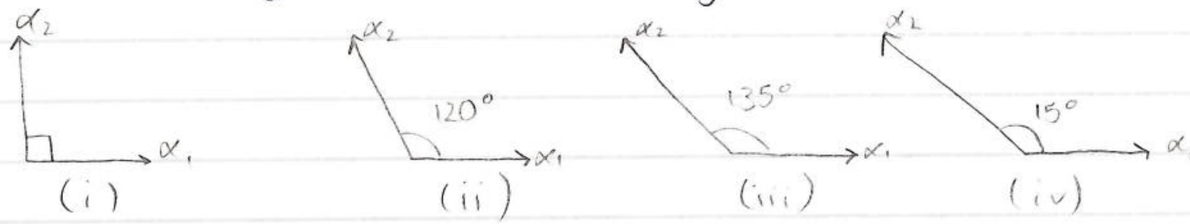
Algebra of Rank 1 : $SU(2)$

Algebra of Rank 2 :

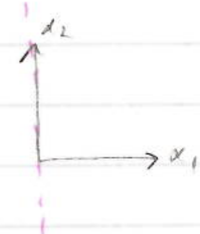
(i) $SO(4)$
(ii) $SU(3)$
(iii) $Sp(2) \simeq SO(5)$
(iv) G_2.

Graphical rep^n for rank 2 roots are separated by a specific angle



→ all possible angles.

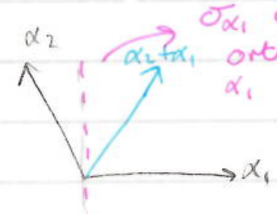
Axis of reflection



(i) $\sigma_{\alpha_1}(\alpha_2) = \alpha_2$
$= \alpha_2 + 0\alpha_1$

axis of reflection wrt α_1 (ortho.)

σ_{α_1} ie. axis orthogonal to α_1



(ii) $\sigma_{\alpha_1}(\alpha_2) = \alpha_2 + \alpha_1 = \alpha_2 + \alpha_1$

(iii) $\sigma_{\alpha_1}(\alpha_2) = \alpha_2 + 2\alpha_1$

(iv) $\sigma_{\alpha_1}(\alpha_2) = \alpha_2 + 3\alpha_1$

$\Rightarrow$ $\sigma_{\alpha_1}(\alpha_2) = \alpha_2 + n\alpha_1$; $n = 0, 1, 2, 3$
$\leftrightarrow 90°, 120°, 135°, 150°$

- Graphical notation for the algebras are called Dynkin diagrams
  → show the angle properties.

Dynkin Diagrams

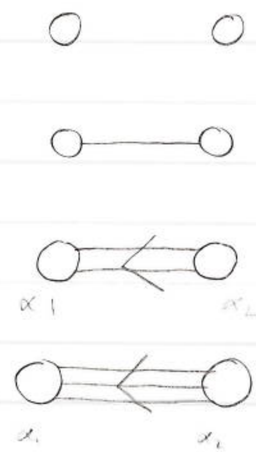
No. of lines ⟷ to how many times you get α_1 when you reflect. ie $n \equiv$ no. of lines

SO(4)

SU(3)

Sp(2)

G2



α_1, α_2 equal length

$<$ as $\alpha_2 > \alpha_1$

again as $\alpha_2 > \alpha_1$

Encodes data on <u>all</u> properties of the algebra eg. angle, no. of roots, length of roots etc.

- Another important property: relation between simple roots and fundamental weights:

SO(4) $\quad \left.\begin{matrix} \alpha_1 = 2\lambda_1 \\ \alpha_2 = 2\lambda_2 \end{matrix}\right\} \rightarrow \quad C = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

Cartan Matrix

Aside: [SU(2) has one simple root α and 1 fundamental weight $\lambda = \frac{1}{2}\alpha$.

λ, highest weight
$\alpha, -\alpha$ are raising/lowering op.]

<u>Cartan Matrix</u>

SO(4) $\longrightarrow \quad C = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

SU(3) $\alpha_1 = 2\lambda_1 - \lambda_2$ $\longrightarrow$ $C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

$\alpha_2 = -\lambda_1 + 2\lambda_1$

Sp (2) $\alpha_1 = 2\lambda_1 - \lambda_2$ $\longrightarrow$ $C = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$

$\alpha_2 = -2\lambda_1 + 2\lambda_2$

G2 $\alpha_1 = 2\lambda_1 - \lambda_2$ $\longrightarrow$ $C = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$

$\alpha_2 = -3\lambda_1 + 2\lambda_2$

- Can associate C to Dynkin diagram.

  $-n$ in matrix $\Longrightarrow$ specifies the angle / n

- We can generalise for finite dim Lie algebras
  $\longrightarrow$ classification of Cartan
  (and Killing).
  ↳ did most of the classification, but made a mistake at rank 4, which Cartan corrected.

* <u>Classification</u> $C_{ij} = \dfrac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$

A_n n nodes 

Neighbouring nodes have 120°
Non-neighbouring nodes have 90°

Rank n algebra $\Rightarrow$ SU(n+1)

$$C = \begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & & \\ & & 2 & -1 \\ 0 & & -1 & 2 \end{pmatrix}$$

B_n n nodes

Like $SU(n+1)$ for the first $n-1$ nodes, except for last node $\Rightarrow 135°$.

$\Rightarrow SO(2n+1)$

N.B. for rank 2, B_n and C_n coincide as each have one long and one short root.

$$C = \begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & 2 & -2 \\ 0 & & -1 & 2 \end{pmatrix}$$

$$\begin{array}{c|ccc} & n-2 & n-1 & n \\ \hline n-2 & 2 & -1 & 0 \\ n-1 & -1 & 2 & -2 \\ n & 0 & -1 & 2 \end{array}$$

So, $B_2 \Rightarrow C_{B_2} = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$ $\equiv$ transpose of $C_{Sp(2)}$

C_n n nodes

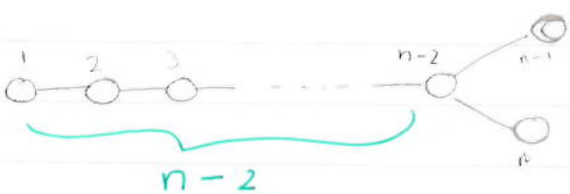
$\Rightarrow Sp(n)$

As before, but arrow other way. C_n, transpose of B_n

$$C = \begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & & \\ & & 2 & -1 \\ 0 & & -2 & 2 \end{pmatrix}$$

$$\begin{array}{c|ccc} & n-2 & n-1 & n \\ \hline n-2 & 2 & -1 & 0 \\ n-1 & -1 & 2 & -1 \\ n & 0 & -2 & 2 \end{array}$$

D_n



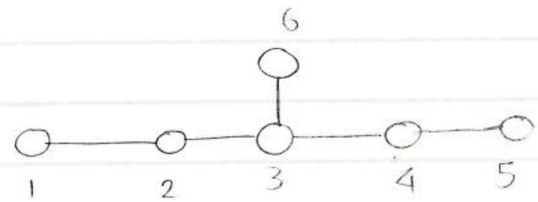
90° between last 2 $\Rightarrow SO(2n)$

$$
\begin{array}{c}
C = \begin{pmatrix} 2 & -1 & \cdots & 0 & 0 \\ -1 & \ddots & \ddots & & \vdots \\ & \ddots & 2 & -1 & -1 \\ 0 & \cdots & -1 & 2 & 0 \\ 0 & \cdots & -1 & 0 & 2 \end{pmatrix}
\begin{array}{l} \\ \\ n-2 \\ n-1 \\ n \end{array} \\
\quad\quad\quad\quad n-2 \;\; n-1 \;\; n
\end{array}
\longrightarrow
\begin{array}{c}
\begin{array}{ccc} n-2 & n-1 & n \end{array} \\
\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}
\end{array}
$$

E_8 - natural gauge group in 10 dim.

