- Continuous symmetries

Constitutes the Poincaré invariance → rotations, boosts, translations

Scale invariance [↗]

Conformal invariance [↗]

Gauge symmetries, Standard Model: $SU(3) \times SU(2) \times U(1)$ [↗]

Global symmetries, $SU(N_f)$, Baryon symmetry, lepton symmetry,... [↗]

(N_f: Flavors)

- Discrete Symmetries

C, P, T

Group Theory

- Lie group, continuous
- Non-abelian
- Lie-algebras
- representation theory

- Group G is a set of elements $g \in G$ with a binary composition law with 4 properties:

1. Closure $g, g' \in G$ $\quad g \cdot g' \in G$
2. Associativity $g, g', g'' \in G$ then $(g \cdot g') \cdot g'' = g \cdot (g' \cdot g'')$
3. Identity $I \in G$ s.t. $I \cdot g = g \cdot I = g$
4. Inverse $g \in G$, $\exists g^{-1}$, s.t. $g^{-1} \cdot g = g \cdot g^{-1} = I$

Examples

rotations in various dimensions

integers with addition

$GL(n, \mathbb{R})$ group of general linear $n \times n$ matrices with real entries

$GL(n, \mathbb{C})$

boosts and rotations

translations

Scale transformations

⋮

• Consider a vector space in $\mathbb{R}^n$, $V = (x_1, \dots, x_n)^T$

length $\sum_{i=1}^{n} x_i^2 = L^2$ $\qquad L^2 = V^T V$

Consider the set of matrices M that preserve the length [↗]

$V' = MV$ where $M \in GL(n, \mathbb{R})$

$(V')^T V' = V^T M^T M V = V^T V$ for any V

$\Rightarrow M^T M = \mathbb{I}_n \in GL(n, \mathbb{R})$

By definition, M is an orthogonal matrix $M \in O(n)$

$n=2 \qquad V = (x, y)^T \qquad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$M^T M = \mathbb{1} \qquad \det M^T M = \mathbb{1} = \det M^T \det M = (\det M)^2$ [↗]

$\det M = \pm 1$

• divide into two cases, one for n odd
n even

<u>$n=2$</u>

$$M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \Rightarrow \det M = 1$$

Another discrete parameter $M = JM'$

define $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

• definition the set of all matrices $M \in O(n)$ s.t. $\det M = 1$
is called $SO(n)$

$\theta \in [0, 2\pi]$, the set of all possible elements of $SO(2)$ is parameterized by a circle

Group manifold: the set of all values that the parameters of the group can admit

group manifold of $SO(2)$ is $S^1 \xrightarrow{}$ the circle

Aside: more generally S^n is a rotation for the n-dimensional sphere, the set of all x_i $i = 1, \dots, n+1$ s.t.

$$\sum_{i=1}^{n+1} x_i^2 = r^2$$

for r the radius of the n-sphere

• $n=3$, $O(3)$ - the set of 3×3 orthogonal matrices

$M \in O(3)$ $\quad M^T M = I_3$

Compute the number of continuous parameters

M has n^2 real entries and is subject to $\frac{n(n+1)}{2}$ equations

leaving $\binom{n}{2}$ parameters

off diagonal entries is $\binom{n}{2}$

add n diagonal entries

For $n=3$ we have 3 parameters, specialising to $SO(3)$

Example:

Euler angles

$$\begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \to [\alpha]$$

Problem: Not unique

$$\to [\alpha] \begin{pmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix} \cdot \to [\beta]$$

$$\to [\beta] \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_3 & -\sin\theta_3 \\ 0 & \sin\theta_3 & \cos\theta_3 \end{pmatrix} = M$$

5. Observables $A = A^\dagger$ Lecture 2 FQM 12.10.2011

[copied to correct location]

$\hat{x}, \hat{p}$ etc. $[\hat{x}, \hat{p}] = i\hbar$

(quantization $\{x, p\} = 1 \longrightarrow [\hat{x}, \hat{p}] = i\hbar$)

6. Measurement $\psi \xrightarrow[A]{\text{measurement}}$ eigenstate of A

"collapse of the wavefunction"

7. Composite systems $\psi(\underline{r}_1, \underline{r}_2) = \psi_A(\underline{r}_1)\, \psi_B(\underline{r}_2)$ if uncorrelated

