Imperial College London

MSc TEST Jan 2017

PARTICLE SYMMETRIES AND UNIFICATION

For Students in Quantum Fields and Fundamental Forces

Thursday 12 Jan 2017: 10:00 to 12:00

Answer all questions in Part A and Part B
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the 2 answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in 2 answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

Go to the next page for questions

SECTION A: Particle Symmetries

1. This problem is about the Poincaré group in two dimensions.

(i) By considering light-cone coordinates $x^\pm = \frac{1}{2}(x^0 \pm x^1)$, show that the metric $\mathrm{d}s^2 = -(\mathrm{d}x^0)^2 + (\mathrm{d}x^1)^2$ is invariant under the transformation

$$x^\pm \mapsto \mathrm{e}^{\pm\phi} x^\pm + a^\pm,$$

where $\phi, a^+, a^- \in \mathbb{R}$. [3 marks]

(ii) Consider matrices and vectors of the form

$$P = \left\{ \begin{pmatrix} \mathrm{e}^{\phi} & 0 & a^+ \\ 0 & \mathrm{e}^{-\phi} & a^- \\ 0 & 0 & 1 \end{pmatrix} : \phi, a^+, a^- \in \mathbb{R} \right\}, \qquad v = \begin{pmatrix} x^+ \\ x^- \\ 1 \end{pmatrix}.$$

Give the definition of an abstract group and show that P forms a group under matrix multiplication. Show that elements of P act on v by transforming $x^\pm$ as in question (i). [3 marks]

(iii) Give the Lie algebra $\mathfrak{p}$ corresponding to the Lie group P. Choose a suitable basis and give the structure constants defining $\mathfrak{p}$. Hence show that the translation subalgebra is an ideal. [3 marks]

(iv) Let e^X be the matrix exponential. Show explicitly that

$$\left\{ \mathrm{e}^X : X \in \mathfrak{p} \right\} = P.$$

Does such a relation always hold for any Lie group G with Lie algebra $\mathfrak{g}$?
[4 marks]

(v) Following Wigner, one can construct a unitary representation $S(\phi, a^+, a^-)$ of P on a Hilbert space $\mathcal{H}$ with basis vectors $|p^+, p^-\rangle \in \mathcal{H}$ such that

$$S(\phi, a^+, a^-)|p^+, p^-\rangle = \mathrm{e}^{-\mathrm{i}\mathrm{e}^{\phi}p^+a^- - \mathrm{i}\mathrm{e}^{-\phi}p^-a^+}|\mathrm{e}^{\phi}p^+, \mathrm{e}^{-\phi}p^-\rangle,$$

where $p^\pm = p^0 \pm p^1$. Show that $S(\phi, a^+, a^-)$ satisfies the conditions necessary for it to be a representation. [3 marks]

(vi) Show that this representation is reducible and find *all* the irreducible subspaces. How are theses subspaces interpreted physically? Show that they can be associated to spaces of solutions of the two-dimensional Klein–Gordon equation. What happens when the mass is zero?

What is the little group for states in the different irreducible subspaces and what does this imply physically? [4 marks]

[Total 20 marks]