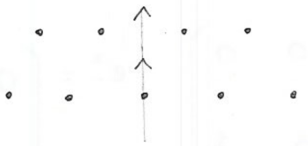
Lecture 6 Particle Symmetries 24.10.2011

SU(2) Lie algebra : representations : highest weight $n$, and $-\frac{n}{2} \leq m \leq \frac{n}{2}$

reflexion sym (on $\varepsilon$ representation of highest weight $n$, so is $-m$)

weight diagrams : For $n=3$

For $n=4$



[↗]

SU(3): recall : def = 3×3 matrices $U \in GL(3, \mathbb{C})$ with $UU^\dagger = \mathbb{1}$, $\det U = 1$

dim SU(3) = 8 : Lie algebra generated by 8 elements $\lambda_i$ (traceless)

$$\lambda_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \lambda_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \lambda_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \lambda_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\lambda_5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix} \quad \lambda_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \lambda_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Gell-Mann matrices

Prop
- $\mathrm{Tr}(\lambda_a \lambda_b) = 2\delta_{ab}$
- $\lambda_a^\dagger = \lambda_a$ (hermitian)

[but] but we can form an anti-hermitian algebra: $T_a = \frac{-i}{2}\lambda_a \Rightarrow T_a^\dagger = -T_a$

giving $[T_a, T_b] = f_{ab}{}^{c}\, T_c$ ($f_{ab}^{c}$ the structure constant)

$(\lambda_1, \lambda_2, \lambda_3)$ form a copy of the Pauli matrices, and have the same properties
e.g. it forms an SU(2) subalgebra of SU(3)

$(\lambda_4, \lambda_5, \frac{1}{2}(\sqrt{3}\lambda_8 + \lambda_3))$ also forms an SU(2) subalgebra (in the (1.3) direction)

$(\lambda_6, \lambda_7, \frac{1}{2}(\sqrt{3}\lambda_8 - \lambda_3))$ " " " " " (" " (2.3) " )

This is a 3D representation of the SU(3) algebra
Let us generalise via these covariant matrices:

$$h_1 = \begin{bmatrix} +\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad h_2 = \begin{bmatrix} \frac{1}{2}\sqrt{3} & 0 & 0 \\ 0 & \frac{1}{2}\sqrt{3} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{3}} \end{bmatrix} \quad e_+^1 = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad e_-^1 = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$e_+^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \end{bmatrix} \quad e_-^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \quad e_+^3 = \begin{bmatrix} 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad e_-^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix}$$

Replace these matrices by $d \times d$ matrices:

$H_{\frac{1}{2}}, E_\pm^{1_{2_3}}$ : it satisfies the relations of commutation again defining again the same algebra

$$[H_1, H_2] = 0 \qquad [H_1, E_\pm^1] = \pm E_\pm^1 \ , \quad [H_2, E_\pm^1] = 0$$

$$[H_1, E_\pm^2] = \mp\frac{1}{2} E_\pm^2 \ , \quad [H_2, E_\pm^2] = \pm\frac{\sqrt{3}}{2} E_\pm^2$$

$$[H_1, E_\pm^3] = \pm\frac{1}{2} E_\pm^3 \ , \quad [H_2, E_\pm^3] = \pm\frac{\sqrt{3}}{2} E_\pm^3$$

$$[E_+^1, E_-^1] = H_1 \qquad [E_+^2, E_-^2] = \frac{\sqrt{3}}{2} H_2 - \frac{1}{2} H_1$$

$$[E_+^3, E_-^3] = \frac{\sqrt{3}}{2} H_1 + \frac{1}{2} H_1$$

Giving the <u>maximal commuting subalgebra</u> $\equiv$ Cartan algebra

$$[E_+^1, E_+^2] = +\frac{1}{\sqrt{2}} E_+^3 \ , \quad [E_+^1, E_-^3] = -\frac{1}{2} E_-^2 \ , \quad [E_+^2, E_-^3] = \frac{1}{\sqrt{2}} E_-^1$$

$$[E_-^1, E_-^2] = -\frac{1}{\sqrt{2}} E_-^3 \ , \quad [E_-^1, E_+^3] = \frac{1}{\sqrt{2}} E_+^2 \ , \quad [E_-^2, E_+^3] = \frac{-1}{\sqrt{3}} E_+^1$$

