n] = [2n] + [2n-4] + [2n-8] + \cdots$

Recall : $\sum_{n=0}^{\infty} [n] t^n = (1 - tx)^{-1}(1 - t/x)^{-1}$ use to prove

character of SU(2)

set $[n] = x^n + x^{n-2} + \ldots x^{-n}$

Tensor product

— Add^n of ang. momentum. in QM.

$$[n_1][n_2] = \sum_{n = |n_1 - n_2|}^{n_1 + n_2} [n] \quad \overset{-}{\underset{\Delta}{=}}$$

$\Delta n = 2$ ie jumps of 2.

$$\underset{\Delta}{=} [n_1+n_2] + [n_1+n_2-2] + \cdots [|n_1-n_2|]$$

Consider now, $\Lambda^2[1] = [0]$.

- Products of characters compute tensor products.

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- Tensor product
- Sym. product
- Antisym. product

Develop analytic tools to evaluate these.

- Consider the character of a rep^n $\chi_R(x_1,\dots,x_r)$, r is the rank of the algebra.

Rank r = no. of generators in the Cartan subalgebra.

$$\chi(x_1,\dots,x_r) = \sum_{\lambda \in \text{weights}} x_1^{\lambda_1} \cdots x_r^{\lambda_r}$$

E.g. SU(2), the set of irrep^{n}s $[n]$, $n = 0, 1\dots$.
The set of weights λ of $[n]$, Λ_n

$$\Lambda_n = \{n, n-2, n-4, \dots, -n\}.$$

So, $$\chi_n(x) = \sum_{\lambda \in \Lambda_n} x^\lambda$$

$$= \sum_{\substack{m=-n \\ \Delta m = 2}}^{n} x^m = \frac{x^{n+1} - x^{-n-1}}{x - x^{-1}}.$$

$\chi_1(x) = x + x^{-1} \longleftrightarrow$ weights $1, -1$.

$\chi_2(x) = x^2 + 1 + x^{-2} \longleftrightarrow 2, 0, -2$.

ie. the power of x is the weight.

Fugacity: we discuss for no. of particles. Here in ref. to 3rd projection of spin. — Both no. of particles + ang. mom. are conserved.

In stat. mech. x is called the fugacity.

x is the conjugate variable to the weight.
→ weight $SU(2)$ is ε-value of J^3.
→ related to conserved Noether charge.

N.B. $\mathrm{Tr}\, e^{iJ_3\theta} = \chi_n(x)$; $x = e^{i\theta/2}$

$(n+1)\times(n+1)$ matrix   angle θ

The character is invariant under conjugation:

If $M_{(n+1)\times(n+1)} \in GL(n+1, \mathbb{C})$ then,

$$\mathrm{Tr}\left(M e^{iJ_3\theta} M^{-1}\right) = \mathrm{Tr}\, e^{iJ_3\theta}.$$

- How do we use characters to compute products?

Characters and Products

Tensor Product

- Just take the product of characters.

Character : $\chi_n(X)$ for rep^n $[n]$

The set of weights for rep^n $[n]$ is denoted by Λ_n.

$$\Lambda_n = \{n, n-2, n-4, \dots, -n\}$$

$$\chi_n(X) = \sum_{\substack{m=-n \\ \Delta m = 2}}^{n} X^m$$

$$= X^n + X^{n-2} + X^{n-4} + \dots + X^{-n+4} + X^{-n+2} + X^{-n}$$

i.e. multiple by 1

$$(X^n + X^{n-2} + X^{n-4} + \dots + X^{-n+4} + X^{-n+2} + X^{-n}) \frac{(X - X^{-1})}{(X - X^{-1})}$$

$$= (X^{n+1} + X^{n-1} + X^{n-3} + \dots X^{-n+5} + X^{-n+3} + X^{-n+1} - X^{n-1} - X^{n-3} \dots - X^{-n+3}$$

$$- X^{-n+1} - X^{-n-1}) \times \frac{1}{(X - X^{-1})}$$

$$= \frac{X^{n+1} - X^{-n-1}}{X - X^{-1}}$$ as required

E.g $[3][2] = [5] + [3] + [1]$

$4 \times 3 \qquad 6 + 4 + 2$

$$\chi_3(x)\,\chi_2(x) = (x^3 + x^1 + x^{-1} + x^{-3}) \times$$
$$\times\,(x^2 + 1 + x^{-2})$$

12 weights as required.

$$\left\{\begin{aligned} &= x^5 + x^3 + x^1 + x^3 + x^1 + x^{-1} \\ &\quad + x^1 + x^{-1} + x^{-3} + x^{-1} + x^{-3} + x^{-5} \end{aligned}\right.$$

$$= \chi_5(x) + \chi_3(x) + \chi_1(x)$$

Symmetric Product / Antisymmetric Product

- Recall: the generator of symmetrisations for n variables X_i is

$$\prod_{i=1}^{n} = \frac{1}{1 - X_i}.$$

Bose-Einstein distribution in Stat. Mech.

For a single object we have $1/_{1-x} = \sum_{n=0}^{\infty} x^n$ → all possible powers

(i.e. because we have only one highest weight of SU(2). Need two objects for SU(3))

- Similarly for antisym. product, we have generator of antisymmetrisations of n variables Y_i,

$$1 + \sum_{i=1}^{m} Y_i + \sum_{i<j} Y_i Y_j + \sum_{i<j<k} Y_i Y_j Y_k + \cdots Y_1 \cdots Y_m$$

$$= \prod_{i=1}^{m} (1 + Y_i)$$

↳ powers of zero or one only

Fermi-Dirac distribution

$$\frac{\prod_{i=1} (1+Y_i)}{\prod_{j=1}^{n} (1-X_j)} \Rightarrow \text{all possible combinations. Antisym. } Y_i, \text{ Sym. } X_j.$$

Set $Y_i = Y$ (ie all the same), then

$$(1+Y)^m = \sum_{k=0}^{m} \binom{m}{k} Y^k$$

↳ antisym. m objects k times.

E.g. $Sym^2[1] = [2]$.

$$\overset{(1+xt+(xt)^2+\cdots)(1+(x^{-1}t)+(x^{-1}t)^2}{\frac{1}{1-xt} \cdot \frac{1}{1-x^{-1}t}} \overset{\text{Taylor}}{=} 1 + (x+x^{-1})t + \underbrace{(x^2+1+x^{-2})}_{[2]}t^2 + \cdots$$

keeps track of no. of weights

symmetrised twice

t is the fugacity, which counts the order of symmetrisation.

→ ie. how many times we symmetrise.

$Sym^3[1] \longrightarrow$ go to t^3

ie. $(x^3 + x + x^{-1} + x^{-3})t^3$ (carried on from above).

* So, $Sym^n[1] = \sum_{n=0}^{\infty} \chi_n(x) t^n$ *

$= [n]$.

\- Now, $Sym^2[2] = [4] + [0]$.

$$\chi_2(x) = x^2 + 1 + x^{-2}$$

$$\frac{1}{(1-tx^2)(1-t)(1-tx^{-2})} = (1+tx^2+t^2x^4+\dots)(1+t+t^2+\dots)(1-tx^{-2}+t^2x^{-4}+)$$

$$= 1 + \chi_2(x)t + (\underbrace{x^4 + x^2 + 1 + 1 + x^{-2} + x^{-4}}_{[4]+[0]})t^2$$

$$+ \dots$$

i.e. $\mathrm{Sym}^2\,[2] = \chi_4(x) + \chi_0(x)$.

$\sim$

- Consider $\prod_{i=1}^{n} \frac{1}{1-x_i} \triangleq$, sym. of n objects

$$\triangleq \exp\left\{\log\left(\prod_{i=1}^{n}\frac{1}{1-x_i}\right)\right\}$$

$$= \exp\left\{-\sum_{i=1}^{n}\log(1-x_i)\right\}$$

$$= \exp\left\{\sum_{i=1}^{n}\sum_{k=1}^{\infty}\frac{x_i^k}{k}\right\} \triangleq$$

Note: n is the rank of the algebra. So for $SU(2)$ we have $n=1 \Rightarrow 1/1-x$. For $SU(3)$ we have $n=2$, so we have $\frac{1}{1-x_1}\cdot\frac{1}{1-x_2}$.

