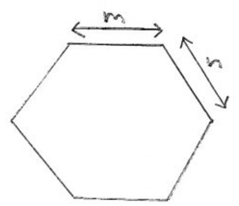
Characters of $SU(3)$ representations



representation of highest weight (m,n) of dimension
$$\frac{(m+1)(n+1)(m+n+2)}{2}$$

$$\chi_{(m,n)}(Y_1,Y_2) = \sum_{(a,b)} Y_1^a Y_2^b$$ sum over all possible weights of representation

(a,b) sum over all weights of the representation

$$\chi_{(m,n)}(Y_1,Y_2) = \mathrm{Tr}\, e^{i\theta_1 h_1 + i\theta_2 h_2}$$

Examples: For 3

(a,b): $(1,0)$, $(-1,1)$, $(0,-1)$

$$\Rightarrow \chi_{(1,0)}(Y_1,Y_2) = Y_1 + \frac{Y_2}{Y_1} + \frac{1}{Y_2}$$ need to be able to do this

Examples: For $\bar{3}$

(a,b): $(-1,0)$, $(1,-1)$, $(0,1)$

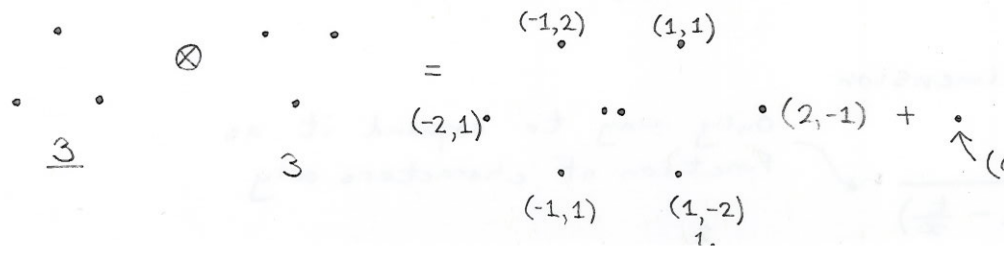
$$\chi_{(0,1)}(Y_1,Y_2) = Y_2 + \frac{Y_1}{Y_2} + \frac{1}{Y_1}$$

For each (m,n), we can • draw a lattice from this highest weight or • compute a character function

Example

$$\chi_{(1,0)}\chi_{(0,1)} = \left(Y_1 + \frac{Y_2}{Y_1} + \frac{1}{Y_2}\right)\left(Y_2 + \frac{Y_1}{Y_2} + \frac{1}{Y_1}\right)$$

$$= Y_1Y_2 + \frac{Y_1^2}{Y_2} + 1 + \frac{Y_2^2}{Y_1} + 1 + \frac{Y_2}{Y_1^2} + 1 + \frac{Y_1}{Y_2^2} + \frac{1}{Y_1Y_2}$$



$$3 \otimes \bar{3} = 8 \oplus 1$$

*NB: tensor products of representations correspond to products of characters

$$\chi_{(1,1)}(Y_1, Y_2) = Y_1 Y_2 + \frac{Y_1^2}{Y_2} + \frac{Y_2^2}{Y_1} + \frac{Y_1}{Y_2^2} + \frac{Y_2}{Y_1^2} + \frac{1}{Y_1 Y_2} + 2$$

$$\chi_{(0,0)}(Y_1, Y_2) = 1$$

$$\chi_{(1,0)}\chi_{(0,1)} = \chi_{(1,1)} + \chi_{(0,0)}$$

$$\Leftrightarrow \chi_{(1,1)}(Y_1,Y_2) + \chi_{(0,0)}(Y_1,Y_2) = \chi_{(0,1)}(Y_1,Y_2)\cdot\chi_{(1,0)}(Y_1,Y_2)$$