all others are null

Let's form the SU(2) subalgebras:

$$[H_1, E^1_\pm] = \pm E^1_\pm, \quad [E^1_+, E^1_-] = H_1 \qquad \text{Form a SU(2) subalg.}$$

$$\left[\frac{\sqrt{3}}{2}H_2 - \frac{1}{2}H_1, E^2_\pm\right] = \pm E^2_\pm, \quad [E^2_+, E^2_-] = \frac{\sqrt{3}}{2}H_2 - \frac{1}{2}H_1 \qquad \text{" " " "}$$

and $$\left[\frac{\sqrt{3}}{2}H_2 + \frac{1}{2}H_1, E^3_\pm\right] = \pm E^3_\pm, \quad [E^3_+, E^3_-] = \frac{\sqrt{3}}{2}H_2 + \frac{1}{2}H_1 \qquad \text{" " " "}$$

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Lecture 7 | Particle Symmetries | 25.10.2011

## SU(3) algebra

### irreducible representations

8 generators, 2 maximally commuting $H_1, H_2$, 6 lowering/raising generators $E_\pm{}^m$

3 SU(2) sub algebras

- Recall for SU(2) $|j, m\rangle$

for SU(3) $H_1 |\lambda, p, q\rangle = p|\lambda, p, q\rangle$ $\lambda$ is the highest weight

$H_2 |\lambda, p, q\rangle = q|\lambda, p, q\rangle$

$*^1$ $E^1_\pm |\lambda, p, q\rangle = C_{\lambda, p, q} |\lambda, p \pm 1, q\rangle$

$E^1_\pm$ changes $(p, q)$ to $(p \pm 1, q)$ ← do these with the commutators

$*^2$ $E^2_\pm$ changes $(p,q)$ to $(p, q) \pm \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$*^3$ $E^3_\pm$ " " " $(p, q) \pm \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

### Finite dimensional representations:

$\exists$ some $p_0, q_0$ for which $E_+{}^m |\lambda_1, p_0, q_0\rangle = 0$

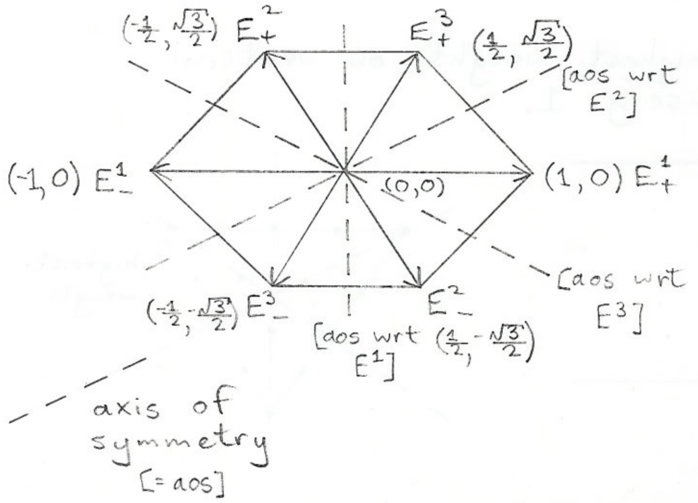
by definition $(p_0, q_0)$ are the highest weights of the representation.

- Recall the reflection symmetry of SU(2):
  if $m$ is a weight then $-m$ is a weight

$*^1$ implies that $2p \in \mathbb{Z}$ from first subalgebra of SU(3)

$*^{2,3}$ $\sqrt{3}q - p \in \mathbb{Z}$ [from 2nd subalg] $\sqrt{3}q + p \in \mathbb{Z}$ [from 3rd subalg]

$(p, q)$ are points on a two dimensional lattice, triangular



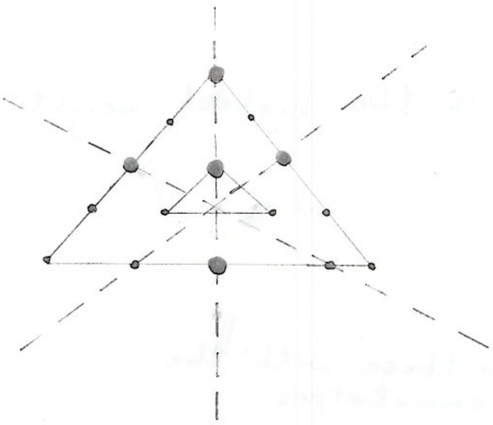

3 reflection symmetries:

1. If $(p, q)$ weight, $(-p, q)$ is a weight
2. If $(p, q)$ weight then $\left(\frac{1}{2}p + \frac{\sqrt{3}}{2}q, \frac{\sqrt{3}}{2}p - \frac{1}{2}q\right)$ is too
3. If $(p, q)$ is a weight then $\left(\frac{1}{2}p - \frac{\sqrt{3}}{2}q, -\frac{\sqrt{3}}{2}p - \frac{1}{2}q\right)$ is too

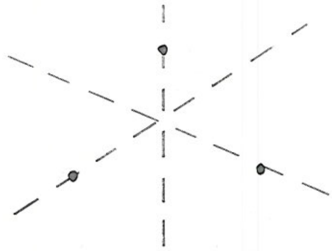
Order 6 symmetry which implies that the weight along all Lattice from Knowledge of weights on the wedge.

The simplest representation has highest weights $(0,0)$, of dimension 1, trivial representation.

Case a: highest weight is along the vertical axis.



15 dimensional representation of $SU(3)$



3 dimensional representation of $SU(3)$

Multiplicity of weights?

$$[E_-^1, E_-^2] = \frac{-1}{\sqrt{2}} E_-^3$$

$$E_-^1 (E_-^2)^n |\psi\rangle = \ldots = \frac{-n}{\sqrt{2}} E_-^3 (E_-^2)^{n-1} |\psi\rangle$$

$$[E_-^2, E_-^3] = 0$$

this argument shows that there is at least one state at each weight/site

- For triangular representations with highest weight on vertical axis multiplicity of points is precisely 1.

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- Highest weight on the other boundary of the wedge
- Highest weight in between?

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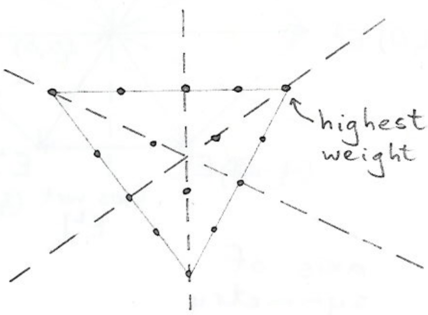
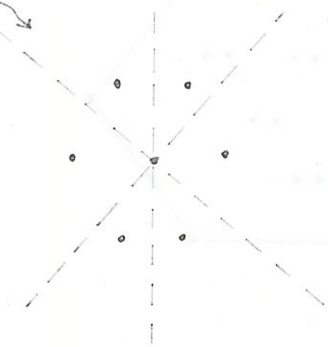


## representations of SU(3)

Weights are points on a 2d triangular lattice

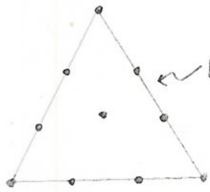
[should be equal angles between dotted lines]



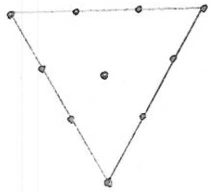
[called a wedge]

3 reflection symmetries 3 SU(2) subgroups of SU(3)

2 triangular lattices



[this SU(2) mixes states with another SU(2)]



[10d rep$^{\underline{n}}$ → decompose rep$^{\underline{n}}$ 4 points × 3]

For every triangle with highest weight on the vertical line there is a triangle with highest weight on the $\theta = 30°$ line

If the triangle has length $n$, the number of points $\binom{n+2}{2}$

1, 3, 6, 10, 15, 21, 28

[weights in 2nd triangle are −weights in 1st]

Recall the defining representation of SU(3) is 3 dim

Consider a vector in $\mathbb{C}^3$ $V = \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}$ $V^\dagger = (\cdot\,,\cdot\,,\cdot)$

$\bigtriangledown$ fundamental rep of SU(3) $3$

$\bigtriangleup$ antifundamental rep $\bar{3}$

[realization of rep$^{\underline{n}}$s
- lattice pts
- tensor products]