Taylor expand:

$$\log(1-z) = \sum_{k=1}^{\infty}\frac{z^k}{k}$$

$$\log(1+z) = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}z^k}{k}$$

- Define $f(x_i) = \sum_{i=1}^{n} x_i$

$$\triangleq \exp\left\{\sum_{k=1}^{\infty}\frac{\sum_{i=1}^{n}x_i^k}{k}\right\} = \exp\left\{\sum_{k=\phi}^{\infty}\frac{f(x_i^k)}{k}\right\}.$$

[illegible] : $\prod^{m}(1-Y_i) = \exp\{\log(\prod(1-Y_i))\} = \exp\{\sum^{m}\log(1-Y_i)\}$

- We have a definition for a func^n $f(x_i)$ s.t. $f(x_i)\Big|_{x_i=0} = 0$.

↱ Greek origin for many/plural } from πληθυσμος, meaning "multiplication".

The plethystic exp. of f is,

$$PE[f] = \exp\left\{ \sum_{k=1}^{\infty} \frac{f(x_i^k)}{k} \right\}.$$

This func^n generates symmetrisations.

- To generate antisymmetrisations,

$$PE_F[f] = \exp\left\{ \sum_{k=1}^{\infty} \frac{(-1)^{k+1} f(x_i^k)}{k} \right\}.$$

Exercise: show $\prod_{i=1}^{n} (1+y_i)$ is equal to the above.

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<u>Plethystic Exponential:</u>

$f(x_i)$ s.t. $f(x_i = 0) = 0$

generates sym. $$PE[f] = \exp\left\{ \sum_{k=1}^{\infty} \frac{f(x_i^k)}{k} \right\}.$$ bosons

generates antisym. $$PE_F[f] = \exp\left\{ \sum_{k=1}^{\infty} \frac{(-1)^{k+1} f(x_i^k)}{k} \right\}.$$ fermions

- What does this mean in practice?

Power of t gives no. of sym., as before.

Suppose $f(x_i) = \sum_{i=1}^{n} x_i$, then

$$PE[f(x_i)t] = \sum_{n=0}^{\infty} Sym^n[f]\, t^n$$

counts no. of time we sym.

$\Rightarrow$ <u>generates</u> sym.

- Similarly,

$$PE_F[f(x_i)t] = \sum_{n=0}^{\infty} \Lambda^n[f]\, t^n.$$

$\Rightarrow$ generates antisym.

- Take $PE[f(x_i)t] \triangleq$

$$\triangleq \exp\left\{ \sum_{k=1}^{\infty} \frac{f(x_i^k)\, t^k}{k} \right\}$$

$e^{\underset{k=1}{f(x_i)t} + \underset{k=2}{f(x_i^2)t^2/2}}$

$\rightarrow$ expand to get t^2 terms.

$$= 1 + \underbrace{f(x_i)t}_{k=1} + \underbrace{\left[\overset{k=1}{\frac{f(x_i)^2}{2}} + \overset{k=2}{\frac{1}{2} f(x_i^2)} \right]}_{Sym^2 f(x_i)} t^2$$

$\rightarrow$ sym. once.

$f(x_i)^2 \equiv$ character of tensor product

Examples : $SU(2)$

$$Sym^2[n] = [2n] + [2n-4] + [2n-8] + \cdots$$

Recall : $[n]$ has $\chi_n(x) = \dfrac{x^{n+1} - x^{-n-1}}{x - x^{-1}}$

$$\mathrm{Sym}^2[n] = \frac{1}{2}\left(\frac{x^{n+1}-x^{-n-1}}{x-x^{-1}}\right)^2 + \frac{1}{2}\left(\frac{x^{2n+2}-x^{-2n-2}}{x^2-x^{-2}}\right)$$

Set $n=3$,

$$\mathrm{Sym}^2[3] = \frac{1}{2}\left(x^3+x+x^{-1}+x^{-3}\right)^2 + \frac{1}{2}\left(x^6+x^2+x^{-2}+x^{-6}\right).$$

$$= x^6 + x^2 + x^{-2} + x^{-6}$$

mix terms $\left\{\begin{array}{l} + x^4 + x^2 + 1 \\ + 1 + x^{-2} + x^{-4} \end{array}\right.$

$$= [6] + [2]$$

- For $\Lambda^2 f(x_i) = \frac{1}{2} f(x_i)^2 - \frac{1}{2} f(x_i^2)$.

$$\Lambda^2[3] = x^4 + x^2 + 1 + 1 + x^{-2} + x^{-4}$$

$$= [4] + [0]$$

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- Now, what about a sym. product of any order?
  → introduce notion of partitions.

$$\mathrm{PE}[f\,t] = \exp\left\{\sum_{k=1}^{\infty} \frac{f(x_i^k)\,t^k}{k}\right\} \stackrel{\Delta}{=}$$

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$PE\left(f(x_i)t\right) = \exp\left\{ \sum_{k=1}^{\infty} \frac{f(x_i^k)\, t^k}{k} \right\}$$

$$= \exp\left(f(x_i)t \right) \exp\left(\frac{f(x_i^2)t^2}{2} \right) \exp\left(\frac{f(x_i^3)t^3}{3} \right) \times$$

$$= \left(1 + f(x_i)t + \frac{f(x_i)^2 t^2}{2} + \frac{f(x_i)^3 t^3}{3!} \right) \left(1 + \frac{f(x_i^2)t^2}{2} + \frac{f(x_i^2)^2 t^4}{4} \right)$$
$$\times \left(1 + \frac{f(x_i^3)t^3}{3} + \frac{f(x_i^3)^2 t^6}{2!} \right)$$

$$= \left(1 + \frac{f(x_i^2)t^2}{2} + f(x_i)t + \frac{1}{2} f(x_i) f(x_i^2) t^3 + \frac{f(x_i)^2 t^2}{2} + \frac{f(x_i)^3 t^3}{3!} + O(t^4) \right) \times \left(1 + \frac{f(x_i^3)t^3}{3} + \cdots \right)$$

$$= 1 + f(x_i)t + \frac{f(x_i^2)t^2}{2} + \frac{1}{2} f(x_i) f(x_i^2) t^3 + \frac{f(x_i)^2 t^2}{2} + \frac{f(x_i)^3 t^3}{3!} + \frac{f(x_i^3)t^3}{3} + O(t^4)$$

$$= 1 + f(x_i)t + \left(\frac{1}{2} f(x_i)^2 + \frac{1}{2} f(x_i^2) \right) t^2 + \left(\frac{1}{6} f(x_i)^3 + \frac{1}{2} f(x_i) f(x_i^2) + \frac{f(x_i^3)}{3} \right) t^3 + O(t^4)$$

$$= \sum_{k=0}^{\infty} \mathrm{Sym}^k [f]\, t^k$$

E.g. $\mathrm{Sym}^2 [n]$, where $[n]$ has

$$\chi_n(X) = \frac{X^{n+1} - X^{-n-1}}{X - X^{-1}}$$

From the expansion, $\mathrm{Sym}^2[f] = \frac{f(X_i)^2}{2} + \frac{1}{2} f(X_i^2)$

$$\text{So, } \mathrm{Sym}^2[n] = \frac{1}{2}\left(\frac{X^{n+1} - X^{-n-1}}{X - X^{-1}}\right)^2 + \frac{1}{2}\left(\frac{X^{2n+2} - X^{-2n-2}}{X^2 - X^{-2}}\right)$$