$\sim$

E6

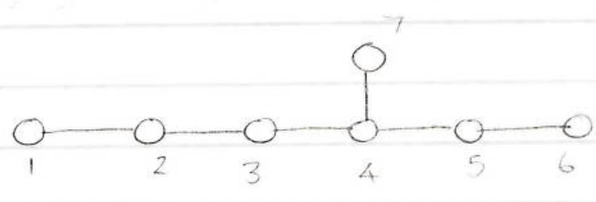


lengths of roots are equal $\Rightarrow$ sym. C

Symmetric

$$
C = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix} \end{array}
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
$$

E7



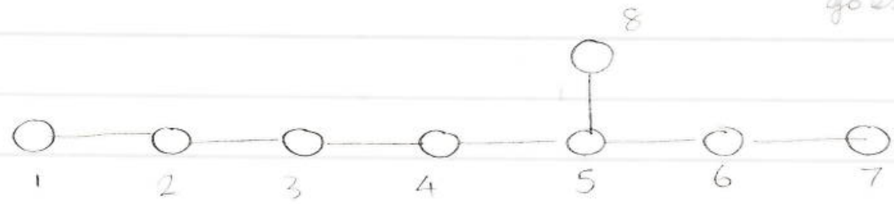
Could put node 7 on 3 an relabel.

$$
C = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ & 2 & -1 & 0 & 0 & 0 & 0 \\ & & 2 & -1 & 0 & 0 & 0 \\ & & & 2 & -1 & 0 & -1 \\ & & & & 2 & -1 & 0 \\ & & & & & 2 & 0 \\ & & & & & & 2 \end{pmatrix}
$$

Sym.

goes is important

E8

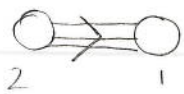


$\Rightarrow$ For E_n, extra node attached to 3rd before last.

F4

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \end{array}$$

G2



$\Rightarrow$ There are ~~A~~ 5 finite dim. exceptional gps.

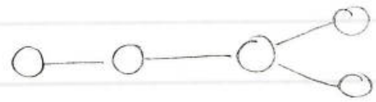
There are 4 ∞ dim gps. A_n, B_n, C_n, D_n. For fixed n, dim <u>is</u> finite, but n can be as large as you want.

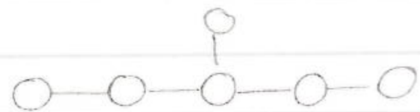
<u>Recap</u>: classification of Lie Algebras (finite dimensional).

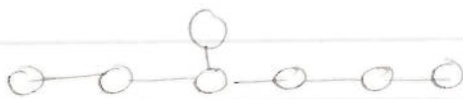
A_n 

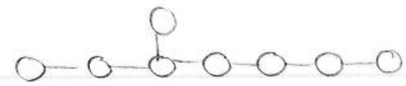
B_n 

C_n 

D_n 

E_6 

E_7 

E_8 

F_4 

G_2 

- Dynkin diagram $\longrightarrow$ compute root system from this.

- Defn of notion of roots:

Let V be a finite dim. vector space (V was 1 dim / 2 dim before), with the standard Euclid. vec. product $(\cdot, \cdot)$.

- A root system inside V is a finite set Φ of non-zero vectors in V.
  $\longrightarrow$ these vectors are the roots.

They satisfy the following conditions:

(i) The roots span the vec. space V.

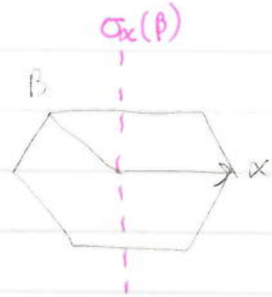
(ii) If $\alpha \in \Phi$ then $-\alpha \in \Phi$
$\rightarrow$ Weyl reflection
Crucially, no other scalar multiple of α is in V.

E.g. SU(3) 

but no other multiple of α.

(iii) Weyl reflection : $\alpha, \beta \in \phi$, define

$$\sigma_\alpha(\beta) = \beta - \underbrace{\frac{2(\alpha,\beta)}{(\alpha,\alpha)}}_{\text{integer}} . \alpha$$



(iv) For any $\alpha, \beta \in \Phi$, the quantity,

$$\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \text{ is an integer.}$$

This condition gives the condition for all possible angles :

$$\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \cdot \frac{2(\beta,\alpha)}{(\beta,\beta)} = 4(\alpha,\beta)^2 \frac{1}{(\alpha,\alpha)(\beta,\beta)}$$

both integers $\Rightarrow$ product an integer

$$= 4 \cos^2 \theta_{\alpha\beta},$$

angle between α, β

which is an integer.

$\Rightarrow$ $2 \cos \theta_{\alpha\beta} \in [-2, 2]$, so only finite cases for this to be an integer.

A note on the relative lengths of α and β:

From (iv), for any $\alpha, \beta \in \Phi$,

$$\frac{2(\alpha,\beta)}{(\alpha,\alpha)} = n_\alpha, \qquad ①$$

where n_α is an integer.

Similarly, $\frac{2(\beta,\alpha)}{(\beta,\beta)} = n_\beta,$ ②

where n_β is also an integer.

① $\Rightarrow$ $2(\alpha,\beta) = \alpha^2 n_\alpha$

② $\Rightarrow$ $2(\beta,\alpha) = \beta^2 n_\beta$

As the inner product is symmetric,

$$2(\beta,\alpha) = 2(\alpha,\beta)$$

$$\Rightarrow \qquad \beta^2 n_\beta = \alpha^2 n_\alpha.$$

$$\Rightarrow \qquad |\beta| = \sqrt{\frac{n_\alpha}{n_\beta}}\,|\alpha|$$

However, as

$$\cos\theta = \frac{(\alpha,\beta)}{(\alpha,\alpha)(\beta,\beta)},$$

n_α and n_β must also satisfy

$$\cos\theta = \frac{1}{2}(n_\alpha n_\beta)^{1/2}. \quad ; \quad n_\alpha n_\beta \text{ integer}$$

(i) For $n_\alpha = n_\beta = 0$, $\theta = \pi/2$ and there is no restriction on the relative lengths of α and β.

(ii) For $n_\alpha = n_\beta = 1$, $\theta = \pi/3$ and $|\alpha| = |\beta|$.

(iii) For $n_\alpha = 2$, $n_\beta = 1$, $\theta = \pi/4$ and $|\beta| = \sqrt{2}\,|\alpha|$

(iv) For $n_\alpha = 3$, $n_\alpha = 1$, $\theta = \pi/6$ and $|\beta| = \sqrt{3}\,|\alpha|$.

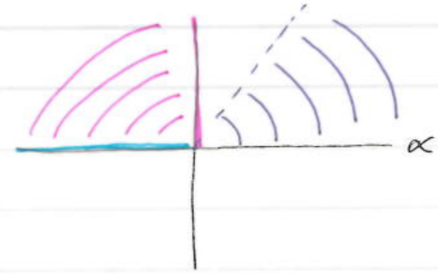
(i) - (iv) gives angles for the linear combinations of roots. To get the simple roots, we just need to go to larger angles that gives the same $\frac{1}{2}(n_\alpha n_\beta)^{1/2}$.

linear comb. of simple roots.

simple roots have angles of at least 90° wrt α.

trivial case as takes $\alpha \to -\infty$

Hence, $\Theta = \begin{cases} 0, & 90, & 180 \\ 30, & 120 \\ 45, & 135 \\ 60, & 150 \end{cases}$



True for any root $\alpha, \beta \in \Phi$.

— trivial case $\alpha \to -\alpha$

— linear comb. of simple roots

— simple roots

$\sim$

- Take $A_2 \simeq SU(3)$. ○—○

Choose vec. space to be $V = \mathbb{R}^3$ s.t. the sum of entries is equal to zero.

If we have $v \in \mathbb{R}^3 \Rightarrow (v_1, v_2, v_3)$ then

$$\sum_{i=1}^{3} v_i = 0.$$

N.B. Before we used a 2 dim vec. space. Now we used 2 dim. embedded in $\mathbb{R}^3$.

The roots are the set of vectors of length 2 w/ a Cartan matrix,

$$C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

Also want v with integer entries.

- v_i is an integer and need $\sum_{i=1}^{3} v_i^2 = 2$.

- Not much freedom:

one $v_i = 0$, other 2 are ± 1.

$\Rightarrow$ 6 sol^{n}s

+ve roots

1. $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \alpha_1$ 2. $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \alpha_2$ 3. $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \alpha_1 + \alpha_2$

inverses
ie $= -\alpha_1$

-ve roots

4. $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ 5. $\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ 6. $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

- $(\alpha_1, \alpha_2) = -1 \iff 120^\circ$ between the 2.