$V_i V_j$ $i = 1,2,3$

6 components

$V_i V_j V_k$

10 components

$\bigtriangledown^m$ $V_{i_1} \cdots V_{i_n}$ ⇐ n-th rank symmetric representation of SU(3)

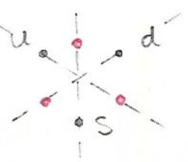
$\binom{n+2}{2}$ components

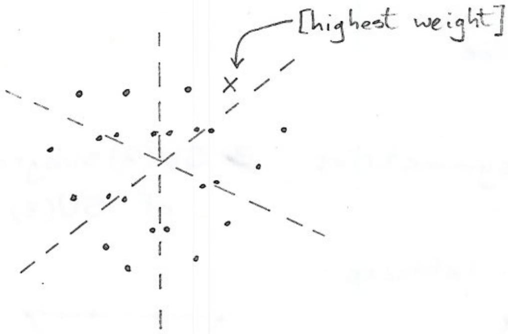
$\bigtriangleup^n$ $(V^\dagger)^{i_1} \cdots (V^\dagger)^{i_n}$ ⇐ n-th rank cplx conjugate rep (conj)

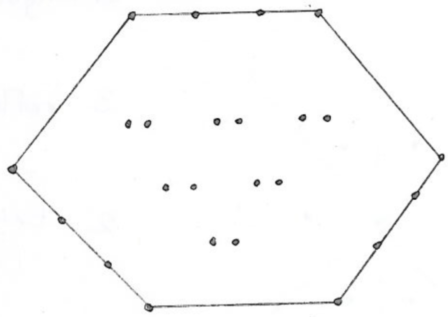
$\binom{n+2}{2}$ components

[use upper, lower indices to characterise dif$^{\underline{t}}$ rep$^{\underline{n}}$s]

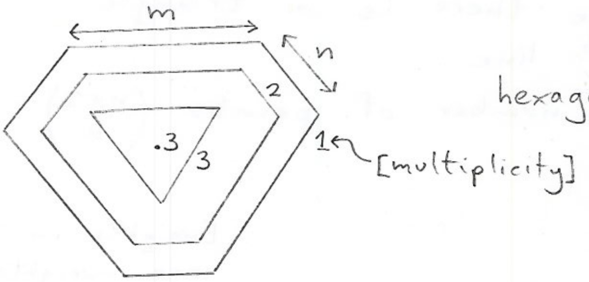
[subalgebra of given algebra ⇒ can form every rep$^{\underline{n}}$ of algebra by subalg ≠ anything about reducibility]





hexagonal lattice
$4 \cdot 3 + 6 \cdot 2 = 24$




hexagon $(m, 0)$ $\bigtriangledown^{m}$ m is the order of symmetrization of 3

$(0, n)$ $\bigtriangleup^{n}$ n is the order of sym of $\bar{3}$

Each hexagonal layer has multiplicity 1 higher than outer one
Grow until first triangular layer, stays fixed

A hexagonal representation with lengths $(m, n)$ has a highest weight
$\left(\frac{m}{2}, \frac{1}{2\sqrt{3}}(m+2n)\right)$

[weights live on triangular lattice]
[most general rep^n of SU(3)]

Of dimension

$m \geq n$: $\binom{m-n+2}{2}(n+1)$ [dim^n of (rep^n) highest weight in terms of 2 dim^ns of lattice]

(# points on inner triangle) (multiplicity of each point)

$$+ \sum_{K=1}^{n} 3K(m+n+2-2K) = \frac{(m+1)(n+1)(m+n+2)}{2}$$

[exchange m<->n get complex conjugate: symmetric to exchange as we would expect]

[Can define any point on lattice by combination 2 vectors - so integers]

An irreducible representation of SU(3) is characterized by 2 non-negative integers $(m, n)$ of dimension

$$\frac{(m+1)(n+1)(m+n+2)}{2}$$

Compare: for SU(2): an irrep has 1 non-negative integer $n$, with weights $\frac{-n}{2} \leq m \leq \frac{n}{2}$