$SO(n)$, $O(n)$

$\downarrow$ special for det = 1 ; det ± 1

# of parameters in $SO(n)$

$$\binom{n}{2} = \frac{n(n-1)}{2}, \quad \text{called the dimension of } SO(n)$$

Look at infinitesimal transformations

$$\psi(x), \quad \psi(x+a) = \psi(x) + a\frac{\partial \psi}{\partial x} + \frac{a^2}{2}\frac{\partial^2 \psi}{\partial x^2} + \dots + \frac{a^n}{n!}\frac{\partial^n \psi}{\partial x^n} + \dots$$

$$a\frac{\partial}{\partial x} \qquad = \left(1 + a\frac{\partial}{\partial x} + \frac{a^2}{2}\frac{\partial^2}{\partial x^2} + \dots\right)\psi$$

$$= e^{a\frac{\partial}{\partial x}}\psi$$

$$= e^{\frac{iap}{\hbar}}\psi(x)$$

$p = -i\hbar\frac{\partial}{\partial x}$, p is the generator of space translations

Conservation law: translation invariance $\longrightarrow$ conserved current, conserved charge, p

This conserved charge generates symmetry we started with

time translations, H, $\quad e^{\frac{iHt_0}{\hbar}}\psi(x,t) = \psi(x, t+t_0)$

rotations, generators of rotations: angular momenta, $\vec{L}$ [3 vectors]

$$SO(2), \quad \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = M \in SO(2)$$

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{X}\theta + \mathcal{O}(\theta^2)$$

X is the generator of infinitesimal rotations in $SO(2)$

$$e^{\theta X} = \sum_{n=0}^{\infty}\frac{(\theta X)^n}{n!} = \sum_{n=0}^{\infty}\frac{(\theta X)^{2n}}{(2n)!} + \sum_{n=0}^{\infty}\frac{(\theta X)^{2n+1}}{(2n+1)!} = [\alpha]$$

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad X^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

[Not insist X hermitian]
[not need i, not insist physical]

$$[\alpha] = I \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!} + X \sum \frac{(-1)^n \theta^{2n+1}}{(2n+1)!} =$$

$$\psi(x+a) = e^{a\frac{\partial}{\partial x}} \psi(x)$$

$$I \cos\theta + X \sin\theta$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Look at $SO(3)$,

$$L_x = Y \frac{\partial}{\partial Z} - Z \frac{\partial}{\partial Y}$$

$$L_i = \varepsilon_{ijk} X_j \frac{\partial}{\partial X_k}$$

[anti-Hermitian: exercise to show don't need i]

$$[L_x, L_y] = L_z$$

$$\begin{pmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \theta_3 \underbrace{\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{X_3} + \mathcal{O}(\theta_3^2)$$

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

[Xs generate infinitesimal rot$^{\underline{n}}$s but not group elements]

Def. a $n \times n$ matrix that represents a group element is called an n dimensional representation of the group.

$$e^{\theta_1 X_1 + \theta_2 X_2 + \theta_3 X_3}$$

$e^{A+B} \neq e^A e^B$, Baker Hausdorff Formula [how to exponentiate matrices]

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A,[A,B]]+\frac{1}{12}[[A,B],B]+\dots}$$

$[X_1, X_2] = X_3$

$[X_2, X_3] = X_1$

$[X_3, X_1] = X_2$

$[X_i, X_j] = \varepsilon_{ijk} X_k$

3 X's that form an algebra, Lie algebra

3 dimensional representation of this algebra

Lie Algebras? A Lie Algebra is a set of elements $\mathcal{A}$ which form a vector space, with a binary operation $[\underset{\mathcal{A}}{\underset{\Uparrow}{\cdot}}, \underset{\mathcal{A}}{\underset{\Uparrow}{\cdot}}] \in \mathcal{A}$

(i) Closure: $A, B \in \mathcal{A}, \quad [A, B] \in \mathcal{A}$

(ii) Bilinearity: $[\alpha A + \beta B, C] = \alpha [A, C] + \beta [B, C]$

$$[C, \alpha A + \beta B] = \alpha [C, A] + \beta [C, B], \quad \forall A, B, C \in \mathcal{A}$$
$$\forall \alpha, \beta \in \mathbb{C}$$

(iii) $[A, B] = -[B, A] \quad \forall A, B \in \mathcal{A}$

(iv) Jacobi identity $[[A, B], C] + [[B, C], A] + [[C, A], B] = 0$

$$\forall A, B, C \in \mathcal{A}$$

Set of all Lie Algebras?