$(\alpha_1, \alpha_1) = 2$; $(\alpha_2, \alpha_2) = 2$

$\longrightarrow$ precisely the entries of C matrix

$SU(4) \simeq A_3 \overset{\sim}{=} D_3 \simeq SO(6)$

$V = \mathbb{R}^4$
$+ \sum v_i = 0$
condition.

$\Rightarrow SU(4) \simeq SO(6)$ – have the same Lie algebra.

dim $SU(4) = 15$; rank 3, 3 simple roots
$\Rightarrow$ 12 roots total.
(dim - rank).

- 6 +ve roots
- 6 -ve roots

$\alpha_1+\alpha_3$ not a root — look at length + entries must sum to zero.

simple roots

$$\alpha_1 = (1,-1,0,0)$$
$$\alpha_2 = (0,1,-1,0)$$
$$\alpha_3 = (0,0,1,-1)$$

Note: $\alpha_1+\alpha_3$ is <u>not</u> a root.

$$\alpha_1+\alpha_3 = (1,-1,1,-1)$$
$$\Rightarrow (\alpha_1+\alpha_3)^2 \neq 2$$

ie. $\alpha_1+\alpha_3$ does not meet the requirement of a root.

6 +ve roots

$$\alpha_1+\alpha_2 = (1,0,-1,0)$$
$$\alpha_2+\alpha_3 = (0,1,0,-1)$$
$$\alpha_1+\alpha_2+\alpha_3 = (1,0,0,-1)$$

Weyl dimension formula:

fundamental weights

C always has 2 on diag.. Need to modify to keep this property when considering roots that aren't the same length.

$$C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \qquad \alpha_i = C_{ij}\lambda_j$$

$$C_{ij} = (\alpha_i, \alpha_j)$$

Inverse of C: $\lambda_i = (C^{-1})_{ij}\alpha_j$

n_i are the Dynkin labels

Any irrep^n of SU(4) has highest weight:

$$\lambda = \sum_{i=1}^{3} n_i\lambda_i,$$

of dimension:

$$\Big\{ \overbrace{(n_1+1)}^{\alpha_1}\overbrace{(n_2+1)}^{\alpha_2}\overbrace{(n_3+1)}^{\alpha_3}\overbrace{(n_1+n_2+2)}^{\alpha_1+\alpha_2}\overbrace{(n_2+n_3+2)}^{\alpha_2+\alpha_3} \times$$
$$\times \overbrace{(n_1+n_2+n_3+3)}^{\alpha_1+\alpha_2+\alpha_3}\Big\} \times \frac{1}{2\cdot 2\cdot 3}$$

- ~~Defining~~ $\dim[1,0,0] = \frac{\cancel{2}\cdot 1\cdot \cancel{3}\cdot \cancel{2}\cdot 4}{\cancel{2}\cdot\cancel{2}\cdot\cancel{3}} = 4$

SO(6) this is $\Rightarrow$ defining / fundamental rep^n

$\dim\,[0,1,0] = \frac{1\cdot 2\cdot 1\cdot 3\cdot \cancel{3}\,\cancel{4}}{12} = 6$ defining / vec. rep^n of SO(6)

What is this in SU(4)?

$\binom{4}{2} = \frac{4!}{2!2!} = \frac{4\times3\times2}{4} = 6$

Consider $\underbrace{\Lambda^2 \underbrace{[1,0,0]}_{4}}_{6} = [0,1,0]$

$\Rightarrow$ for SU(4) $[0,1,0]$ is the 2nd rank antisym. tensor.

$\dim\,[0,0,1] = \frac{1\cdot 1\cdot 2\cdot 2\cdot 3\cdot 4}{12} = 4$ antifundamental of ~~SU(4)~~ SU(4).

spinor of SO(6), $\underline{4}$, complex conjugated. $\longrightarrow$ could call $[1,0,0]$ and $[0,0,1]$ left and right.

Last time: $SU(4) \simeq SO(6)$

$$\dim\,[n_1,n_2,n_3] = \frac{(n_1+1)(n_2+1)(n_3+1)(n_1+n_2+2)(n_2+n_3+2)(n_1+n_2+n_3+3)}{12}$$

First 3 rep^{n}s

	[1 0 0]	[0 1 0]	[0 0 1]
SU(4)	fundamental	2nd rank antisym.	antifundamental / 3rd rank antisym.
SO(6)	spinor	vector	c.c. spinor

$\underbrace{\Lambda^3 \underbrace{[1\,0\,0]}_{4}}_{\binom{4}{3}=4} = [0\,0\,1]$ $\longrightarrow$ why $[0\,0\,1]$ and not $[1\,0\,0]$? See later

- Cartan matrix of $SU(4)$:

$$C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \quad , \quad \det C = 4$$

$\Rightarrow \frac{\Lambda_w}{\Lambda_r}$ must be of order 4.

Could have been $\mathbb{Z}_2 \times \mathbb{Z}_2$, but not because $[100] \neq [001]$

$\Rightarrow \quad \underbrace{\frac{\Lambda_w}{\Lambda_r} = \mathbb{Z}_4}_{\text{order } 4.}$ as fundamental + anti are complex/inequivalent.

order 4. $\rightarrow$ lattice divides into 4 sublattices.

$\mathbb{Z}_4$ is preserved in any tensor product.
$\longrightarrow$ quadrality

- For $[100]$ we assign the $\mathbb{Z}_4$ no. 1
$[010]$ no. 2
$[001]$ no. $3 = -1$ as modulo 4

$[n_1 n_2 n_3]$ has quadrality: $(n_1 + 2n_2 - n_3) \bmod 4$
$\longrightarrow$ selection rule.

- ~~Compute~~ If we have highest weight $\lambda = n_1\lambda_1 + n_2\lambda_2 + n_3\lambda_3$.

$\rightarrow \lambda_1$ lives in quadrality of $[100]$, then $n_1\lambda_1$ carries the quadrality $n_1 \times 1$.
$\rightarrow$ Similarly, $n_2\lambda_2$ has quadrality $n_2 \times 2$.
etc.

E.g. For $SU(2)$ $\frac{\Lambda_w}{\Lambda_r} = \mathbb{Z}_2$ ie. have 2 sublattices - one w/ integer spin, the other w/ ½-integer spin.

- Have simple roots $\alpha_1, \alpha_2, \alpha_3$ and the fundamental weights $\lambda_1, \lambda_2, \lambda_3$. These are related by,

$$\alpha_i = C_{ij} \lambda_j .$$

α_i are the generators of Λ_r
λ_i are the generators of Λ_ω.

$\Rightarrow$ Λ_r is a linear integer combination of the α_i

Λ_ω is a linear com. of the λ_i.

- Λ_r is a sublattice of the weight lattice Λ_ω of index 4 for SU(4).
  - $\longrightarrow$ Λ_ω divided into 4 sublattices.
  - $\longrightarrow$ rep^{n}s come in 4 different "colours", which add up modulo 4.

N.B. SU(5) we had real and pseudoreal rep^{n}s.

- $$\underbrace{\mathrm{Sym}^n \underbrace{[1,0,0]}_{4}}_{\binom{n+3}{3}} = \underbrace{[n,0,0]}_{\frac{(n+1)(n+2)\cdot 2\cdot(n+3)}{12}} = \binom{n+3}{3}$$

$$= \frac{(n+3)!}{3!\,n!} = \frac{(n+3)(n+2)(n+1)}{6}$$

$\longrightarrow$ get the n^{th} rank Sym. rep^n of SU(4).

Quadrality $\longrightarrow$ 1 and $n \bmod 4$.

- Can write down the plethystic expon.

$$PE\left[[1,0,0]t\right] = \sum_{n=0}^{\infty} [n,0,0]t^n$$

- $\wedge$ product does <u>not</u> $\uparrow$ the Dyn. label - just shifts it.

  Sym. <u>does</u> $\uparrow$ the Dyn. label.

- Now consider

$$\underbrace{\text{Sym}^2 \underbrace{[0,1,0]}_{6}}_{21} = \text{Sym}^2\left(\wedge^2 [1,0,0]\right)$$

$$= \underbrace{[0,2,0]}_{\frac{1\cdot3\cdot1\cdot4\cdot4\cdot5}{12} = 20} + \underbrace{[0,0,0]}_{1}$$

$\binom{7}{2} = \frac{7!}{5!\,2!} = \frac{7\times 6}{2} = 21$

Quad. 2 for $[0,1,0]$ $\Rightarrow$ Sym^2 $[0,1,0]$ has quad. zero. $\longrightarrow$ quad. for both $[0,2,0]$ and $[0,0,0]$ is zero

$\Rightarrow$ Quad. preserved.