Set $n = 3$, then

$$\begin{aligned}
\mathrm{Sym}^2[3] &= \frac{1}{2}\left(X^3 + X + X^{-1} + X^{-3}\right)^2 + \frac{1}{2}\left(X^6 + X^2 + X^{-2} + X^{-6}\right) \\
&= \frac{1}{2}\Big(X^6 + X^4 + X^2 + 1 \\
&\qquad + X^4 + X^2 + 1 + X^{-2} \\
&\qquad + X^2 + 1 + X^{-2} + X^{-4} \\
&\qquad + 1 + X^{-2} + X^{-4} + X^{-6}\Big) + \frac{1}{2}\left(X^6 + X^2 + X^{-2} + X^{-6}\right) \\
&= X^6 + X^2 + X^{-2} + X^{-6} + X^4 + X^{-4} + 1 + 1 + X^2 + X^{-2} \\
&= [6] + [2]
\end{aligned}$$

n_k where $k = 1, \ldots, \infty$

Expand each exponential

$$\stackrel{\Delta}{=} \underbrace{\sum_{n_1=0} \frac{f(x_i)^{n_1} t^{n_1}}{n_1!}}_{k=1} \underbrace{\sum_{n_2=0}^{\infty} \frac{f(x_i)^{n_2} t^{2n_2}}{2^{n_2} n_2!}}_{k=2} \cdots$$

$$= \sum_{\{n_k\}=0}^{\infty} \prod_{k=1}^{\infty} \frac{[f(x_i^k)]^{n_k} t^{kn_k}}{k^{n_k} n_k!} \stackrel{\Delta}{=}$$

- We want to extract the n^{th} power.

$$\stackrel{\Delta}{=} \sum_{\{n_k\}=0}^{\infty} \prod_{k=1}^{\infty} \left(\frac{[f(x_i^k)]^{n_k}}{k^{n_k} n_k!} \right) \underbrace{t^{\sum_{k=1}^{\infty} k n_k}}_{\text{partition of } n}$$

$$= \sum_{n=0}^{\infty} \mathrm{Sym}^n [f] t^n$$

compare these two.

$$\Rightarrow \mathrm{Sym}^n [f] = \prod_{k=1}^{\infty} \sum_{n_k} \frac{[f(x_i^k)]^{n_k}}{k^{n_k} n_k!},$$

where $\sum_{k=1}^{\infty} k n_k = n.$

Examples

$\mathrm{Sym}^3 [f]$

Paritions of 3 : 2 + 1 ②, 1 + 1 + 1 ③

3 ①

$\Rightarrow$ 3 partitions so 3 contributions.

① $\Rightarrow n_3 = 1, \quad n_i = 0$ $\qquad t^3 \qquad k=3$

② $\Rightarrow n_1 = n_2 = 1, \quad n_i = 0$ $\qquad t^2 \qquad k=1,\ k=2$

③ $\Rightarrow n_1 = 3, \quad n_i = 0$ $\qquad t \qquad k=1$

n_i counts no of times a given element appears

$$\mathrm{Sym}^3[f] = \underbrace{\frac{1}{3} f(x_i^3)}_{k=3} + \underbrace{\frac{1}{2} f(x_i) f(x_i^2)}_{k=1,\ k=2}$$

$$+ \underbrace{\frac{1}{6} f(x_i)^3}_{k=1}$$

So,

$$\mathrm{Sym}^n[f] = \prod_{k=1}^{\infty} \sum_{n_k} \frac{[f(x_i^k)]^{n_k}}{k^{n_k} n_k!}$$ * Important!! *

- Can generalise to antisym.

$$\Lambda^n[f] = \sum_{n_k} \prod_{k=1}^{\infty} \frac{(-1)^{k+1} [f(x_i^k)]^{n_k}}{k^{n_k} n_k!}$$

<u>Partitions</u>

- How do we compute these?

$$\sum_{\{n_k\}=0}^{\infty} \prod_{k=1}^{\infty} \left(\frac{[f(X_i^k)]^{n_k}}{k^{n_k}\, n_k!} \right) t^{\sum_{k=1}^{\infty} k n_k},$$

$$n = \sum_{k=1}^{\infty} k n_k$$

- $Sym^3[f]$; 3 partitions : ① 3
  ② $2+1$
  ③ $1+1+1$

$$\sum_k k n_k = n = 3$$

For $k=1$, $\displaystyle\sum_{n_1=0}^{\infty} \frac{f(X_i)^{n_1}}{n_1!}$

For $k=2$, $\displaystyle\sum_{n_2=0}^{\infty} \frac{f(X_i^2)^{n_2}}{2^{n_2} n_2!}$

For $k=3$, $\displaystyle\sum_{n_3=0}^{\infty} \frac{f(X_i^3)^{n_3}}{3^{n_3} \cdot n_3!}$

Need to multiply these togethe as $\prod_{k=1}^{\infty}$.

Then, we can set n_1, n_2, n_3 s.t. we have highest power X_i^3, which correspond to t^3.

- For $k=1$; $n_1 = 3$, $n_i = 0$ $\Rightarrow \frac{1}{6} f(X_i)^3$

For $k=2, k=1$; $n_1 = n_2 = 1$, $n_i = 0$ $\Rightarrow \frac{1}{2} f(X_i) f(X_i^2)$

For $k=3$; $n_3 = 1$, $n_i = 0$ $\Rightarrow \frac{1}{3} f(X_i^3)$

$$\Rightarrow Sym^3[f] = \frac{1}{6} f(X_i)^3 + \frac{1}{2} f(X_i) f(X_i^2) + \frac{1}{3} f(X_i^3)$$

Taylor

$$\text{Take} \quad \prod_{k=1}^{\infty} \frac{1}{1-t^k} = \sum_{n=0}^{\infty} a_n t^n$$

→ a_n = no. of partitions of n.

→ generating func^n for partitions

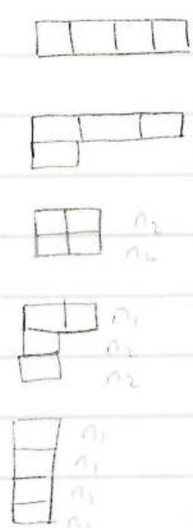
E.g. $a_4 = 5$

$$\left.\begin{aligned} 4 &= 4 \\ &\quad 3+1 \\ &\quad 2+2 \\ &\quad 2+1+1 \\ &\quad 1+1+1+1 \end{aligned}\right\} \text{5 partitions}$$

→ 5 sym. contributions

Young Diagram:

A collection of boxes pushed to the top left



	No. of boxes per row ($=k$)	
n_4	4	$n_4 = 1$
n_3, n_1	$3+1$	$n_3 = 1$, $n_1 = 1$
n_2, n_2	$2+2$	$n_2 = 2$
n_1, n_1, n_2	$2+1+1$	$n_1 = 2$, $n_2 = 1$
n_1, n_1, n_1, n_1	$1+1+1+1$	$n_1 = 4$

Disallowed: , instead

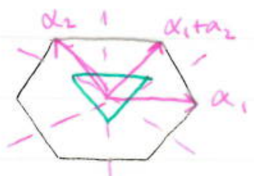
$$\mathrm{Sym}^4[f] = \frac{1}{24} f(x_i)^4 + \frac{1}{4} f(x_i)^2 f(x_i^2) + \frac{1}{8} f(x_i^2)^2 + \frac{1}{3} f(x_i^3) f(x_i) + \frac{1}{4} f(x_i^4)$$

- We can now symmetrise as much as we want!

- Proceed to SU(3)
  - have a graphical repⁿ, but need to give this up as we go to higher rank.

Character of SU(3)

- Start w/ the root system



regular hexagon.
α_1, α_2 simple roots.

$\chi = [1] = z_1 + z_2$
s.t. $z_1 z_2 = 1$
by weyl reflection
let $z_1 = x$, $z_2 = x^{-1}$
to recover $[1] = x + x^{-1}$
in SU(2)

[1,0] [0,1]

e.g. quarks

We write character $\chi_{[1,0]} = [1,0]$

3 weights in 3dim. repⁿ

$$= z_1 + z_2 + z_3$$

anti-quarks

$$[0,1] = z_1^{-1} + z_2^{-1} + z_3^{-1}$$

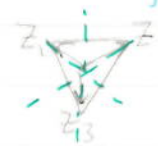
reflection of highest weight

Have condition that

$$\prod_{i=1}^{3} z_i = 1.$$

i.e. due to weyl sym.