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An irreducible representation of $SU(3)$ is characterised by 2 non-negative integers $(n,m)$ of dimension

$$\frac{(n+1)(m+1)(n+m+2)}{2}$$

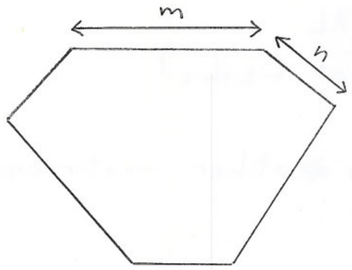
Irreps for $SU(2)$ are resp. by 1 non-negative integer $n$, of spin

$$\frac{n}{2} \quad ( = \frac{n}{2}$$

SU(3) representations

highest weight $(m, n)$ $\quad m, n = 0, 1, \ldots$

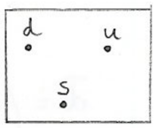
of dimension $\dfrac{(m+1)(n+1)(m+n+1)}{2}$



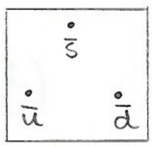

we have seen that the multiplicity increases by one when we go to the centre

simple cases $(0,0)$, singlet 1

next case $(1,0)$, triplet 3 ↳ fundamental rep

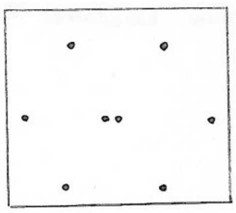



antifundamental rep ← $\bar{3}$
$(0,1)$ triplet




the weights are opposite of

adjoint octet 8
$(1,1)$



<u>the 3-d representation</u> and its vector space

$$h_1 = \begin{pmatrix} \frac{1}{2} & & \\ & -\frac{1}{2} & \\ & & 0 \end{pmatrix} \qquad h_2 = \begin{pmatrix} \frac{1}{2\sqrt{3}} & & \\ & \frac{1}{2\sqrt{3}} & \\ & & -\frac{1}{\sqrt{3}} \end{pmatrix}$$

$$h_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad h_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

[$h_1$ & $h_2$ act on a 3D vector space]
[$\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)$ is an eigenstate for $h_1, h_2$ with eigenvalues $\frac{1}{2}, \frac{1}{2\sqrt{3}}$]

vector weight

$$u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \left(\frac{1}{2}, \frac{1}{2\sqrt{3}}\right) \qquad d = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \left(-\frac{1}{2}, \frac{1}{2\sqrt{3}}\right) \qquad s = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \left(0, -\frac{1}{\sqrt{3}}\right)$$

weight of u

<u>remember</u>: $e^1$ acts horizontally $\quad d = \sqrt{2}\, e^1_- u$

Clebsch-Gordon coefficients of $SU(3)$

$$e^1_- = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

be careful:
it is the 3-d rep for the FUNDAMENTAL
(also true for $L_1, L_2$)

$\downarrow$

other cases $\Rightarrow$ other matrices

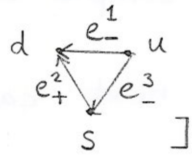
Similarly we can also compute

$$S = \sqrt{2}\, e^3_- u \qquad [(e^i_-)^{-1} = \alpha . e^i_+ \; : \; d = \sqrt{2}\, e^2_+ s$$

$$S = \sqrt{2}\, e^2_- d$$

or visually

[cf diagram lec. 8 p. 1]



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<u>The 3 rep</u> $\quad \bar{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \bar{d} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \bar{s} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

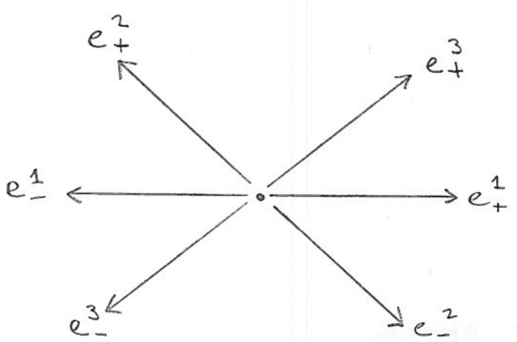
eigenvalues $\left(-\frac{1}{2}, \frac{-1}{2\sqrt{3}}\right) \quad \left(\frac{1}{2}, \frac{-1}{2\sqrt{3}}\right) \quad \left(0, \frac{1}{2\sqrt{3}}\right)$

<u>note</u> $h_1 \neq \bar{h}_1 = \begin{bmatrix} -1/2 & & \\ & 1/2 & \\ & & 0 \end{bmatrix}$