How many finite dimensional Lie Algebras are there?

reps

[Ex: 2×2 matrices to satisfy Lie Algebra - can they be found? 4×4?]

Recap

$SO(n)$, $MM^T = \mathbb{1}_n$ $\det M = 1$

$M \in SO(n)$ preserves the length of an n-dimensional vector

$\mathbb{R} \ni X_i$ $i = 1, \dots, n$ $\sum_{i=1}^{n} X_i^2$ $X_i' = M_{ij} X_j$ $\sum_{i=1}^{n} X_i^2 = \sum_{i=1}^{n} X_i'^2$

Algebra of $SO(3)$ $[X_i X_j] = \varepsilon_{ijk} X_k$ X_i are 3×3 matrices

For unitary groups, take a complex n-dimensional vector

$$V = \begin{pmatrix} V_1 \\ \vdots \\ V_n \end{pmatrix}$$

$V_i \in \mathbb{C}$, a length $V^\dagger V$, $V' = UV$ $U \in GL(n, \mathbb{C})$

$(V')^\dagger (V') = V^\dagger U^\dagger U V = V^\dagger V$, $U^\dagger U = \mathbb{1}_n$

$U(n)$ is the set of all matrices in $GL(n, \mathbb{C})$ s.t. $UU^\dagger = \mathbb{1}_n$

$\det(UU^\dagger) = 1$ $\det(UU^\dagger) = \det(U)\underbrace{\det(U^\dagger)}_{=(\det U)^*} = |\det U|^2$

So $\det U = e^{i\theta}$ for some θ, a phase

for $U(1)$, the matrix is $|X|$ can be written as $e^{i\theta}$, $\theta \in [0, 2\pi]$

group manifold is a circle S^1

$U(1) \simeq SO(2)$ [$SU(n)$ is the set of unitary matrices s.t. $\det U = 1$]

One can construct for any matrix in $U(n)$ a unique decomposition

$U = e^{i\theta} \tilde{U}$ $U \in U(n)$, $\tilde{U} \in SU(n)$

$\rightarrow U(n) = U(1) \times SU(n)$

[$SU(2)$ i.e. $SU(n)$ with $n=2$]

Take $n = 2$ 2×2 matrices $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $a, b, c, d \in \mathbb{C}$

[What conditions must this matrix satisfy to belong to $SU(2)$?]

$U^{-1} \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} = U^\dagger = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$ $\Rightarrow$ $c = -b^*$, $d = a^*$

$\Rightarrow U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$

$$UU^\dagger = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}\begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix} \qquad |a|^2+|b|^2=1$$

$a = Y_0 + iY_3$

$b = Y_2 - iY_1$

$$\Rightarrow (Y_0)^2+(Y_1)^2+(Y_2)^2+(Y_3)^2=1$$

group manifold of $SU(2)$ is S^3

$$U = \begin{pmatrix} Y_0+iY_3 & i(Y_1-iY_2) \\ -iY_1+Y_2 & Y_0-iY_3 \end{pmatrix} = Y_0 \underset{\text{[expansion]}}{\mathbb{1}_2} + iY_i\sigma_i$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_i\sigma_j = \delta_{ij}\mathbb{1}_2 + i\varepsilon_{ijk}\sigma_k$$

$$T_i = -\frac{i}{2}\sigma_i \qquad [T_i, T_j] = \varepsilon_{ijk}T_k$$

2 representations of $SU(2)$ 2 dimensional

3 "

→ Lie algebra of $SU(2)$ is the same as the Lie algebra of $SO(3)$

For any matrix $U \in SU(2)$

consider the 3×3 matrix, $O_{ij} = \frac{1}{3}\mathrm{Tr}(\sigma_i U \sigma_j U^{-1})$