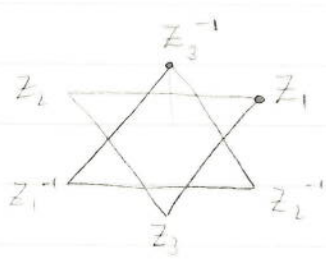
as



$\sum_{i=1}^{3} z_i = 0$

∴ weyl reflection

z_3^{-1}, z_1 highest weights of [0,1], [1,0] respectively



y_1, y_2 are fugacities for the highest weights.

Call highest weight for $[1,0]$ $z_1 = y_1$ and for $[0,1]$ $z_3^{-1} = y_2$.

So from condition, $z_2 = z_1^{-1} z_3^{-1} = \frac{y_2}{y_1}$.

$[1,0] = y_1 + \frac{y_2}{y_1} + \frac{1}{y_2}$.

$[0,1] = y_2 + \frac{y_1}{y_2} + \frac{1}{y_1}$.

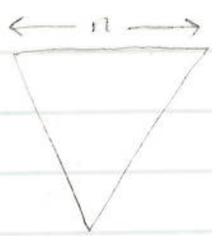
Use these to generate characters for all $SU(3)$ rep^n.

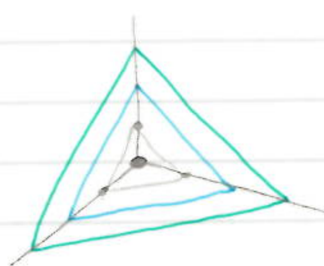
- Can now construct the other rep^n using

$\mathrm{Sym}^n [1,0] = [n,0]$.

$\mathrm{Sym}^n [0,1] = [0,n]$

can then get $\chi_{[n,0]}$ and $\chi_{[0,n]}$.





3 pts

6 pts.

10 pts.

Origin $[0,0]$ rep^n, followed by $[1,0]$ rep^n

Fill all pts in the +ve octant.

$$PE\left[[1,0]t\right] = \sum_{n=0}^{\infty} \mathrm{Sym}^n [1,0] t^n$$

$$= \sum_{n=0}^{\infty} [n,0] t^n$$

this counts the pts. on the lattice $\longrightarrow$

$$= \frac{1}{(1-y_1 t)(1-\frac{y_2}{y_1}t)(1-t/y_2)}$$

generating func^n for the $[n,0]$

$$= \sum_{\substack{k_1=0 \\ k_2+k_3=0}} (y_1 t)^{k_1} \left(\frac{y_2}{y_1} t\right)^{k_2} \left(\frac{t}{y_2}\right)^{k_3}$$ ... of other rep's.

$$PE\Big[[0,1]t\Big] = \sum_{n=0}^{\infty} [0,n] t^n$$

$$= \frac{1}{(1-y_2 t)\left(1-\frac{y_1}{y_2} t\right)\left(1-t/y_1\right)}$$

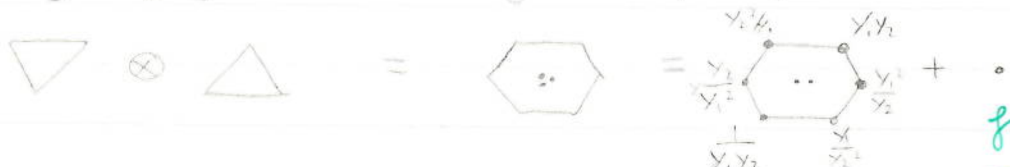
∴ we can now get triangular rep^{n}s. Try to get hexagonal now.

Consider tensor product

<u>Tensor Product</u>

$$[1,0]\,[0,1] = [1,1] + [0,0]$$

3 × 3 = 8 + 1

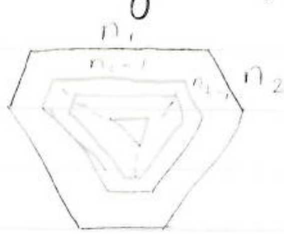


from $[0,0]$

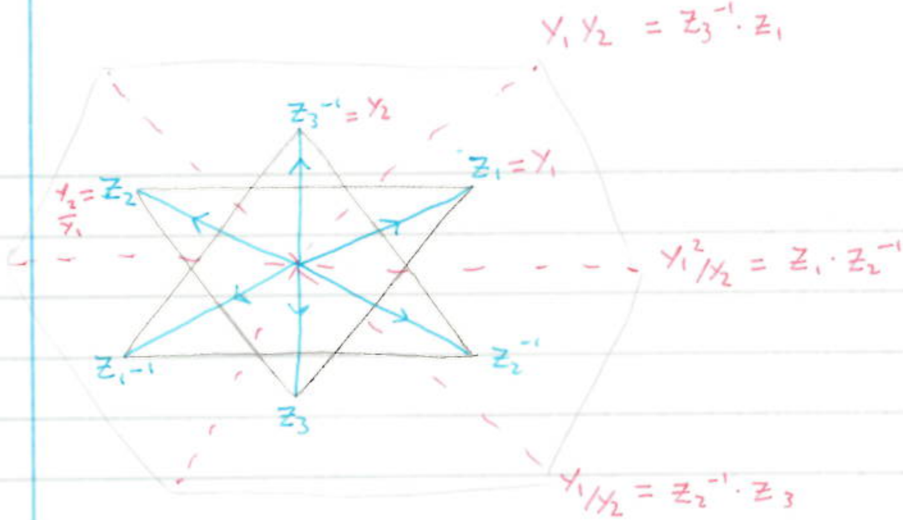
$$\chi_{[1,1]} = \left(y_1 + \frac{y_2}{y_1} + \frac{1}{y_2}\right)\left(y_2 + \frac{y_1}{y_2} + \frac{1}{y_1}\right) - 1$$

$$= \underbrace{y_1 y_2}_{\text{represents a weight}} + \frac{y_1^2}{y_2} + \frac{y_2^2}{y_1} + 1 + \frac{y_2}{y_1^2} + 1 + \frac{y_1}{y_2^2} + \frac{1}{y_1 y_2}$$

Recall: the structure of $[n_1, n_2]$



reduce by 1 until we hit a △, then the multiplicity stays the same



- Try to write down a generating func^n $\forall$ irrep^{n}s of $SU(3)$.

Introduce 2 fugacities for highest weights, t_1 and t_2. Want to evaluate:

$$f(t_1,t_2,Y_1,Y_2) \equiv \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \underbrace{[n_1,n_2](Y_1,Y_2)}_{\equiv \chi_{[n_1,n_2]}(Y_1,Y_2)} t_1^{n_1} t_2^{n_2}$$

First evaluate when $Y_1 = Y_2 = 1$ → setting it like this counts the no. of weights.

$$f(t_1,t_2,1,1) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \underbrace{\frac{(n_1+1)(n_2+1)(n_1+n_2+2)}{2}}_{\text{dim. of rep}^n} t_1^{n_1} t_2^{n_2}$$

$$\stackrel{=}{\triangle}$$

N.B. t_1, t_2 fugacities for <u>highest</u> weights.
Y_1, Y_2 are fugacities for weights. → keeps track of all the weights.

$$\stackrel{=}{\triangle} \frac{\partial}{\partial t_1} \frac{\partial}{\partial t_2} \frac{\partial}{\partial \lambda} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} t_1^{n_1} t_2^{n_2} \lambda^{n_1+n_2} \Big|_{\lambda=1}$$

$$= \frac{1 - t_1 t_2}{(1-t_1)^3 (1-t_2)^3}$$

$n_2 = 0$ $\quad$ $n_1 = 0$

From $[1,0][0,1]$

$$= (1 - t_1 t_2) \, PE\left[3t_1 + 3t_2\right]$$

restore Y depend.

Generates all χ of $SU(3)$.

$$f(t_1, t_2, Y_1, Y_2) = (1 - t_1 t_2) \, PE\left[[1,0] t_1 + [0,1] t_2\right]$$

$$= \frac{1 - t_1 t_2}{(1 - t_1 Y_1)(1 - t_1 Y_2/Y_1)(1 - t_1/Y_2)(1 - t_2 Y_2)(1 - t_2 Y_1/Y_2)(1 - t_2/Y_1)}$$

Setting $t_1 = 0 \Rightarrow$ get all the $[0,n]$ rep^{n}s (i.e. sym product of antifundamental).