$h_2 \neq \bar{h}_2 = \begin{bmatrix} -1/2\sqrt{3} & & \\ & -1/2\sqrt{3} & \\ & & 1/\sqrt{3} \end{bmatrix}$

$e^i_\pm \neq \bar{e}^i_\pm$

$\bar{e}^i_\pm = -(e^i_\mp)^\dagger$
[in fact]




[CF diagram Lecture 7 p. 1]

different e to the one used in the fundamental repn

$\downarrow$

[see [$\alpha$]]

$$\bar{u} = -\sqrt{2}\, \bar{e}^3_- \bar{s} \qquad \bar{d} = -\sqrt{2}\, \bar{e}^2_- \bar{s} \qquad \bar{u} = -\sqrt{2}\, \bar{e}^1_- \bar{d}$$

$$[\alpha] \quad \bar{e}^m_\pm = -\left(e^m_\mp\right)^\dagger$$

<u>Exercise</u>: verify that
e.g. other e's acting on different $u, d, s = 0$

Now, let's do the d-dimensional rep: the adjoint.
It's the simplest hexagon

$h_1, h_2$ have weight $(0,0)$

$e^1_+$ has weight [one] $(1,0)$

$e^1_-$ " " $(-1,0)$

$e^2_+$ " " $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$e^2_-$ " " $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

$e^3_+$ " " $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$e^3_-$ " " $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

adjoint action, commutator

$[h_1, e^1_+] = e^1_+$ $\qquad$ $[h_1, e^2_+] = -\frac{1}{2} e^2_+$

$[h_2, e^1_+] = 0$ $\qquad$ $[h_2, e^2_+] = \frac{\sqrt{3}}{2} e^2_+$

... with eigenvalues as results

Now, we can do <u>tensor products</u>!

$V_i V_j$ = second rank symmetric rep of $SU(3)$

$u \otimes u = \left(1, \frac{1}{\sqrt{3}}\right)$ $\qquad$ weights are <u>additive</u>

$d \otimes u = u \otimes d = \left(0, \frac{1}{\sqrt{3}}\right)$

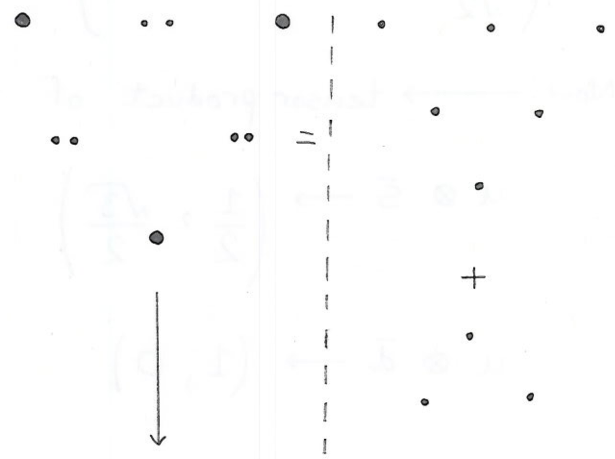
$d \otimes d = \left(-1, \frac{1}{\sqrt{3}}\right)$

$s \otimes s = \left(0, \frac{-2}{\sqrt{3}}\right)$

$s \otimes u = u \otimes s = \left(\frac{1}{2}, -\frac{1}{2\sqrt{3}}\right)$

$s \otimes d = d \otimes s = \left(\frac{-1}{2}, \frac{-1}{2\sqrt{3}}\right)$

draw these weights on lattice



We can find something interesting
$\rightarrow$ it is not irreducible rep$^{\underline{n}}$

$\rightarrow 3 \otimes 3 = 6_s \oplus \underline{3}_A$

(symmetric) (antisymmetric)

$\Uparrow$

these numbers are dimensions

$V_i \otimes V_j \quad (V_i V_j) \quad (\varepsilon^{ijk} V_j V_k)$

How to construct with 2 vectors?