- Tr of a generator in the algebra is 0

$$\boxed{\begin{aligned} &\det U = 1 \\ &U = \exp(\theta_i T_i) \\ &\Rightarrow \mathrm{Tr}\, T_i = 0 \end{aligned}}$$

* <u>Exercise</u>: show that $O_{ij} \in SO(3)$

- note that U and $-U$ give the same element in $SO(3)$ therefore we have a 2 to 1 mapping from $SU(2)$ to $SO(3)$
- recall the group manifold of $SU(2)$ is S^3

" $SO(3)$ " $S^3/\mathbb{Z}_2 = \mathbb{R}P^3$

- If Y_0, Y_1, Y_2, Y_3 s.t. $\sum_{i=0}^{3} Y_i^2 = 1$, is a point in S^3 or represents an element in $SU(2)$, then $Y_i, -Y_i$ represent a single point in $SO(3)$
- What is the set of all possible representations of $SU(2)$?

For an integer $n = 2j$, $n = 0, 1, 2, \dots$, n is non-negative
there exists a representation of dimension $n+1 = 2j+1$, j is the spin.

- A matrix of dimension $(n+1) \times (n+1)$
each such representation has a weight m $-j \leq m \leq j$

For an integer $n = 2j$, $n = 0, 1, 2, \ldots$, n is non-negative,
there exists a representation of dimension $n+1 = 2j+1$, i.e. the spin-
j matrix of dimension $(n+1) \times (n+1)$.
Each such representation has a weight m, $-j \leq m \leq j$.

Representation Theory

SU(2) $[L_i, L_j] = \varepsilon_{ijk} L_k$ $L_i = -\frac{i}{2}\sigma_i$ ← Pauli matrices

2d representation of this algebra

$J_3 = iL_3$ $J\pm = \frac{i}{\sqrt{2}}(L_1 \pm iL_2)$

$[J_3, J\pm] = \pm J_\pm$

$[J_+, J_-] = J_3$

J's act as a finite dimensional vector space

$J_3|\phi\rangle = m|\phi\rangle$

$J_3 J_\pm|\phi\rangle = ([J_3, J\pm] + J\pm J_3)|\phi\rangle$

$= (\pm J_\pm + mJ_\pm)|\phi\rangle$

$= (m \pm 1) J_\pm |0\rangle$

• Highest weight j s.t. for m=j $J^+|m=j\rangle = 0$ → changed notation from $|\phi\rangle$ to $J_3|m\rangle = n|m\rangle$

↓ requirement for finite dimensional

• There is a lowest weight state j` s.t. for $J^-|m=j`\rangle = 0$

The number of steps from the lowest weight j` to the highest weight j is a non-negative integer, call it n, j-j`=n

There is an operator which commutes with all 3 operators:

$J^2 = \frac{1}{2}(J_+J_- + J_-J_+) + J_3^2$ $[J^2, J\pm] = 0$

$[J^2, J_3] = 0$

$J^2 J^+|m\rangle = J^+ \underbrace{J^2|m\rangle}_{\alpha|m\rangle} = \alpha J^+|m\rangle$

$J^2|m\rangle = \alpha|m\rangle$ ⇒ α is the eigenvalue of J^2 for all m

$J^2(J^+|m\rangle) = \alpha J^+|m\rangle$

For: $J^2|m=j\rangle = (\underbrace{J_+J_- + J_-J_+}_{[J_+,J_-]+J_-J_+} + J_3^2)|m=j\rangle$

$[J_+,J_-] = J_3$, $\quad J_3^2 \to j^2$

but $J_+|m=j\rangle = 0$

$\Rightarrow J^2|m=j\rangle = ([J_+,J_-] + \cancel{J_-J_+}^{\,0})|m=j\rangle + j^2|m=j\rangle$

$\Rightarrow J^2|m=j\rangle = (j^2+j)|m=j\rangle \quad \Rightarrow \alpha = j^2+j$ second casimir operator of SU(2)

Similarly:

$J^2|m=j'\rangle = (\cancel{J_+J_-} + J_-J_+ + J_3^2)|m=j'\rangle = (j'^2 - j')|m=j'\rangle$