- General comment about rotation gps:

  In general, for $SO(n)$ gp., take V_i $i=1,\dots,n$ (n elements of defining rep^n).

  $(V_1,\dots V_n) \in \mathbb{R}^n$ ie. vec. in $\mathbb{R}^n$.

  Construct tensors out of V eg. V_iV_j.

  We have 2 invariant tensors : δ_{ij}
  For $SO(3)$ also have ε_{ijk} $\Big\}$ invariant tensors. Used to contract

For $SO(n)$ ε generalises to $\varepsilon_{i_1 \dots i_n}$.

Scalar product : $V_i V_j \delta_{ij}$

eg. $A^1{}_{i_1} A^2{}_{i_2} \dots A^n{}_{i_n} \varepsilon_{i_1 \dots i_n}$,

$\Big\}$ no indices (free) $\Rightarrow$ transf. under $[000]$.

Going back to :

$$\text{Sym}^2 [0,1,0] = [0,2,0] + [0,0,0]$$

$SO(6)$ $\quad$ 2^{nd} rank. sym. $\quad$ singlet

$a = V_i V_j \delta_{ij} \rightarrow [0,0,0]$

$b_{ij} = V_i V_j - \frac{1}{6} \delta_{ij} a \Rightarrow b_{ij} \delta_{ij} = 0$

These are the 2 products we can construct from the 2^{nd} rank sym.
$\longrightarrow$ generalises to n dim.

Now consider : $C_{ijk} = V_i V_j V_k - \frac{\cancel{a}}{18}\left(\delta_{ij} V_k + \delta_{ik} V_j + \delta_{jk} V_i\right)$

Check: $\cancel{\text{Sym}}$ $PE\left[[0,1,0]t\right] = \frac{1}{1-t^2} \sum_{n=0}^{\infty} [0,n,0] t^n$

So, $\text{Sym}^3 [0,1,0] = [0,3,0] + [0,1,0]$.

For Sym^3, need to expand PE to order t^3. On LHS, set $n=3$.

this is exactly like reducing the indices of a tensor by 2 by contracting w/ a δ_{ij}.

So, $\mathrm{Sym}^n [0,1,0] = \sum_k [0, n-2k, 0]$

where $k \mid (n-2k) \geq 0$ i.e. $k = 0 \to \infty$.

$\longrightarrow$ need to divide into cases n even and n odd to get the correct final term.

So, $C_{ijk} = v_i v_j v_k - \frac{a}{18}\left(\delta_{ij} v_k + \delta_{ik} v_j + \delta_{jk} v_i \right)$

and $a v_i$ are both 3^{rd} rank sym. tensors.

- So, $[0, n, 0]$ is called the traceless n^{th} rank tenser of $SO(6)$.

  $\longrightarrow C_{ijk}$.

~

<u>Last time</u>: $SU(4) \simeq SO(6)$

- 3 rep^{n}s

$\dim [1,0,0] = 4$

$\dim [0,1,0] = 6 \qquad \Lambda^2 [1,0,0]$

$\dim [0,0,1] = 4 \qquad \Lambda^3 [1,0,0]$

- Any $[n_1, n_2, n_3]$ will be in the tensor product $\mathrm{Sym}^{n_1} [100] \otimes \mathrm{Sym}^{n_2} [0,1,0] \otimes \mathrm{Sym}^{n_3} [0,0,1]$.

- Start w/ fundamental, antisym. to get rep^n that have highest weight that is the other fundamental weights, then symmetrise

Characters

- $[1,0,0] = \underset{\text{highest weight}}{z_1} + z_2 + z_3 + z_4$

with condition $\prod_{i=1}^{4} z_i = 1$. ①

- Weyl gp is S_4 of order 24.
  ↳ ~~cyclic~~ permutation gp.

- $[0,1,0] = \sum_{i<j} z_i z_j$ → as $\Lambda^2 [1,0,0]$

$= z_1 z_2 + \cdots$ (5 more terms)

highest weight

$[0,0,1] = \sum_{i<j<k} z_i z_j z_k$ → As $\Lambda^3 [1,0,0]$

$= z_1 z_2 z_3 + \cdots$ (3 more terms

highest weight

- Now let $z_1 = Y_1$ fugacity for 1^{st} fundamental weight, λ_1

$z_1 z_2 = Y_2$ fugacity for 2^{nd} f. weight, λ_2

$z_1 z_2 z_3 = Y_3$ fugacity for 3^{rd} f. weight, λ_3

$\Rightarrow$ $z_2 = \frac{Y_2}{Y_1}$ $z_3 = \frac{Y_3}{Y_2}$ $z_4 = \frac{1}{Y_3}$ $z_1 = Y_1$

from ④

So, $[100] = y_1 + \frac{y_2}{y_1} + \frac{y_3}{y_2} + \frac{1}{y_3}$ complex

$$[010] = y_2 + \frac{y_1 y_3}{y_2} + \frac{y_1}{y_3} + \frac{y_3}{y_1} + \frac{y_2}{y_1 y_3} + \frac{1}{y_2}$$

real

$$[001] = y_3 + \frac{y_2}{y_3} + \frac{y_1}{y_2} + \frac{1}{y_1}$$

complex

Recall: c.c. rep^n of rep^n R w/ weights w_i is given by the set of all weights $-w_i$.

E.g. Fundamental rep^n of SU(4) has weights

$$\underset{y_1}{(1,0,0)} \quad \underset{\frac{1}{y_1}\cdot y_2}{(-1,+1,0)} \quad \underset{\frac{1}{y_2}\cdot y_3}{(0,-1,1)} \quad \underset{\frac{1}{y_3}}{(0,0,-1)}$$

$$\xrightarrow{c.c.} \underset{1/y_1}{(-1,0,0)} \quad \underset{y_1\cdot\frac{1}{y_2}}{(1,-1,0)} \quad \underset{y_2\cdot\frac{1}{y_3}}{(0,1,-1)} \quad \underset{y_3}{(0,0,1)}$$

which is just the antifundamental [001]

This is why [100] and [001] are denoted by 4 and $\underline{4}$.

For [010], its c.c. rep^n is itself!
→ it is a real rep^n i.e. self-conjugate.

- Is $[010]$ real or pseudoreal?

Compute $\mathrm{Sym}^2\,[010] = [020] + [000]$

$\uparrow$ singlet in Sym^2 $\Rightarrow$ $[010]$ is real.

$$\underbrace{\Lambda^2\,\underbrace{[010]}_{6}}_{15} = [1,0,1] \qquad \frac{2\cdot 1\cdot 2\cdot 3\cdot 3\cdot 5}{12} = 15$$

Quadality zero

$\Rightarrow [1,0,1]$ is the adjoint repⁿ (has dim 15).

Consider $[1\,0\,0] \times [0\,0\,1]$

$$= [1\,0\,1] + [0\,0\,0]$$

- SU(4) has no pseudoreal – only complex and real.

For a repⁿ $[n_1, n_2, n_3]$, the c.c. is $[n_3, n_2, n_1]$.

If $n_1 = n_3$, the repⁿ is real
If $n_1 \neq n_3$, the repⁿ is complex.

~

- Can generalise to higher rank.

SU(n)

$\dim SU(n) = n^2 - 1 \longrightarrow$ dim. of adjoint

Rank $n-1$

No. of roots : $n^2 - n$

No of +ve roots $= \binom{n}{2}$

$(n-1)$ simple roots

- Dynkin diagram $\underbrace{\circ - \circ - \cdots - \circ - \circ}_{n-1 \text{ nodes}}$

$n-1$ nodes, 120° between roots (or 90°).

$SU(n) \cong A_{n-1}$

- Root system: $n^2 - n$ roots

Let e_i be a set of orthonormal vec. in an n-dim. vec. space V.