Setting $t_2 = 0 \Rightarrow$ sym. product of fundamental rep^n.

$$\frac{1}{(1-t)^3} = (1-t)^{-3} =$$

$$= 1 + 3t + 6t^2 + 10t^3 + \dots$$

$n_2 = 0$ $\quad \sum \frac{(n_1+1)(n_1+2)}{2} t_1^{n_1} = \cancel{\frac{1}{(t}} \sum \frac{(n_1+2)!}{2!\,n_1!} t_1^{n_1}$

$$= \frac{1}{(1-t_1)^3}$$

n_1: $\quad \sum \frac{(n_2+1)(n_1+2)}{2} t_2^{n_2} = \frac{1}{(1-t_2)^3}$

$n_1 = n_2 = 1$ $\quad \frac{\cancel{2} \cdot 2 \cdot 4}{\cancel{2}} t_1 t_2 = 8 t_1 t_2$

Expansion of $\sum\sum (n_1+1)\dots t_1^{n_1} t_2^{n_2}$ gives $8t_1t_2$. (1)

$\sim$

$$(1 + 3t_1 + 6t_1^2 + \dots)(1 + 3t_2 + 6t_2^2)$$

$$= 1 + 3t_1 + 9t_1t_2 + 3t_2 + \dots$$

$\longrightarrow$ Expansion of $(1-t_1)^{-3}(1-t_2)^{-3}$ gives $9t_1t_2$. (2)

To match one (1) and (2), (2) needs a factor of $(1 - t_1t_2)$ s.t. $9t_1t_2 \longrightarrow 8t_1t_2$.

This is $\equiv$ to the tensor product $[1;0] \times [0,1]$. We needed to remove the singlet in order to get the $\chi_{[1,1]}$.

Last time : $SU(3)$ rep^{n}s

$[n_1, n_2]$ - two highest weights.

dim : $\dfrac{(n_1+1)(n_1+2)(n_1+n_2+2)}{2}$.

two fugacities

characters $\chi_{[1,0]} = [1,0]$

$$= Y_1 + \frac{Y_2}{Y_1} + \frac{1}{Y_2}$$

Exponents of Y are weights of rep^n.

Complex conj. rep^n has opp weight :

$$[0,1] = Y_2 + \frac{Y_1}{Y_2} + \frac{1}{Y_1}.$$

Def^n. For a rep^n of weights $\{\lambda\}$, the complex conj. rep^n has weights $\{-\lambda\}$.

$[1,0]$ $\quad \bigtriangledown$

$[0,1]$ $\quad \triangle$

- Tensor product :

$$\underset{3}{[1,0]} \otimes \underset{\bar{3}}{[0,1]} = \underset{8}{[1,1]} + \underset{1}{[0,0]}$$

Generally,

$$\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} [n_1, n_2] t_1^{n_1} t_2^{n_2} = (1 - t_1 t_2)\, PE\Big[[1,0]t_1 + [0,1]t_2\Big]$$

$\rightarrow$ use this to commute tensor products of ∞ many rep^{n}s.

Suppose $[n_1, 0][0, n_2]$:

Take : $[2,0][0,1] \stackrel{?}{=}$

dim $\quad 6 \quad\quad 3$

$= [2,1] \quad + \quad [1,0]$

$\frac{3 \cdot 2 \cdot 5}{2} = 15$

need to show it's this and not $[0,1]$.

Use $*$ to show it is $[1,0]$:

$PE\left[[1,0]t_1\right] PE\left[[0,1]t_2\right] \rightarrow PE\left([0,1]t_2 + [1,0]t_1\right)$ exp.

rewrite left hand side.

Expand to show contribution from $n_1 = 1, n_2 = 0$

$$= \frac{1}{1 - t_1 t_2} \sum_{n_1, n_2} [n_1, n_2] t_1^{n_1} t_2^{n_2}$$

$$= \underbrace{\sum_{n_1=0}^{\infty} [n_1, 0] t_1^{n_1}}_{PE \equiv Sym^{n_1}} \sum_{n_2=0}^{\infty} [0, n_2] t_2^{n_2}.$$

Compare powers of t_1, t_2 on both sides

$$\Rightarrow [n_1, 0][0, n_2] = [n_1, n_2] + [n_1 - 1, n_2 - 1]$$

$$+ [n_1 - 2, n_2 - 2] + \cdots$$

$$= \sum_{k=0}^{\min(n_1, n_2)} [n_1 - k, n_2 - k].$$

SO(4)

- Consider rep of SO(4) – rank 2.
  - ST signature. ⇒ SO(3,1) / Lorentz gp
- Algebra of SO(4):

$\dim SO(4) = 6$ "3 boosts + 3 rot."

<u>Note</u>: $SO(4) \simeq SU(2) \times SU(2)$.

only correct on level of algebra (NOT gp). Need to include spin for gp.

N.B. $SO(3,1)$ is locally iso. to $SL(2,\mathbb{C})$ due to the ST signature.

one highest weight → one from each SU(2).

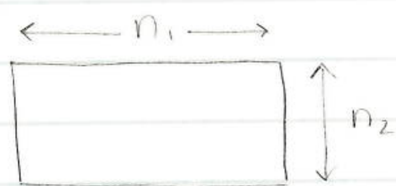
Denoted by $[n_1, n_2]_{SO(4)}$.

→ what is the dim. of the rep^n?

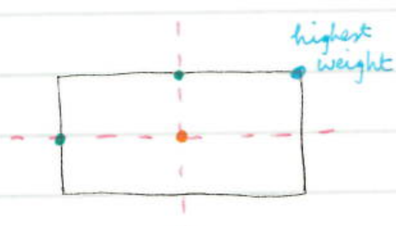
$$\dim [n_1, n_2]_{SO(4)} = \dim [n_1] \dim [n_2]$$

$$= (n_1+1)(n_2+1)$$ ie. product of 2 SU(2) dim.

Graphical rep^n:



Rectangle of $(n_1+1) \times (n_2+1)$ pts.



Weyl gp. $= \mathbb{Z}_2 \times \mathbb{Z}_2$, order 4.

- orbit size 2
- orbit size 4 → ie. pt. has 3 images, which forms a gp. of 4.

Dominant weyl chamber: 

There are 2 basic rep^n:

L $[1, 0]$ dim 2 } spinor rep^{n}s L and R.

R $[0, 1]$ "

$\longrightarrow$ Weyl spinor. (irrep).

N.B. Dirac spinor - red. rep^n of the Weyl spinors.

Maj. spinor - red., real spinor.

- Other rep^{n}s:

$[1, 1]$ $\longrightarrow$ $\boxed{\text{Vector rep}^n}$ defining rep^n of SO(4) $\to$ adjoint rep^n

dim 4

$\longrightarrow$ made out of 2 spinors!
(tensor product of spinors).

(i) V_μ $\mu = 0, 1, 2, 3.$ space-time

(ii) $V_{\alpha\dot{\alpha}}$ $\alpha, \dot{\alpha} = 1, 2$ gp. properties of $SU(2) \times SU(2)$

i.e. can write a 4-vec. w/ spinor indices.

Note: Dirac spinor is a red. rep^n : $[1,0] \oplus [0,1]$
$\Rightarrow$ still a spinor

4-vec : $[1,0] \otimes [0,1]$ $\Rightarrow$ not a spinor.

$[2,0] \Rightarrow$ self dual.. why?

dim 3

$\Lambda^2[1,1] \longrightarrow$ antisym. 2 4-vec.

$$= \underbrace{\Lambda^2\underbrace{\{(-1,1),\ (1,1),\ (-1,-1),\ (1,-1)\}}_{4}}_{6} = \underbrace{[2,0]}_{3} + \underbrace{[0,2]}_{3}$$

in the $[2,0] \oplus [0,2]$ rep^n.

$F^{\mu\nu}$ transf. as a reducible rep^n

$\longrightarrow$ what do $[0,2]$ and $[2,0]$ represent?

Given $F_{\mu\nu}$, can construct $\underbrace{\varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}}_{F^*}$

$\Rightarrow F$

$F^{\pm}_{\mu\nu}$ are linear combinations of $\underline{B}$ and $\underline{E}$.

