Imperial College London

QFFF MSc TESTS Jan 2014

PARTICLE SYMMETRIES AND UNIFICATION TEST

For Students in Quantum Fields and Fundamental Forces

Thursday 16 Jan 2014: 10:00 to 12:00

Answer all questions in Part A and Part B
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the 2 answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

PUT YOUR PART A AND PART B ANSWERS INTO *SEPARATE* ANSWER BOOKS.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in 2 answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

PART A: Particle Symmetries

1. This question is about the quark model of baryons. We write the quarks as $q^{i\alpha a}$, such that

$$\begin{pmatrix} q^{1\alpha a} \\ q^{2\alpha a} \\ q^{3\alpha a} \end{pmatrix} = \begin{pmatrix} u^{\alpha a} \\ d^{\alpha a} \\ s^{\alpha a} \end{pmatrix}, \qquad q^{i\alpha a} \mapsto \rho^i{}_j q^{j\alpha a}, \qquad q^{i\alpha a} \mapsto \rho^\alpha{}_\beta q^{i\beta\alpha}, \qquad q^{i\alpha a} \mapsto \rho^a{}_b q^{i\alpha b},$$

where $\rho^i{}_j$ is the defining representation of the $\mathrm{SU}(3)_f$ of flavour, $\rho^\alpha{}_\beta$ is the defining representation of the $\mathrm{SU}(2)_s$ of spin, and $\rho^a{}_b$ is the defining representation of the $\mathrm{SU}(3)_c$ of colour.

(i) What are the dimensions of the defining representations of $\mathrm{SU}(3)_f$ flavour, $\mathrm{SU}(2)_s$ spin and $\mathrm{SU}(3)_c$ colour? Given the baryons are constructed from three quarks, as a tensor of $\mathrm{SU}(3)_f \times \mathrm{SU}(2)_s \times \mathrm{SU}(3)_c$ what index structure does the baryon have?

Consider the colour representation of the baryon. Show that there is an invariant subspace that transforms as a singlet under $\mathrm{SU}(3)_c$. What symmetry properties on the colour indices does this singlet tensor have? [7 marks]

(ii) The Cartan matrix for $\mathfrak{sl}(3, \mathbb{C}) \simeq \mathfrak{su}(3)_{\mathbb{C}}$ is

$$A_{ij} = \frac{2\langle\alpha_i, \alpha_j\rangle}{\langle\alpha_i, \alpha_i\rangle} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

How many fundamental roots α_i are there? What are their relative lengths? What is the angle between them? Draw the full root diagram and indicate the sets of positive and negative roots.

How is the Weyl group generated and how many elements does it have in this case?

Draw the fundamental weights w_i and define the weight lattice. What is the relation between w_i and α_i? [12 marks]

(iii) Draw the weight diagram for the defining representation V_1 of $\mathfrak{su}(3)$ which has highest weight $\lambda = w_1$. Do the same for the conjugate of the defining representation and identify the corresponding highest weight.

Now consider the adjoint representation V_2. What is its dimension? Draw the weight diagram and comment on how it is related to the root diagram. What are the dimensions of the weight spaces?

Finally consider the representation V_3 corresponding to the highest weight $\lambda = 3w_1$. Draw the weight diagram and show how V_3 decomposes into $\mathfrak{sl}(2, \mathbb{C})$ representations for the different $\mathfrak{sl}(2, \mathbb{C})$ sub-algebras defined by each positive root. [12 marks]

(iv) Assuming each weight space in the V_3 representation is one-dimensional, write down the characters of V_1, V_2 and V_3.

Hence, using the properties of characters, give the decomposition of $V_1 \otimes V_1 \otimes V_1$ into irreducible representations. [12 marks]

(v) What does the decomposition described in the last question tell us about the classification of baryons? Given the largest and smallest representations in the decomposition are totally symmetric and totally antisymmetric tensors respectively, what do we know about the spin of the corresponding baryons? [7 marks]

[Total 50 marks]

PART B: Unification and the Standard Model

2. In this question, general theorems relating to broken symmetry systems may simply be quoted; they do not have to be proved. Consider a theory with two complex scalar fields $\Phi(x), \chi(x) \in \mathbb{C}$ and a real vector field $A_\mu(x) \in \mathbb{R}$, described by the Lagrangian

$$\mathcal{L} = -D^\mu\Phi^* D_\mu\Phi - D^\mu\chi^* D_\mu\chi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - V(|\Phi|^2, |\chi|^2) \; , \tag{2.1}$$

where

$$D_\mu\Phi = (\partial_\mu - \mathrm{i}gA_\mu)\Phi \qquad D_\mu\chi = (\partial_\mu + \mathrm{i}gA_\mu)\chi \tag{2.2}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.3}$$

$$V(B, C) = 2\alpha B + 2\beta C + \lambda_1 B^2 + \lambda_2 C^2 + \lambda_{12} BC \; , \tag{2.4}$$

in which α, β, λ_1, λ_2 and λ_{12} are all real constants, and where λ_1, λ_2 and λ_{12} are all positive.

(i) What is the rigid internal (non-spacetime) symmetry group of this theory? What is the local symmetry? Give explicit expressions for the corresponding transformations of each field for each symmetry.

[8 marks]

(ii) What are the unbroken rigid and local symmetries if $\alpha > 0$ and $\beta > 0$? Give the masses of all the physical degrees of freedom in this case. How many degrees of freedom per space-time point are there?

[10 marks]

(iii) If $\alpha < 0$ and $\beta > 0$ together with positive values of the λ_1, λ_2 and λ_{12} parameters, show that the vacuum values of the scalar fields satisfy

$$|\Phi|^2 = -\frac{\alpha}{\lambda_1} > 0 \; , \quad \chi = 0 \; . \tag{2.5}$$

Is the $\chi = 0$ vacuum stable for these values of $\alpha, \beta, \lambda_1, \lambda_2$ and λ_{12}? What are the unbroken rigid and local symmetries in this case? Give the transformations of the fields with respect to the unbroken symmetries. Give the masses of the physical degrees of freedom and the charges with respect to the unbroken symmetry in this case and show that the overall number of degrees of freedom per spacetime point is the same as in part (ii).

[20 marks]

(iv) Now consider a different choice of α and β, where, for suitable values of the $\lambda_1, \lambda_2 \& \lambda_{12}$ parameters, the vacuum values of the scalar fields satisfy

$$|\Phi|^2 > 0 \; , \quad |\chi|^2 > 0 \; . \tag{2.6}$$

What are the unbroken rigid and local symmetries in this case? Give the massive/massless character of the physical degrees of freedom in this case and show that the overall number of degrees of freedom per spacetime point is the same as in part (ii). You do not have to give detailed expressions for the masses in this case.

[12 marks]

[Total 50 marks]

End of examination paper