Natural scalar product,

$$(e_i, e_j) = \delta_{ij}.$$

Let $\alpha_1 = e_1 - e_2$; $\alpha_2 = e_2 - e_3$, $\cdots$ $\alpha_{n-1} = e_{n-1} - e_n$

Construct +ve roots:

$e_i - e_j$ for $i < j$

-ve roots: $e_i - e_j$ for $i > j$

- Cartan matrix

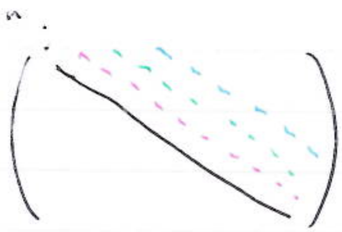
$$C_{ij} = (\alpha_i, \alpha_j)$$

$(n-1)\times(n-1)$ matrix

$$= \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & \ddots & & 0 \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ 0 & & & -1 & 2 \end{pmatrix}$$

Consider $(\alpha_1, \alpha_3) = (e_1 - e_2, e_3 - e_4) = 0$.

Fundamental rep^n:



see $SU(3)$ matrices had 2 diagonal, 6 off diagonal. They have this simple structure.

- - - - simple roots

- - - - +ve roots; sum of 2

- - - - +ve roots sum of 3 α's

$$\left.\begin{aligned} \text{eg. } \alpha_1 + \alpha_2 &= e_1 - e_3 \\ \alpha_2 + \alpha_3 &= e_2 - e_4 \\ &\vdots \\ \alpha_{n-2} + \alpha_{n-1} &= e_{n-2} - e_n \end{aligned}\right\} \text{all +ve roots}$$

Now take 3 α's:

eg. $\alpha_1 + \alpha_2 + \alpha_3 = e_1 - e_4$

$\vdots$

Then we get:

$$i < j \qquad e_i - e_j = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$$

$$= \sum_{k=i}^{j-1} \alpha_k$$

- Fundamental weights:

$$\alpha_i = C_{ij} \lambda_j \quad \Rightarrow \quad \lambda_i = (C^{-1})_{ij} \alpha_j$$

- A generic rep^n of highest weight λ,

$$\lambda = \sum_{i=1}^{n-1} n_i \lambda_i \, ; \quad n_i \text{ non-neg. integers.}$$

- Dim. formula: Dynkin labels n_i,

$\dim [n_1, \cdots, n_{n-1}]$

$$= \Big\{ (n_1+1)(n_2+1) \cdots (n_{n-1}-1)(n_1+n_2+2) \cdots (n_{n-2}+n_{n-1}+2) \times$$

$$\times (n_1+n_2+n_3+3) \cdots \Big(\sum n_i + n-1 \Big) \Big\} \frac{1}{\underbrace{1 \cdots 1}_{\substack{n-1 \text{ of} \\ \text{these}}} \cdot \underbrace{2 \cdots 2}_{n-2} \cdot \underbrace{3 \cdots 3}_{n-3} \cdots \underbrace{(n-1)}_{\substack{1 \text{ of} \\ \text{this}}}}$$

Last time: Root system of $SU(N)$, A_{N-1}.

Dim. formula: $[n_1, \dots n_{N-1}]$

Special / common rep"s:

- fundamental $\dim[1, 0, \dots 0] = N$

k^th rank sym.

- $\mathrm{Sym}^k [1, 0, 0 \dots 0]_{SU(N)} = [k, 0, 0, \dots 0]$

$\longrightarrow$ verify using dim. formula

$$\dim[k, \dots 0] = \binom{N+k-1}{k}$$

k^th rank antisym.

- * $\Lambda^k [1, \dots, 0] = [0, \dots, 1, 0, \dots 0]$ has N-ality k.

k^th pos^n

$\longrightarrow$ verify using dim.

- $PE[[1,0]t]$ $\qquad PE_F[[1,0, \dots 0]t]$

$$= \sum_{n=0}^{\infty} [n, 0, \dots 0] t^n \qquad = \sum_{k=0}^{N} [0, 0 \dots, 0, 1, 0 \dots] t^k$$

k^th pos^n

~

* The character: $[1, 0 \dots 0] = \sum_{i=1}^{N} z_i = y_1 + \frac{y_2}{y_1} + \frac{y_3}{y_2} + \dots \frac{y_{N-1}}{y_{N-2}} + \frac{1}{y_{N-1}}$

$$[0, 1, 0 \dots 0] = \sum_{i<j} z_i z_j$$

$$[0010\cdots 0] = \sum_{i<j<k} z_i z_j z_k \quad \text{etc.}$$

$\rightarrow$ match these dimensionally to the antisym. products.

$[n_1, \cdots n_{N-1}]$ is in the tensor product of

$$\prod_{k=1}^{N-1} \mathrm{Sym}^{n_k}\left[\Lambda^k[1,0,\cdots 0]\right].$$

- Weyl gp: set of permutations S_N.

$\dfrac{\Lambda_w}{\Lambda_r} = \mathbb{Z}_N$; "N-ality" is preserved in tensor products.

- The adjoint rep^n:

$$\underbrace{[1,0,0\cdots 0]}_{N} \otimes \underbrace{[0,0,\cdots 0,1]}_{N^2}$$

$$= \underbrace{[1,0\cdots 0,1]}_{N^2-1} + \underbrace{[0,0\cdots 0]}_{\text{dim } 1}$$

Look at "N-ality" $-$ $[1,000\cdots 0] \otimes [0,0\cdots 01]$

N-ality $\longrightarrow$ 1 $\quad$ $(N-1) = -1$

$\Rightarrow 1+(-1)=0 \quad \Rightarrow \left.\begin{matrix}[1,0\cdots 01]\\ [0,0\cdots 0]\end{matrix}\right\}$ both have N-ality zero.

- The N-ality of a rep^n $[n_1, \dots n_{N-1}]$ is

$$\sum_{k=1}^{N-1} k n_k \bmod N,$$

for $k = N-1$, $(N-1) \bmod N = -1$

- Complex conjugate: $[n_1, \dots n_{N-1}]$

$$\downarrow \text{c.c.}$$

$[n_{N-1}, \dots, n_1]$

palindromic expression.

N.B. No pseudo real for A_n except $n=1$ (ie $SU(2)$).

→ $SU(n)$ is generally complex, with a subset of real.

→ easier to think of $Sp(1) \simeq SU(2)$.

~

<u>Root System of B_n Algebra</u>

$B_n \simeq SO(2n+1)$, $n=1 \Rightarrow SO(3) \simeq SU(2)$.

$n=2 \Rightarrow SO(5)$.

- Next is $B_3 \simeq SO(7)$.

- Start w/ vec. space; let e_i be an

$i=1,\dots,n$ orthonormal basis for an n-dim vec. space.

Scalar product (bilinear) $(e_i, e_j) = \delta_{ij}$

- Dynkin diagram: 

- last root shorter than the rest.

simple roots

Roots: $\alpha_1 = e_1 - e_2$ $\quad (\alpha_i, \alpha_i) = 2 \quad i = 1, \dots n-1$

$\alpha_2 = e_2 - e_3$ $\quad (\alpha_n, \alpha_n) = 1$

$\vdots$ $\quad \Rightarrow$ shorter root.

$\alpha_{n-1} = e_{n-1} - e_n$

long simple roots

$\alpha_n = e_n$

short simple root

$$\dim SO(2n+1) = \binom{2n+1}{2} = n(2n+1)$$

Rank $= n$

$\Rightarrow$ No. of roots $= 2n^2$

$\longrightarrow n^2$ +ve

n^2 -ve.

$\longrightarrow$
- n short roots
- $n(n-1)$ long roots

} +ve

- same for the -ve roots.

- Set of +ve roots are

$e_i - e_j$	$i < j$	$\frac{n^2 - n}{2} = \binom{n}{2}$
$e_i + e_j$	$i < j$	$\binom{n}{2}$
e_i	$i = 1, \dots n$	n

E.g. $\alpha_{n-1} + 2\alpha_n$

as can rotate $\alpha_n = e_n$ to e_i for any $i = 1, \dots, n$

- Cartan matrix :

$$C = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & \ddots & & 0 \\ & -1 & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & 0 & & 2 & -2 \\ & & & -1 & 2 \end{pmatrix}$$

E.g. B_2 has $C = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$.

Check: $C_{ij} = \dfrac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$; $\alpha_i = C_{ij} \lambda_j$.
$\lambda_i = (C_{ij})^{-1} \alpha_j$.

Example

B_3 : $\left.\begin{aligned} \alpha_1 &= e_1 - e_2 \\ \alpha_2 &= e_2 - e_3 \\ \alpha_3 &= e_3 \end{aligned}\right\}$ simple roots need 6 more +ve roots

$$C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}$$

Alternatively, we can get C from the Dynkin diagram and then find the set of roots that satisfy C.