Then construct : $F_{\mu\nu} \pm \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} = F_{\mu\nu}^{\pm}$

$\longrightarrow F^+$ is self-dual $[2,0]$

$$F^+_{\mu\nu} = \varepsilon_{\mu\nu\lambda\sigma} F^{+\lambda\sigma}$$

$$F^-_{\mu\nu} = -\varepsilon_{\mu\nu\lambda\sigma} F^{-\lambda\sigma} \Longrightarrow \text{anti-self dual.} \quad [0,2].$$

~

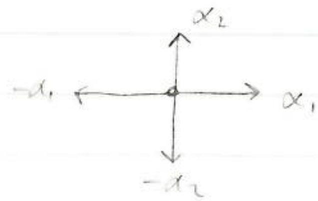
$$\underset{\text{dim } 9}{[2,2]} \Rightarrow \underbrace{\text{Sym}^2 \underbrace{[1,1]}_{\text{dim } 4}}_{10} = \underset{\text{dim } 9}{[2,2]} + \underset{1}{[0,0]}$$

$[2,2] \longrightarrow$ second rank symmetric tensor, traceless (as taking away singlet).

Consider $T_{\mu\nu} = T_{\nu\mu}$ 2^{nd} rank sym.
$\rightarrow$ 10. compts.

If traceless $T_{\mu\nu} g^{\mu\nu} = 0$ $\Rightarrow$ 9 compt.

- Tensor products
- Sym. products
- Antisym. products



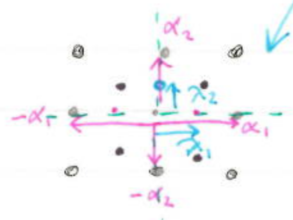
4 roots $\downarrow$ 2 +ve 2 −ve $\downarrow$ these 2 +ve roots are also simple roots.

~

Last time: Algebra of SO(4) - dim 6

dominant chamber

Root system:



weyl reflection - - - -
$W = \mathbb{Z}_2 \times \mathbb{Z}_2$.

- $\Rightarrow \lambda_1 + \lambda_2$

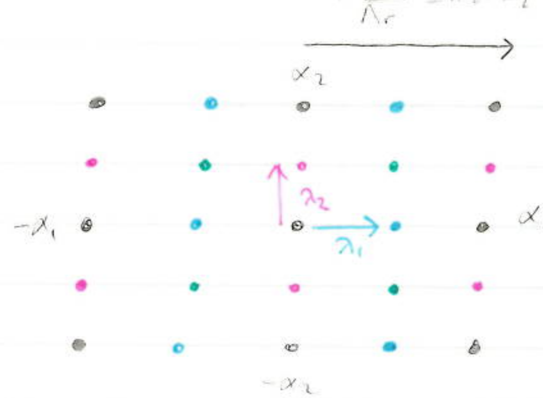
Highest weight (vector) : $\lambda_1 = 1/2\, \alpha_1$
$\lambda_2 = 1/2\, \alpha_2$

An irrep of Dynkin labels n_1, n_2 has highest weight:

$$\lambda = n_1 \lambda_1 + n_2 \lambda_2 . \longrightarrow \text{lives in dom. chamber.}$$

- pts differ by roots

These two pts only differ by a root vector $\Rightarrow \frac{\Lambda_W}{\Lambda_r} = \mathbb{Z}_2 \times \mathbb{Z}_2$



The pts. differ by linear combination of roots.

We denote the weight lattice by Λ_w and root lattice by Λ_r.

4 rep^{n}s given by the • • • • on the lattice.

$$\frac{\Lambda_w}{\Lambda_r} = \mathbb{Z}_2 \times \mathbb{Z}_2.$$

$\mathbb{Z}_2 \times \mathbb{Z}_2$ label of $[n_1, n_2]$ $\Rightarrow (n_1 \bmod 2, n_2 \bmod 2)$.

- 4 different types of rep^n corresponding to $[n_1, n_2]_{SO(4)}$

Odd multiple of λ_1 ie. n_1 odd, n_2 even $\Rightarrow$ • $(1,0)$

Similarly, n_1 even, n_2 odd $\Rightarrow$ • $(0,1)$ rep^n.

• $\Rightarrow$ n_1 odd, n_2 odd $(1,1)$

• $\Rightarrow$ n_1 even, n_2 even $(0,0)$

Selection Rules

In a tensor products, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ labels are conserved.

E.g. $[1,0][0,1] = [1,1]$

$\mathbb{Z}_2$ labels:

$(1,0) \quad (0,1) \qquad (1,1)$

$$\underset{(1,1)}{\Lambda^2[1,1]} = \underset{(0,0)}{[2,0]} + \underset{(0,0)}{[0,2]}$$

- We see these selection rules in QM.
  eg bosonic + femionic states = ferm. state
  bosonic + bosonic = bosonic states etc.

- Recall: selection rule for $SU(2)$ rep^{n}s $[n]_{SU(2)}$

$$\frac{\Lambda_w}{\Lambda_r} = \mathbb{Z}_2 \quad ; \quad \mathbb{Z}_2 \text{ number} = n \bmod 2$$

$\longrightarrow$ conserved in tensor products.

N.B. For $SU(3)$, $\frac{\Lambda_w}{\Lambda_r} = \mathbb{Z}_3$

$[n_1, n_2]_{SU(3)}$ then, $(n_1 - n_2) \bmod 3$ is the conserved quantum no.

~

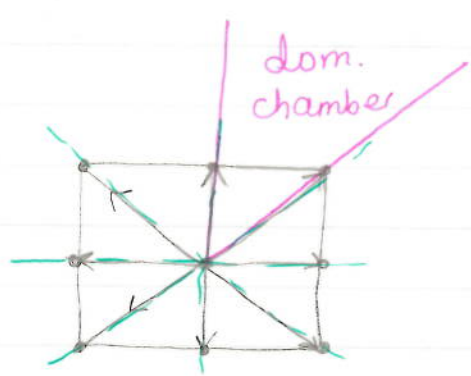
<u>$Sp(2) \simeq SO(5)$</u>

dim 10, rank 2, 8 roots ($\overset{10-2}{\uparrow}$); roots divide into 4 +ve and 4 -ve.

- 4 +ve roots $\Rightarrow$ 2 simple roots.
- Expect octagons for graphical rep^n.

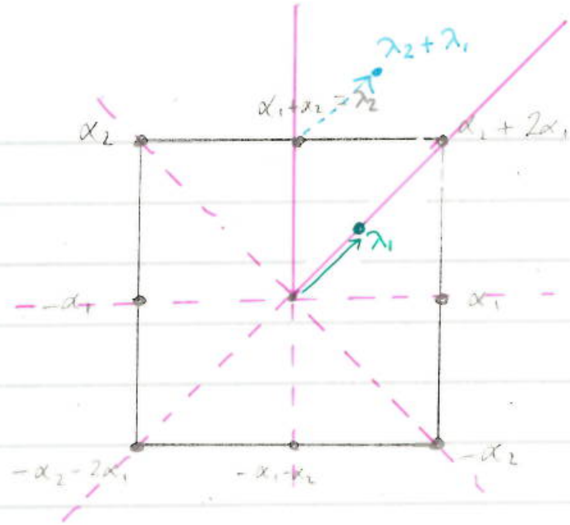
Root system:

Note: $\nwarrow$ and $\rightarrow$ have different length → first time we've encountered this



--- Weyl sym.

Weyl gp., $W = D_4$ (sym gp of square) of order 8



$\lambda_1 = \cancel{\frac{1}{2}(\alpha_2+\alpha_1)}\ \frac{1}{2}\alpha_2+\alpha_1$

$\lambda_2 = \alpha_1 + \alpha_2$

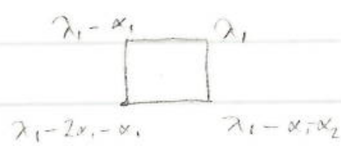
A generic rep^n has a highest weight

$\lambda = n_1\lambda_1 + n_2\lambda_2,$

$n_1 = 0, 1 \ldots\ ;\ n_2 = 0, 1, \ldots$

Simplest rep^n w/ λ_1 is a square:

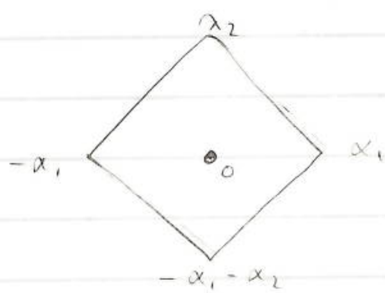
In 5 space dim., find 4 compt. irreducible spinor if we repeat Dirac's exercise. In 3+1 we get a red. spinor of 4 compts.