$J_-J_+ = [J_-,J_+] = -J_3$, $\quad J_3^2 \to j'^2$

$\alpha = j^2 + j = j'^2 - j'$ solutions: $j' = -j$ ①

② put $j = j' + n$

$\Rightarrow \quad j^2 + j = (j-n)^2 - (j-n)$

$= j^2 - 2jn + n^2 - j + n$

$\Rightarrow \quad 2j = n - 2jn + n^2$

$\Rightarrow \quad 2j(n+1) = n(n+1)$

$\Rightarrow \quad j = n/2 \qquad n > 0$

- Any representation of SU(2) is characterized by 2 quantum numbers
  1) spin j (can replace by n - highest weight - an invariant of this representation, constant over all states)
  2) m, eigenvalue of J_3, $-j \leq m \leq j$
- There are infinitely many representations for SU(2) characterised by a highest weight n, $n = 0, 1, 2$
  - $\to$ the dimension of each representation is $n+1 = 2j+1$
- There are $n+1$ weights which are characterised by m where $-j \leq m \leq j$

• Recall $SO(3)$ has the same Lie algebra

All reps of $SO(3)$ are in this set but a rotation operator around the L_3 looks like

$$e^{iJ_3\theta}|m\rangle = e^{im\theta}|m\rangle$$

θ is an angle taking values $[0, 2\pi]$

$\Rightarrow e^{im2\pi} = 1 \quad \therefore m$ is an integer

$\therefore j$ is an integer

$\Rightarrow n$ must be even

$\Rightarrow$ For $SO(3)$ n is even

$SU(2)$: the highest weight n is any non-negative integer

$SO(3)$: " " " " " " even integer

Normalisations of States

- the $SU(2)$ lattice, $n=0$, $j=0$, $m=0$

$n=1$, $j=\frac{1}{2}$, $m=\pm\frac{1}{2}$

$n=2$, $j=1$, $m=-1, 0, 1$

$n=3$, $j=\frac{3}{2}$, $m=\frac{-3}{2}\dots\frac{3}{2}$

$\Rightarrow$ mapping of 2d lattice to all weights of $SU(2)$ (all points in the positive quadrant)

irreducible representations of SU(2)

highest weight $n = 2j$ $\quad n = 0, 1, 2, 3, \ldots$

of dimension $n+1$

$(n+1)$ dimensional vector space $|j, m\rangle$

$-j \leq m \leq j$

$J_3, J_\pm$

Consider the matrix $M_{m,m'} = \langle j, m | J_3 | j', m' \rangle = \mathrm{diag}(j, j-1, j-2, \ldots, -j)$

$\mathrm{Tr}(M)$ $\quad M \quad J_\pm$

$\mathrm{Tr}\ e^{iJ_3\theta} = e^{ij\theta} + e^{i(j-1)\theta} + e^{i(j-2)\theta} + \ldots + e^{i(-j)\theta}$

def if M is a d dimensional representation of a group element then $\mathrm{Tr}(M)$ is called (the) character(s).

Set $X = e^{\frac{i\theta}{2}}$

$$\chi_n(X) = \mathrm{Tr}\ e^{iJ_3\theta} = \sum_{m=-\frac{n}{2}}^{\frac{n}{2}} e^{im\theta} = \sum_{m=-\frac{n}{2}}^{\frac{n}{2}} X^{2m}$$

$n = 2j$

$\chi_n(X)$ [dummy parameter which counts the weights]

(the character of the SU(2) rep of highest weight n)

Ex show that the character is an invariant of the representation

Can replace J_3 by J_+ and still get the same character

Examples for $n=0$ $\quad \chi_0(X) = 1, \chi_1(X) = X + \frac{1}{X}, \chi_2(X) = X^2 + 1 + \frac{1}{X^2}$

$$\chi_3(X) = X^3 + X + \frac{1}{X} + \frac{1}{X^3}$$

Tensor product decomposition, [is realised by] multiplication of characters

$$\chi_1(X)\chi_1(X) = \left(X + \frac{1}{X}\right)^2 = X^2 + 2 + \frac{1}{X^2} = \chi_2(X) + \chi_0(X)$$

[$= \chi_2(X) + 1$]

$\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0 \rightarrow$ [working with characters of the group to obtain the same result]

$$\chi_1(X)\chi_2(X) = \left(X + \frac{1}{X}\right)\left(X^2 + 1 + \frac{1}{X^2}\right) = \underset{i}{X^3} + \underset{ii}{X} + \underset{iii}{\frac{1}{X}} + X + \frac{1}{X} + \underset{iv}{\frac{1}{X^3}}$$

$$= \chi_3(X) + \chi_1(X)$$

[6 weights here and here]