$e_i - e_j$ family $\left\{\begin{aligned} \alpha_1 + \alpha_2 &= e_1 - e_3 \\ \alpha_2 + \alpha_3 &= e_2 \end{aligned}\right.$ $\qquad \alpha_1 + \alpha_2 + \alpha_3 = e_1$

$e_i + e_j$ family $\left\{\begin{aligned} \alpha_2 + 2\alpha_3 &= e_2 + e_3 \\ \alpha_1 + 2\alpha_2 + 2\alpha_3 &= e_1 + e_2 \end{aligned}\right.$ $\qquad \alpha_1 + \alpha_2 + 2\alpha_3 = e_1 + e_3$

- Can now write the dim formula for an irrep^n of $SO(7) \simeq B_3$.

highest weight $\lambda = \sum_{i=1}^{3} n_i \lambda_i$.

Rep^n denoted by $[n_1, n_2, n_3]$ of dim

$$\Big\{ \overbrace{(2n_1+2)}^{\alpha_1} \overbrace{(2n_2+2)}^{\alpha_2} \overbrace{(n_3+1)}^{\alpha_3} \overbrace{(2n_1+2n_2+4)}^{\alpha_1+\alpha_2} \overbrace{(2n_2+n_3+3)}^{\alpha_2+\alpha_3} \overbrace{(2n_1+2n_2+n_3+5)}^{\alpha_1+\alpha_2+\alpha_3} \times$$

$$\times \overbrace{(2n_2+2n_3+4)}^{\alpha_2+2\alpha_3} \overbrace{(2n_1+2n_2+2n_3+6)}^{\alpha_1+\alpha_2+2\alpha_3} \overbrace{(2n_1+4n_2+2n_3+8)}^{\alpha_1+2\alpha_2+2\alpha_3} \Big\}$$

$$\times \frac{1}{2\cdot 2\cdot 4\cdot 3\cdot 5\cdot 4\cdot 6\cdot 8}$$

$$= \Big\{ (n_1+1)(n_2+1)(n_3+1)(n_1+n_2+2)(2n_2+n_3+3)(n_2+n_3+4)$$

$$\times (n_1+2n_2+n_3+4)(2n_1+2n_2+n_3+5)(n_1+n_2+n_3+3) \Big\} \frac{1}{2\cdot 2\cdot 3\cdot 3\cdot 4\cdot 5}$$

$$\dim[1,0,0] = \frac{2\cdot 1\cdot 1\cdot \cancel{3}\cdot \cancel{3}\cdot \cancel{4}\cdot \cancel{5}\cdot 7\cdot \cancel{4}}{\cancel{2}\cdot 2\cdot \cancel{3}\cdot \cancel{3}\cdot \cancel{4}\cdot \cancel{5}} = 7 \quad \text{fundamental}$$

$$\dim[0,1,0] = \frac{1\cdot \cancel{2}\cdot 1\cdot \cancel{3}\cdot 3\cdot \cancel{5}\cdot \overset{4}{\cancel{6}}\cdot 7\cdot \cancel{4}}{\cancel{2}\cdot \cancel{2}\cdot \cancel{3}\cdot 3\cdot \cancel{4}\cdot \cancel{5}} = 21 \quad \text{adjoint}$$

$$\dim[0,0,1] = \frac{1\cdot 1\cdot \cancel{2}\cdot \cancel{2}\cdot \cancel{3}\cdot \cancel{4}\cdot \cancel{5}\cdot \overset{2}{\cancel{6}}\cdot 4}{\cancel{2}\cdot \cancel{2}\cdot \cancel{3}\cdot \cancel{3}\cdot \cancel{4}\cdot \cancel{5}} = 8 \quad \text{spinor}$$

ASIDE: N-ality - gp $\mathbb{Z}_n$.

$SU(N)$, Rep^n R - n-ality a

$\mathrm{Sym}^n R$. - n-ality $na \bmod N$

$\Lambda^n R$ - n-ality $na \bmod N$

Every rep^n on RHS has the same n-ality
→ Because n-ality is additive.
→ group action in $\mathbb{Z}_n$ is additive modulo N.

N-ality ⟷ conserved quantities eg. energy, momentum

So, $\mathbb{Z}_n$ symmetry gives us a "conservation law".

~

Last time: $SO(7)$, B_n root system
→ B_3

$$\left.\begin{array}{l} e_i \\ e_i - e_j \\ e_i + e_j \end{array}\right\} \begin{array}{l}\text{set of +ve roots} \\ \to 9 \text{ in total}\end{array}$$

Have n^2 +ve roots $so(2n+1)$ ⇒ 9 roots as $n=3$

Rep^{n}s: dim $[1\,0\,0] = 7$ (fundamental / vector)
more common

dim $[0\,1\,0] = 21$ adjoint / 2nd rank antisym.

also defining rep^n

dim [0 0 1] = 8. spinor

Dimension formula:

- General feature of rotation gps: adjoint is 2^{nd} rank antisym.
  $\longrightarrow$ Check

$\Lambda^2 [1 0 0] = [0 1 0]$

- Common rep^n's natural one that arises is $\Lambda^3 [1 0 0]$ - what is it.

E.g. index $\mu = 0, 1 \ldots 6$
construct vec. V_μ

can also construct a tensor
$B_{\mu\nu}$ - has 21 compts.

Another tensor $C_{\mu\nu\lambda}$ - antisym.
(8 compts.)

etc ...

Can take any tensor and antisym.
eg $T_{\mu_1 \ldots \mu_4}$.

$$\underbrace{\Lambda^3 \underbrace{[1 0 0]}_{7}}_{\binom{7}{3} = 35} = \underbrace{[0, 0, 2]}_{\text{dim } 35} \approx Sym^2 [0 0 1]$$

Why [0 0 2]? Taking $Sym^2 [0 0 1] = [0 0 2]$

$\Rightarrow$ spinor is a <u>real</u> rep^n.

Generic feature of spinor rep^n:

- Sym product of spinors always give antisymmetric rep^n.

- Antisym. product of spinors always give sym. rep^n?

Spinor rep^ns have a dimension 2^{α}.

$$(1+x)^{\beta} = \sum_{n=0}^{\beta} \binom{\beta}{n} x^n$$

$$\text{Sym}^2[001] = \underset{\Lambda^3[100]}{[002]} + \underset{\Lambda^7[100]}{[000]}$$

$$\xrightarrow[\text{jumps by } 4]{}$$

$$\underbrace{\Lambda^4 \underbrace{[100]}_{7}}_{\binom{7}{4} = \binom{7}{3}} = [002]$$

$(7-4) \rightarrow 3$ form $\Lambda^3[100]$

Given an antisym. tensor, $C_{\mu_1 \cdots \mu_n}$ can construct a $(7-n)$ form using ε symbol

$$\varepsilon^{\mu_1 \cdots \mu_7} C_{\mu_1 \cdots \mu_n} = C^{\mu_{n+1} \cdots \mu_7}$$

So, $\Lambda^7[100]$ ie. 7 form $\longrightarrow (7-7) = 0$-form

- There is a natural extension for all rotation groups.

$\Lambda^5 [100] = [010] \longrightarrow$ 2 form

$\Lambda^6 [100] = [100] \longrightarrow$ 1 form

- Then, $PE_F \left[[100]t \right]$

$$= 1 + [100]t + [010]t^2 + [002]t^3$$

$$+ [002]t^4 + [010]t^5 + [100]t^6 + t^7.$$

i.e. they are dual to one another.

- Comes from the real rep^{n}s of rotation gps.
  $\longrightarrow$ not true for $SU(n)$ as the rep^{n}s are complex.

- Rotation gps have 2 invariant tensors,
  - $\delta_{ij} \longrightarrow A_i B_j$ s.t. $B_i A_j \delta_{ij} = S$
  - $\varepsilon^{i_1 \dots i_n}$ for $SO(n)$.