Rep^n $\overset{n_1\ n_2}{[1, 0]}$ $\rightarrow$ dim 4.

Spinor rep^n.

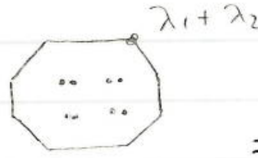
w/ λ_2:



$[0, 1] \rightarrow$ dim 5

Fundamental rep^n (as dim 5) / defining / vector rep^n.

$\lambda_1 + \lambda_2$:



$\rightarrow$ central pts have multiplicity 2.

$[1, 1]$

$\rightarrow$ dim

$= \dfrac{2 \cdot 2 \cdot 4 \cdot 6}{6} = 16$

gravitino (combo. of spinor + vector)

- dimension of $[n_1, n_2]_{Sp(2)}$:

$$\frac{(n_1+1)(n_2+1)(n_1+n_2+2)(n_1+2n_2+3)}{6}$$

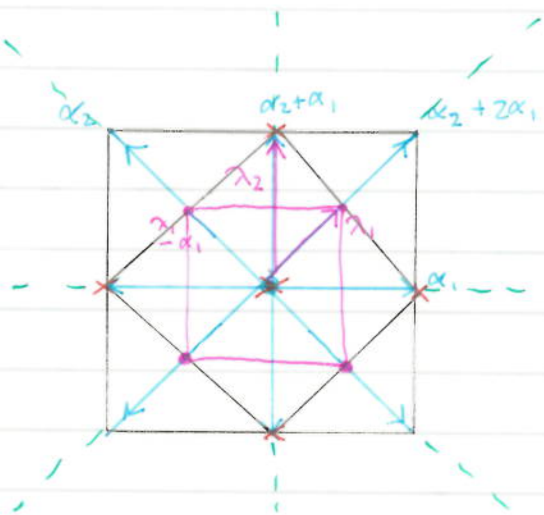
~

Sp(2) names

Dim	Rep^n
4	fundamental.
5	traceless 2^{nd} rank antisym.

Root system of $SO(5) \simeq Sp(2)$

- - - Weyl sym.



• $\chi_{[1,0]}$
× $\chi_{[0,1]}$

Get to pt $\lambda_1 - \alpha_1$ from origin by :

$-\lambda_1$ λ_2

$\Rightarrow \lambda_2 - \lambda_1$

4 short roots - 2 +ve, 2 -ve
4 long roots - " "

1 simple root.

α_1 shorter than α_2 by factor 2.

contribution from ... α_1 α_2 $\alpha_2+\alpha_1$ $\alpha_2+2\alpha_1$

$$\dim [n_1, n_2]_{Sp(2)} = \frac{(n_1+1)(2n_2+2)(n_1+2n_2+3)(2n_2+2n_1+4)}{2\cdot 3\cdot 4}$$

$\bar{\bar{\Delta}}$ double in length $\Rightarrow \times 2$

$$\underset{\Delta}{=} \frac{(n_1+1)(n_2+1)(n_1+n_2+2)(n_1+2n_2+3)}{6}$$

Weyl dim formula

- Fundamental weights

$$\left.\begin{aligned} \lambda_1 &= \alpha_1 + \frac{1}{2}\alpha_2 \\ \lambda_2 &= \alpha_1 + \alpha_2 \end{aligned}\right\}$$ along the boundary of dom. chamber.

So, $\alpha_1 = 2\lambda_1 - \lambda_2$
$\alpha_2 = -2\lambda_1 + 2\lambda_2$

- Can now construct character:

$$\chi_{[1,0]} = \overset{a}{Y_1} + \overset{b}{\frac{Y_2}{Y_1}} + \overset{c}{\frac{Y_1}{Y_2}} + \overset{d}{\frac{1}{Y_1}}$$

• $\lambda_1 - \alpha_1 = \lambda_1 - 2\lambda_1 + \lambda_2 = -\lambda_1 + \lambda_2$ (check for the other pts).

$Y_1 \equiv$ fugacity of 1st weight (λ_1)
$Y_2 =$ " " 2nd " (λ_2)

Powers of Y_i are coefficients of λ_i.
$\Rightarrow -\lambda_1 + \lambda_2 \Rightarrow \frac{Y_2}{Y_1}$.

E.g. for 1st term of $\chi_{[1,0]}$, $a=1$ and $b=0$.

R is a repn.

Then $\chi = \sum_{\omega \in R} Y_1^{a} Y_2^{b}$, where

$\omega \in \Lambda_\omega \Rightarrow$ then can expand: $\omega = a\lambda_1 + b\lambda_2$
(weight in lattice)
$\Rightarrow \lambda_i$ forms a basis for the weight vec.

- Construct $\chi_{[0,1]} = \Lambda^2[1,0] - 1$

ab ac ad bd cd

$$= \underset{①}{y_2} + \underset{②}{\frac{y_1^2}{y_2}} + 1 + \frac{y_2}{y_1^2} + \frac{1}{y_2}$$

-ve weight of ①

-ve of ②

- All irreps of $SO(5) \simeq Sp(2)$ are <u>real</u> (ie. for every weight we get a -ve weight).

Consider $\underbrace{Sym^2 \underbrace{[1,0]}_{4}}_{\dim 10} = \underbrace{[2,0]}_{\dim 10}$ no singlet

$$\underbrace{\Lambda^2 \underbrace{[1,0]}_{4}}_{6} = \underbrace{[0,1]}_{\dim 5} + \underbrace{[0,0]}_{\dim 1}$$

In real rep^n, singlet in either Sym or Λ.

$$\underbrace{Sym^2 \underbrace{[0,1]}_{5}}_{\frac{5\cdot 6}{2} = 15} = \underbrace{[0,2]}_{\frac{1\cdot 3\cdot 4\cdot 7}{6} = 14} + [0,0]$$

singlet now in sym for $sym^2[0,1]$

$$\underbrace{\Lambda^2 \underbrace{[0,1]}_{5}}_{\dim 10} = \underbrace{[2,0]}_{\dim 10}$$

$\Lambda^2[0,1]$ does NOT contain the singlet.

- Real rep^{n}s divide into 2 families
  - If for a real rep^n, R, $sym^2 R$ contains the singlet it is called a real rep^n
  - If for a real R, the $\Lambda^2 R$ contains the singlet it is called a pseudo-real rep^n

- For SU(2), the rep^{n}s $[n]$, for n odd are pseudo real. If n even, then $[n]$ is real.

- Next we compute $\mathrm{Sym}^n[1,0] = [n,0]$.

Then, $PE([1,0]t) = \sum_{n=0}^{\infty} [n,0]\, t^n$

Exercise: prove this!

$$PE([0,1]t) = \frac{1}{1-t^2} \sum_{n=0}^{\infty} [0,n]\, t^n$$

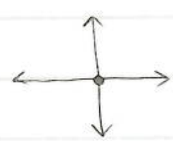
- Constructing $[n_1, n_2]$ characters is harder.

But we know at least

$[n_1, 0][n_2, 0] = [n_1, n_2]$ + rep^{n}s of lower order.

can do at least this

- $SO(4)$: $\mathbb{Z}_2 \times \mathbb{Z}_2 = W$



$SU(3)$: $W = \mathbb{Z}_3$

$SO(5)$: $W = D_4$

There is another one...