$V_i \otimes V_j = V_i V_j \quad \rightarrow$ symmetric

& we can use $\varepsilon^{ijk} V_j V_k \quad \rightarrow$ antisymmetric

upper indices $\Rightarrow$ antifundamental

The 6-d rep $u \otimes u,\ d \otimes d,\ s \otimes s$

$\underline{6_s \oplus \underline{3}_A}$

[result of computation -not a choice]

$$3 \left\{ \begin{array}{l} \frac{1}{\sqrt{2}}(d \otimes u + u \otimes d) \\ \\ \frac{1}{\sqrt{2}}(u \otimes s + s \otimes u) \\ \\ \frac{1}{\sqrt{2}}(d \otimes s + s \otimes d) \end{array} \right\}$$ symmetric part of rep

$$\bar{3} \left\{ \begin{array}{l} \frac{1}{\sqrt{2}}(d \otimes u - u \otimes d) \\ \\ \frac{1}{\sqrt{2}}(d \otimes s - s \otimes d) \\ \\ \frac{1}{\sqrt{2}}(s \otimes u - u \otimes s) \end{array} \right\}$$ antisymmetric part of rep

Now: $\longrightarrow$ tensor product of $3 \otimes \underline{3}$

$$u \otimes \bar{s} \longrightarrow \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

$$u \otimes \bar{d} \longrightarrow (1, 0)$$

$$d \otimes \bar{s} \longrightarrow \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

$$\frac{1}{\sqrt{6}}(d \otimes \bar{d} + u \otimes \bar{u} - 2 s \otimes \bar{s}) \longrightarrow (0,0)$$

$$\frac{1}{\sqrt{2}}(d \otimes \bar{d} - u \otimes \bar{u}) \longrightarrow (0,0)$$

$$d \otimes \bar{u} \longrightarrow (-1, 0)$$

$$s \otimes \bar{u} \longrightarrow \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

$$s \otimes \bar{d} \longrightarrow \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

$3 \otimes 3 = 8 \oplus 1$

$\downarrow$

$$\downarrow$$

$$\frac{1}{\sqrt{3}}\left(u\otimes\bar{u} + d\otimes\bar{d} + s\otimes\bar{s}\right)$$

characters: another way of doing this product

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Let's finish with $3\otimes 3\otimes 3$, helped by $3\otimes 3 = 6_S \oplus \underline{3}_A$
(note that $v_i v_j$ is sym thus gives $6_S$ only
for $3\otimes 3$, we need $v_i w_j$ via $v_i w_j + w_i v_j \sim 6_S$ and $\varepsilon^{ijk} v_j w_k \sim 3_A$)

Recall that:

$$6 = \left\{u\otimes u,\ d\otimes d,\ s\otimes s,\ \frac{1}{\sqrt{2}}(d\otimes u + u\otimes d)\ldots\right\}$$

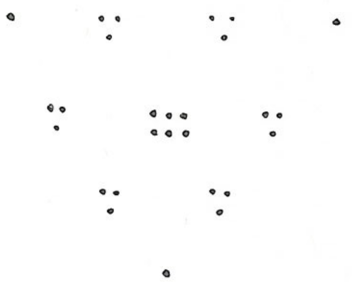
and

$$E^m_\pm = e^m_\pm \otimes \mathbb{1} + \mathbb{1}\otimes e^m_\pm \text{ acting on } u\otimes u$$

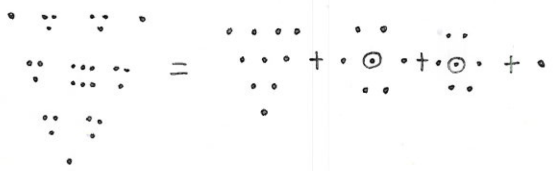
gives

$$\frac{1}{\sqrt{2}}(d\otimes u + u\otimes d)$$

(if we recall $s = \sqrt{2}\, e^3_- u$, $d = \sqrt{2}\, e^1_- u$)