E.g. $\varepsilon^{i_1 \dots i_7} A_{i_1} B_{i_2} \longrightarrow$ 5th rank antisym. tensor

$= C^{i_3 \dots i_7}$

E.g $\underbrace{\Lambda^2 [001]}_{28} = \underbrace{[010]}_{21} + \underbrace{[100]}_{7}$

$= \Lambda^2 [100] + \Lambda^6 [100] \; \stackrel{=}{\scriptscriptstyle\Delta}$

$$\stackrel{\Delta}{=} \Lambda^5[100] + \Lambda^1[100]$$

jump of 4

- More generally, the tensor products of spinors,

This only works for order 2.

$$[001]^2 = [002] + [100] + [010] + [000]$$
$$= \Lambda^3[100] + \Lambda^1[100] + \Lambda^5[100] + \Lambda^7[100]$$

i.e. $\Lambda^1 + \Lambda^3 + \Lambda^5 + \Lambda^7$

jumps of 2

○ contribute to antisym. product of [001]
○ contribute to sym. product of [001].

- So, tensor products of spinor [001] rep^n, are a sum of the antisym. products of [100].

Exercise: $\mathrm{Sym}^n [001]_{SO(7)}$

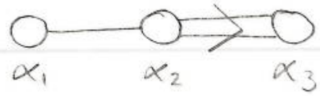
~

- <u>Bn Algebras</u>

- Common Rep^{n}s

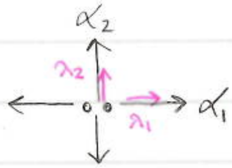
(i) [100] vector
(ii) [010] adjoint
(iii) [001] spinor

Relate these to the Dynkin diagram.



$\rightarrow$ how are these related to the fundamental weights?

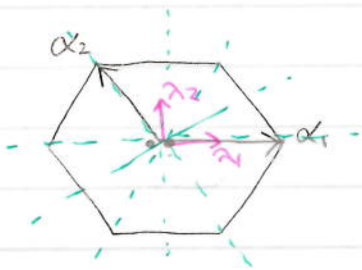
Recall: $D_2 = SO(4)$



$\rightarrow$ fundamental weights

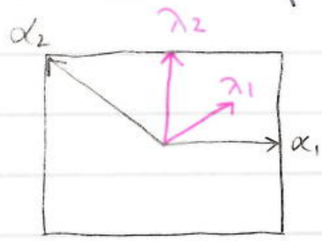
Then, $(\alpha_i, \lambda_j) \sim \delta_{ij}$ ~~a~~ i.e. they are orthogonal.

Now consider $A_2 = SO(3)$,



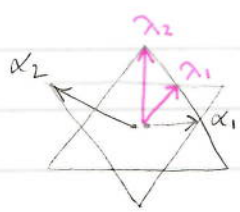
Again $(\alpha_i, \lambda_j) \sim \delta_{ij}$.

$B_2 = C_2 = SO(5) = Sp(2)$



$(\alpha_i, \lambda_j) \sim \delta_{ij}$

G2



$(\alpha_i, \lambda_j) \sim \delta_{ij}$

- Use the Cartan matrix :

$$C_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$$

$\rightarrow$ use C_{ij} to find relation between roots and weights.

$$\text{ie} \quad \alpha_i = C_{ij} \lambda_j$$

Consider $\dfrac{2(\lambda_i, \alpha_j)}{(\alpha_j, \alpha_j)}$

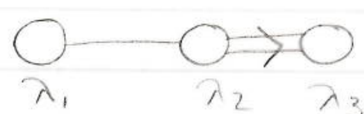
$$= 2 \, (C^{-1})_{ik} \underbrace{\frac{(\alpha_k, \alpha_j)}{(\alpha_j, \alpha_j)}}_{\underline{\underline{C}}}$$

$$= (C^{-1})_{ik} \, C_{kj} = \delta_{ij}$$

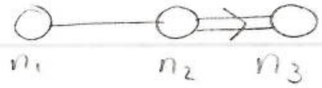
ie. the roots and weights are orthogonal
$\rightarrow$ if they have the same index, then it corresponds to the length of vec.

$\Rightarrow$ Λ_ω is dual to Λ_r.

- Can now associate a node in Dyn. diagram to λ_i as well as α_i.

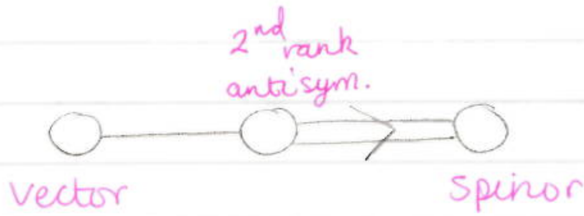


- <u>And</u> we can denote rep^n's using D diagrams!

For $[n_1\, n_2\, n_3] \Rightarrow$ 

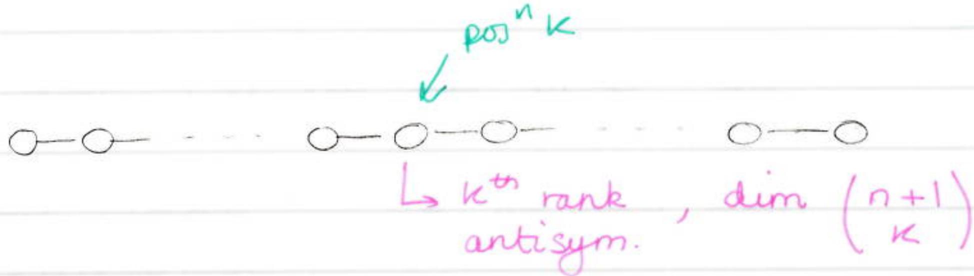
as $\sum_{i=1}^{3} \lambda_i n_i$.

- Consider :

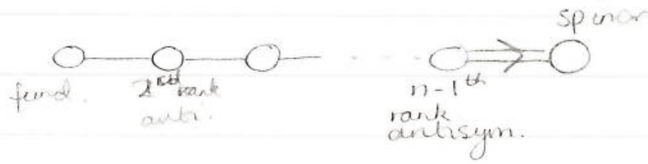


Recall: $\underline{A_n = SU(n+1)}$

Generally, for A_n



$B_n = SO(2n+1)$



Antisym. rep^{n}s : $\Lambda^k([1 \dots 0])$

$$= [0, \dots 1, \dots 0]$$

k^{th} pos^n

$$\text{of dim} \binom{2n+1}{k}.$$

k ranges between 0 and $(2n+1)$, but only account for $k = 0, \dots n-1$ above.

$$\dim \binom{2n+1}{k} = \binom{2n+1}{2n-k+1}$$

$\Rightarrow$ $k = 0, \dots n-1$; $k = 2n+1 \dots n+2$ $\Big\}$ $\Rightarrow$ missing ~~other~~ two.

$k=n$ and $k=n+1$ have same dimension $\Leftrightarrow$ real rep^{n}s

What is the rep^n n^{th} rank antisym. of B_n? ie. what is the highest weight?

$\Rightarrow$ the same rep^n as $(n+1)^{th}$ rank antisym.

N.B We had this for B_3 $\rightarrow$ recall the palindromic structure.

So, for B_n the highest weight is

$[0, \cdots 0, 2]$ – n^{th} rank antisym. of $SO(2n+1)$
ie Sym^2 (spinor rep^n).

Exercise : show that the $[1, \cdots 0]_{B_n}$ character is :

$$\underbrace{Y_1 + \frac{Y_2}{Y_1} + \frac{Y_3}{Y_2} + \cdots + \frac{Y_{n-1}}{Y_{n-2}} + \frac{Y_n^2}{Y_{n-1}}}_{n} +$$

$$\underbrace{+ \frac{Y_{n-1}}{Y_n^2} + \frac{Y_{n-2}}{Y_{n-1}} + \cdots \frac{Y_1}{Y_2} + \frac{1}{Y_1}}_{n} + 1$$

Antisymmeterise the above to get χ of other rep^{n}s.

- Compute det. of C for B_n :

$\det C = 2$ for B_n

$$\Rightarrow \frac{\Lambda_\omega}{\Lambda_r} = \mathbb{Z}_2 \text{, conserved quantum no.}$$

ie. there are two types of rep^{n}s depending on whether they are fermionic or bosonic.

Consider $[n_1, n_2, \ldots \underbrace{n_n}]$
↳ if odd – spinor/fermionic
if even – bosonic

Exercise: $\Lambda^3 [100]_{B_3} = \Lambda^4 [100]_{B_3}$

- Check the reality condition
  - vec. rep^n is real, so its antisym. are also real.

  - is the spinor real or pseudoreal?

For B_2: vector is real
spinor it depends on the rank. (has dim 4 for B_2).