- Properties of $SO(5) \simeq Sp(2)$
   - 4 +ve roots : $\alpha_1, \alpha_2, \alpha_1+\alpha_2 + 2\alpha_1+\alpha_2$
   - Fundamental weights :

$$\left.\begin{aligned} \lambda_1 &= \alpha_1 + \tfrac{1}{2}\alpha_2 \\ \lambda_2 &= \alpha_1 + \alpha_2 \end{aligned}\right\} \longrightarrow \begin{aligned} \alpha_1 &= 2\lambda_1 - \lambda_2 \\ \alpha_2 &= -2\lambda_1 + 2\lambda_2 \end{aligned}$$

generators of the weight lattice

generators of the root lattice

Transition matrix $\Lambda_w \longleftrightarrow \Lambda_r$,

$$C = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix},$$

$\det C = 2 \rightarrow \Lambda_r$ is a sublattice of Λ_w of index 2.

$\Rightarrow \dfrac{\Lambda_w}{\Lambda_r} = \mathbb{Z}_2$: two types of rep^n (irreps) of $Sp(2) \simeq SO(5)$:

(i) real

(ii) pseudoreal

conserved quantum no. Gives selection rules.

$\mathbb{Z}_2$ quantum no. is preserved in tensor products.

For a given rep^n $[n_1, n_2]$:

the no. of these determines if rep^n is real or pseudoreal.

$[1,0]$ has dim 4, spinor of $SO(5)$ fermionic

$[0,1]$ has dim 5, vector of $SO(5)$ bosonic

If n_1 is odd, the rep^n is pseudoreal; if n_1 is even then the rep^n is real.

$\longrightarrow$ independ. of n_2 as (from [0,1]) only gives bosonic contribution.

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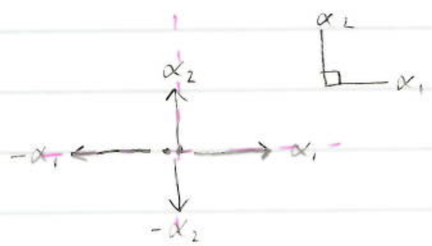
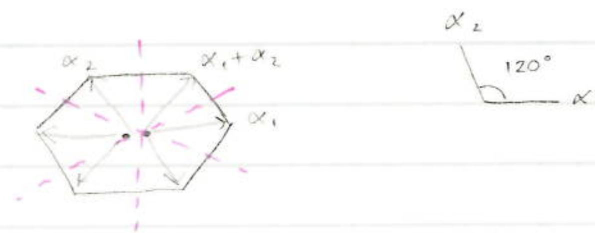
- We've seen 3 root systems in rank 2.





SO(4)

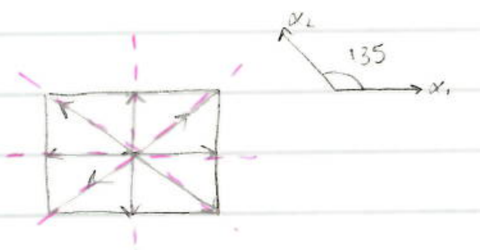
$W = \mathbb{Z}_2 \times \mathbb{Z}_2$

order 4

SU(3)

$W = S_3$

order 6



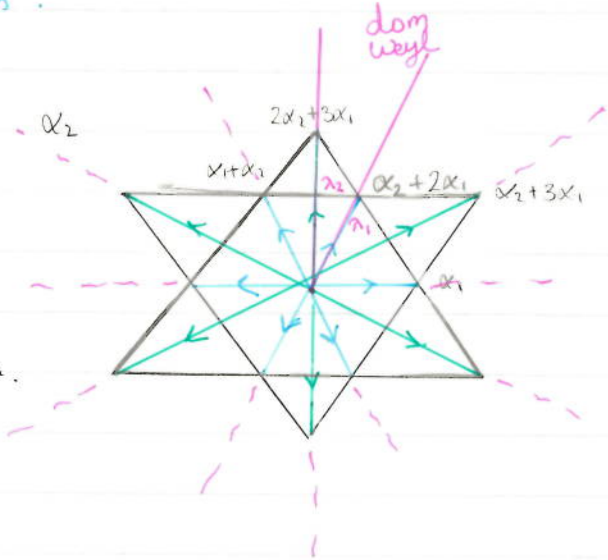
What's next?

$SO(5) \simeq Sp(2)$

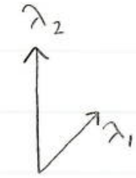
$W = D_4$, order 8

Next is root system for G2, rank 2 :

Check this !



Fuck → your diagram.



$W = D_6$, order 12

6 +ve roots $\begin{cases} 3 \text{ long} \\ 3 \text{ short} \end{cases}$

6 −ve roots $\begin{cases} 3 \text{ long} \\ 3 \text{ short.} \end{cases}$

Dimension of G2 : rank + no +ve roots + no −ve roots

$= 2 + 6 + 6$

$= 12.$

$$\left.\begin{aligned} \lambda_1 &= 2\alpha_1 + \alpha_2 \\ \lambda_2 &= 3\alpha_1 + 2\alpha_2 \end{aligned}\right\} \longrightarrow \begin{aligned} \alpha_1 &= 2\lambda_1 - \lambda_2 \\ \alpha_2 &= -3\lambda_1 + 2\lambda_2 \end{aligned}$$

The matrix $C = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$, det = 1

$\Rightarrow \Lambda_w = \Lambda_r.$

- A highest weight is $n_1\lambda_1 + n_2\lambda_2$.

Consider $[1,0] \Rightarrow \lambda = \lambda_1$ highest weight
$\longrightarrow$ pt. has orbit 6 (weyl reflection)
distance between pts. is α_1

So $\Rightarrow$  $= [1,0]$

N.B. The axis of reflections are orthogonal to one of the roots.
$\longrightarrow$ check this for $SU(3)$, $SO(4)$, $SO(5)$ and G_2.

- The dimension of a rep^n $[n_1 n_2]$.
$\rightarrow$ 6 factors : one for each +ve root

$$\underbrace{\alpha_1 \,,\, \alpha_2}_{\substack{\text{ratio of}\\ \text{lengths is 3}\\ \Rightarrow \alpha_2 \text{ has factor}\\ 3.}} ,\ \alpha_1+\alpha_2 ,\ \alpha_2+2\alpha_1 ,\ \alpha_2+3\alpha_1 ,\ 2\alpha_1+3\alpha_1$$

$\Rightarrow$ factors of 3 appear for n_2 in the dim. formula

Weyl dim. formula

$$\Big\{ \overbrace{(n_1+1)}^{\alpha_1}\overbrace{(3n_2+3)}^{\alpha_2}\overbrace{(n_1+3n_2+4)}^{\alpha_1+\alpha_2}\overbrace{(2n_1+3n_2+5)}^{\alpha_2+2\alpha_1}$$
$$\times \overbrace{(3n_1+3n_2+6)}^{\alpha_2+3\alpha_1}\overbrace{(3n_1+6n_2+9)}^{2\alpha_1+3\alpha_2}\Big\} \frac{1}{3\cdot 4\cdot 6\cdot 9}$$

to give trivial rep^n dim 1

$$\dim[n_1,n_2]_{G_2} = \frac{(n_1+1)(n_2+1)(n_1+n_2+2)(n_1+2n_2+3)(n_1+3n_2+4)(2n_1+3n_2+5)}{5!}$$

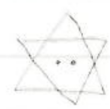
- The dim $[1,0] = \dfrac{2 \cdot 3 \cdot 4 \cdot 5 \cdot 7}{5!} = 7$

$[1,0]$ fundamental rep^n

$$\dim [0,1] = \frac{2 \cdot 3 \cdot 5 \cdot 7 \cdot 8}{5!} = 14$$

$[0,1]$ adjoint rep^n (i.e. ~~the~~ 14 roots = dim of gp).

$[1,0]$
7 pts.



$[0,1]$
14 pts.

- The $\text{Sym}^n [1,0] = ?$ Exercise

- The $\text{Sym}^n [0,1]$ is a complicated expression.

- Then can construct $[n_1, n_2]$ as it's contained in $\text{Sym}^{n_1}[1,0] \otimes \text{Sym}^{n_2}[0,1]$.

- Thus we are done with the rank 2 algebra.

Roots in rank 2 have different angles!
- only get angles $90°$, $120°$, $135°$ and $150°$ for the Lie groups.