thus $3\otimes 3\otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$



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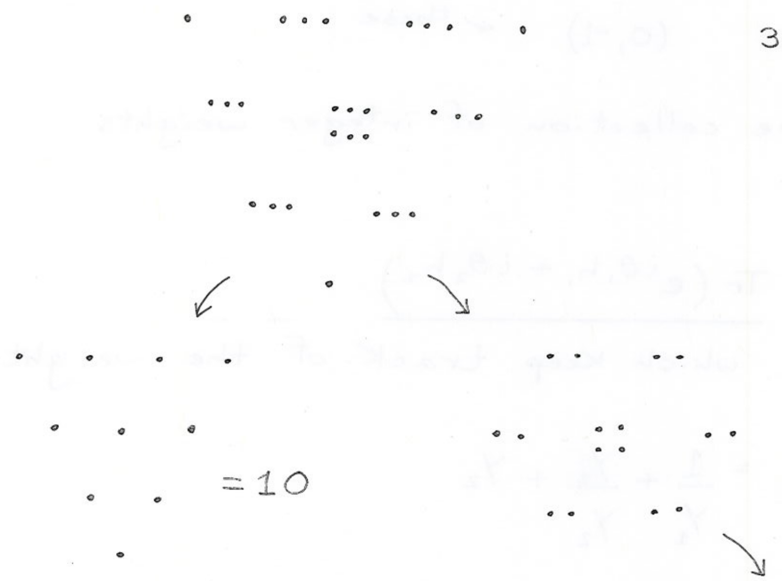
- For SU(2) highest weight n, weights m $-\frac{n}{2} \leq m \leq \frac{n}{2}$

- For SU(3) " " $(m,n)$

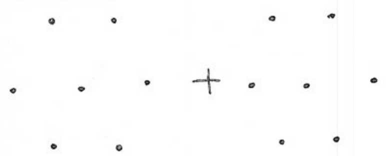


$= 10$

$3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$

↑ 3rd rank symmetric rep of SU(3)

(10) $u \otimes u \otimes u$ , $d \otimes d \otimes d$ , $s \otimes s \otimes s$



= 3 points of multiplicity 1

$\frac{1}{\sqrt{3}}(u \otimes u \otimes s + u \otimes s \otimes u + s \otimes u \otimes u)$

↳ then same thing for uud, sdd, uss, ssd, udd, suu = 6 points with multiplicity 3

at origin: $(0,0)$ $\frac{1}{\sqrt{6}} \frac{(u \otimes d \otimes s + d \otimes u \otimes s + \text{perm.})}{6}$

= 10 points = 3rd rank totally sym rep of SU(3)

[& because exchanging for instance u and s = the same! i.e. sym]

- (8)d representation $3 \otimes 3 \otimes 3$

$$\frac{1}{\sqrt{6}}(d \otimes u \otimes u + u \otimes d \otimes u - 2u \otimes u \otimes d)$$

[orthogonal to the sym uud but with mixed sym]

Note: if we take $3 \otimes 3 \otimes 3$ as

$$(3 \otimes 3) \otimes 3 = (6_S \otimes \underline{3}_A) \otimes 3 = (10_S \oplus 8_{SA}) \oplus (8_{AS} \oplus 1_A)$$

↳ thus we have ↗

- $\frac{1}{2\sqrt{3}}(s \otimes d \otimes u + s \otimes u \otimes d + d \otimes s \otimes u + u \otimes s \otimes d - 2 d \otimes u \otimes s - 2 u \otimes d \otimes s)$

- <u>Characters of irreducible representations of SU(3)</u> recall that a position of a state with highest weight $(m,n)$ is $\left(\frac{m}{2}, \frac{1}{2\sqrt{3}}(m+2m)\right)$

Set a point with weights $(a,b)$ to be at $\left(\frac{a}{2}, \frac{1}{2\sqrt{3}}(a+2b)\right)$

Example

$u\left(\frac{1}{2}, \frac{1}{2\sqrt{3}}\right)$ $d\left(-\frac{1}{2}, \frac{1}{2\sqrt{3}}\right)$ $s\left(0, \frac{-1}{\sqrt{3}}\right)$ ← these points become

$(1,0)$ $(-1,1)$ $(0,-1)$ ← these

The 3d representation is the collection of integer weights $(1,0)$ $(-1,1)$ $(0,-1)$

⟶ character remember is $\underline{\mathrm{Tr}\left(e^{i\theta_1 h_1 + i\theta_2 h_2}\right)}$

Introduce 2 variables $Y_1, Y_2$ which keep track of the weights

$$\chi_3 = Y_1 + \frac{Y_2}{Y_1} + \frac{1}{Y_2}, \quad \Rightarrow \chi_{\underline{3}} = \frac{1}{Y_1} + \frac{Y_1}{Y_2} + Y_2$$