N.B. The spinor of B_n has dim 2^n.
eg. B_3 spinor has dim 8.

For B_2, spinor is pseudo real. as,

singlet in Λ^2.

$$\Lambda^2 [0,1]_{B_2} = [1,0]_{B_2} + [0,0]_{B_2}$$

Note: Notice, $[n_1, n_2]_{B_2} = [n_2, n_1]_{C_2}$

ie. pseudoreal $\longleftrightarrow$ fermionic.

trivial in Sym

For B_3, $\mathrm{Sym}^2 [001]_{B_3} = [002]_{B_3} + [000]$

$\Rightarrow$ all rep^n in B_3 are real. No pseudoreal.

B_1 is also pseudoreal., as

$$\Lambda^2 [1] = [0].$$

Exercise: B_4 – take spinor of B_4 and see if singlet is in Sym^2 or Λ^2?
– Answer: real.

Generally, the spinor of B_n is real if $n = 0, 3 \mod 4$.

It is pseudo real if $n = 1, 2 \mod 4$.

– c.f. Fulton and Harris.

– Last time: rep^{n}s of B_n

Spinor ; dim 2^n.

Fermion / Bosons

$$\mathrm{Sym}^2 [0, \cdots, 1]_{B_n} = \sum_{\substack{k=0 \\ k = n \bmod 4}}^{2n+1} \Lambda^k [1, 0, \dots 0]_{B_n}$$

$$\Lambda^2 [0, \dots 0, 1]_{B_n} = \sum_{\substack{k=0 \\ k = (n+2) \bmod 4}}^{2n+1} \Lambda^k [1, 0, \dots, 0]_{B_n}$$

E.g. $\underbrace{\mathrm{Sym}^2 \underbrace{[0\,0\,0\,1]_{B_4}}_{2^4 = 16}}_{\left(\frac{17 \cdot 16}{2}\right) = 136}$; $k = 0, \dots 9$
$k = 0 \mod 4$

$$= \left(\Lambda^0 + \Lambda^4 + \Lambda^8 \right) [1\,0\,0\,0]_{B_4}$$

$$= [0\,0\,0\,0]_{B_4} + [0\,0\,0\,2]_{B_4} + [1\,0\,0\,0]_{B_4}$$

dim $1 + \binom{9}{4} = \frac{9 \cdot 8 \cdot 7 \cdot 6}{24} = 126 + 9 = 136$

$$\underbrace{\Lambda^2 \underbrace{[0001]_{B_4}}_{16}}_{120} = \left(\Lambda^2 + \Lambda^6\right)[1000]$$

$$= \underbrace{[0,100]}_{36} + \underbrace{[0010]}_{\frac{9\cdot 8\cdot 7}{6} = 84}$$

$k = n \bmod 4$ follows from $(1+x)^\beta = \sum_{k=0}^{\beta} \binom{\beta}{k} x^k$. character of the vec rep^n

→ Sym. and Λ jumps by 4.
→ spinors give antisymmetrisations
⇒ fermions are spinor.

$C_n \simeq Sp(n)$

Bn and Cn are dual to one another.

dim $n(2n+1)$; rank n ⇒ no. of roots $2n^2$.

Adjoint rep^n is Λ^2

- Roots : n^2 +ve — n long ; $n^2 - n$ short
  n^2 −ve

Root System :

$$C = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & \ddots & & 0 & \\ & \ddots & 2 & \ddots & & \\ & 0 & & -1 & 2 & -1 \\ & & & & -2 & 2 \end{pmatrix}$$

Simple Roots

$\alpha_1 = e_1 - e_2$

$\alpha_2 = e_2 - e_3$

$\vdots$

$\alpha_{n-1} = e_{n-1} - e_n$

$\alpha_n = 2e_n$ – long root $|\alpha_n|^2 = 4$; $|\alpha_i|^2 = 2$

$\Rightarrow$ factor 2

+ve Roots

$$\left.\begin{array}{lll} e_i - e_j & i<j & \binom{n}{2} \\ e_i + e_j & i<j & \binom{n}{2} \end{array}\right\} \begin{array}{c} n^2 - n \\ \text{short} \end{array}$$

$2e_i$ $\quad i = 1, \dots n \quad n \quad$ n long

- Pick $n = 3 \Rightarrow C_3$.

$\alpha_1 = e_1 - e_2$ ✓

$\alpha_2 = e_2 - e_3$ ✓

$\alpha_3 = 2e_3$ ✓

$$C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$$

Need 6 more roots ... $\binom{3}{2}$.

$\alpha_1 + \alpha_2 = e_1 - e_3$ ✓ $\qquad 2\alpha_2 + \alpha_3 = 2e_2$ ✓

$\alpha_2 + \alpha_3 = e_2 + e_3$ ✓ $\qquad \alpha_1 + \alpha_2 + \alpha_3 = e_1 + e_3$ ✓

$\alpha_1 + 2\alpha_2 + \alpha_3 = e_1 + e_2$ ✓ $\qquad 2\alpha_1 + 2\alpha_2 + \alpha_3 = 2e_1$ ✓

- Weyl dimension formula: dim of rep^n w/ highest weight λ is equal to

$$\prod_{\alpha \in \Phi^+} \frac{(\lambda + \rho, \alpha)}{(\rho, \alpha)}.$$

$\rho \equiv$ Weyl vector $-$ $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$.

i.e. $1/2$ sum of all +ve roots.

- Set highest weight $\lambda = \sum_{i=1}^{3} n_i \lambda_i$.

and $\lambda_i = (C^{-1})_{ij} \alpha_j$.

Then $\dim [n_1, n_2, n_3]_{C_3} \triangleq$

$$\triangleq \Big\{ (n_1+1)(n_2+1)(2n_3+2)(n_1+n_2+2)(n_2+2n_3+3)(2n_2+2n_3+4) \times$$

$$\times (n_1+n_2+2n_3+4)(n_1+2n_2+2n_3+5)(2n_1+2n_2+2n_3+6) \Big\} \times$$

$$\times \frac{1}{2^2 \cdot 3 \cdot 4^2 \cdot 5 \cdot 6}$$

$$= \Big\{ (n_1+1)(n_2+1)(n_3+1)(n_1+n_2+2)(n_2+n_3+2)(n_2+2n_3+3) \times$$

$$\times (n_1+n_2+n_3+3)(n_1+n_2+2n_3+4)(n_1+2n_2+2n_3+5) \Big\}$$

$\times \frac{1}{2^2 \cdot 3^2 \cdot 4 \cdot 5}$

$\dim[100] = 6$ defining rep^n / fundamental.

$$= \frac{\cancel{2}\cdot 1\cdot 1\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{5}\cdot 6}{\cancel{2}^2\, \cancel{3}^2\, \cancel{4}\, \cancel{5}}$$

Crucial difference between B_n and C_n. $\Lambda^2 \equiv$ adjoint in B_n. But as B_n dual to C_n, Sym^2 is adjoint for C_n.

$$\dim[010] = \frac{1\cdot \cancel{2}\cdot 1\cdot \cancel{3}\cdot \cancel{3}\cdot 4\cdot \cancel{4}\cdot \cancel{5}\cdot 7}{2^{\cancel{2}}\, \cancel{3}^2\, \cancel{4}\, \cancel{5}} = 14$$

2^{nd} rank Λ^2 antisym. $\rightarrow$ traceless

$$\dim[001] = \frac{1\cdot 1\cdot \cancel{2}\cdot \cancel{2}\cdot \cancel{3}\cdot \cancel{4}\cdot \cancel{5}\cdot \cancel{6}^{2}\cdot 7}{\cancel{2}^2\, \cancel{3}^2\, \cancel{4}\cdot \cancel{5}} = 14$$

traceless Λ^3 3^{rd} rank antisym.

Antisymmetric product:

$$\underbrace{\Lambda^2[100]}_{15} = \underset{14}{[010]} + \underset{1}{[000]}$$

$$\underbrace{\Lambda^3[100]}_{20} = \underset{14}{[001]} + \underset{6}{[100]}$$

Take form w/ C_{ijk} tot. antisym. then,

$C_{ijk}\,\varepsilon^{ij} = 0 \implies$ traceless.
$\rightarrow$ left w/ one index.

For a 2-form, $C_{ij}\,\varepsilon^{ij} = 0$.

ε^{ij} invariant tensor in C_n.