$\frac{1}{2} \otimes 1 = \frac{3}{2} \oplus \frac{1}{2}$

[from (i), (ii), (iii), (iv)]

$1 \otimes 1 = 2 \oplus 1 \oplus 0$

[In general:] $\quad j \otimes j' = \bigoplus_{j''=|j'-j|}^{j+j'} j''$

[$\oplus$ = direct sum]

$|jm\rangle$

$$|\psi_m\rangle = (J_-)^m |\psi\rangle$$

[assuming]
$\langle\psi|\psi\rangle = 1$

$$\langle\psi_m|\psi_m\rangle = \frac{(2j)!\, m!}{2^m (2j-m)!} = N_m \qquad J_3^+ = J_3,\ J_\pm^+ = J_\mp$$

[normalization Factor]

$$\langle\psi_m|\psi_m\rangle = \langle\psi|(J_+)^m (J_-)^m|\psi\rangle$$

$$|j,m\rangle = \frac{1}{N_{j-m}}|\psi_{j-m}\rangle$$

Clebsch Jordan coefficients

$$J_-|j,m\rangle = \frac{1}{\sqrt{2}}\sqrt{(j+m)(j-m+1)}\,|j,m-1\rangle$$

$$\left(\tfrac{1}{2}+\tfrac{1}{2}\right)\left(\tfrac{1}{2}-\tfrac{1}{2}+1\right)$$

$$J_+|j,m-1\rangle = \frac{1}{\sqrt{2}}\sqrt{(j+m)(j-m+1)}\,|j,m\rangle$$

[or]

$$J_+|j,m\rangle = \frac{1}{\sqrt{2}}\sqrt{(j+m+1)(j-m)}\,|j,m+1\rangle$$

[equally valid]

$$J_+|\tfrac{1}{2},\tfrac{1}{2}\rangle = 0$$

$$J_+|\tfrac{1}{2},\tfrac{1}{2}\rangle = \frac{1}{\sqrt{2}}|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

$$J_-|\tfrac{1}{2},\tfrac{1}{2}\rangle = \frac{1}{\sqrt{2}}|\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$J_-|\tfrac{1}{2},-\tfrac{1}{2}\rangle = 0$$

[Can use this calculation to evaluate the tensor products]

[Think of]

$$J_- = J_-\otimes 1 \oplus 1\otimes J_-$$

[Can act with J_- on both sides J_- acts on all states]

$$\frac{1}{2}\otimes\frac{1}{2} = 1\oplus 0$$

[highest weights are additive]

$$|1,1\rangle = |\tfrac{1}{2},\tfrac{1}{2}\rangle\otimes|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

$$J_-|1,1\rangle = \left(J_-|\tfrac{1}{2},\tfrac{1}{2}\rangle\right)\otimes|\tfrac{1}{2},\tfrac{1}{2}\rangle + |\tfrac{1}{2},\tfrac{1}{2}\rangle\otimes\left(J_-|\tfrac{1}{2},\tfrac{1}{2}\rangle\right)$$

$$|1,0\rangle = \frac{1}{\sqrt{2}}|\tfrac{1}{2},-\tfrac{1}{2}\rangle\otimes|\tfrac{1}{2},\tfrac{1}{2}\rangle + \frac{1}{\sqrt{2}}|\tfrac{1}{2},\tfrac{1}{2}\rangle\otimes|\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

[RHS]

$$R_{d_1}\otimes R_{d_2}$$

[dim] $d_1\times d_1 \qquad d_2\times d_2$

$$\begin{array}{c} d_1+d_2 \\ \left(\begin{array}{c|c} d_1\times d_1 & d_1\times d_2 \\ \hline d_2\times d_1 & d_2\times d_2 \end{array}\right) \end{array} d_1+d_2$$

[Another way to think of tensor product by matrices]

$$|1,-1\rangle = |\tfrac{1}{2},-\tfrac{1}{2}\rangle\otimes|\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$J_-|1,0\rangle = |1,-1\rangle$$