$$\chi_{(1,0)}\chi_{(1,0)} = \chi_{(2,0)} + \chi_{(0,1)}$$

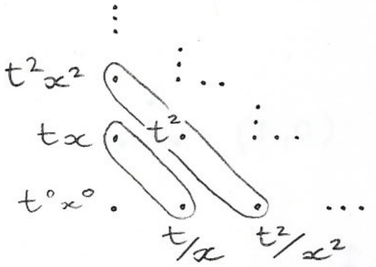
$$3 \otimes 3 = 6 \oplus \bar{3}$$

$$\left(Y_1 + \frac{Y_2}{Y_1} + \frac{1}{Y_2}\right)^2 = Y_1^2 + \frac{Y_2^2}{Y_1^2} + \frac{1}{Y_1^2} + 2Y_2 + 2\frac{Y_1}{Y_2} + 2\frac{1}{Y_1}$$

$$\underline{SU(2)}: \chi_n(x) = \sum_{m=-n/2}^{n/2} x^{2m}$$

$$f(t,x) = \sum_{n=0}^{\infty} \chi_n(x) t^n$$ generating function of all characters of irreps of $SU(2)$

How to add these?
Lattice:



with $t_1 = tx$, $t_2 = t/x$

$$\Rightarrow f(t,x) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} t_1^{n_1} t_2^{n_2} = \frac{1}{(1-t_1)(1-t_2)} = \frac{1}{(1-tx)(1-t/x)}$$

Taylor expansion and get the weights of SU(2)

$$\underbrace{\frac{1}{(1-tx)(1-\frac{t}{x})}}_{=f(t,x)} = \underbrace{1}_{\chi_0(x)} + \underbrace{\left(x + \frac{1}{x}\right)}_{\chi_1(x)} t + \underbrace{\left(x^2 + 1 + \frac{1}{x^2}\right)}_{\chi_2(x)} t^2 + \dots$$

For $x = 1$

$$f(t, x=1) = \sum_{n=0}^{\infty}(n+1)t^n = \frac{d}{dt}\sum_{n=0}^{\infty} t^n = \frac{d}{dt}\frac{1}{1-t} = \frac{1}{(1-t)^2}$$

$$\chi_n(x=1) = n+1$$ dimension

$$F(t,x) = \frac{1}{(1-tx)(1-\frac{t}{x})}$$ Only way to expand it as function of characters only

SU(3):

$$g(t_1,t_2,Y_1,Y_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \chi_{(m,n)}(Y_1,Y_2)\, t_1^n t_2^m$$

generating Function of all irreps of SU(3)

$$g(t_1,t_2,Y_1=1,Y_2=1) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} (m+1)(n+1)\frac{(m+n+2)}{2} t_1^n t_2^n$$ [see note ① below]

$$= \frac{d}{dt_1}\frac{d}{dt_2}\frac{d}{d\lambda}\sum_{m,n} t_1^m t_2^n \lambda^{m+n}\Big|_{\lambda=1} = \frac{1-t_1t_2}{(1-t_1)^3(1-t_2)^3}$$

[see note ② below]

3 dim (SU(3))

$$\underset{\displaystyle 3}{(1-t_1)^3}\;\underset{\displaystyle \underline{3}}{(1-t_2)^3}$$

$$\Rightarrow g(t_1,t_2,Y_1,Y_2) = \frac{1-t_1t_2}{(1-t_1Y_1)(1-t_1\frac{Y_2}{Y_1})(1-t_2Y_2)(1-t_1\frac{1}{Y_2})(1-t_2\frac{Y_1}{Y_2})(1-t_2\frac{1}{Y_1})}$$

Note

① Y Keeps track of the weights ($\sim \exp(\text{angle})$)

t Keeps track of the dim$^{\underline{n}}$? rep$^{\underline{n}}$?

For $Y_1=Y_2=1$, $\chi_{(m,n)}(1,1) = \dim = (m+1)(n+1)\frac{(m+n+2)}{2}$

② $\frac{d}{dt_1}\frac{d}{dt_2}\frac{d}{d\lambda} \Rightarrow$ equ$^{\underline{n}}$ for $g(\ldots)$

$$[\alpha] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \chi_{(m,n)}(Y_1, Y_2)\, t_1^m t_2^n = \frac{1 - t_1 t_2}{(1-t_1Y_1)(1-t_1\frac{Y_2}{Y_1})(1-t_1\frac{1}{Y_2})(1-t_2Y_2)(1-t_2\frac{Y_1}{Y_2})(1-t_2\frac{1}{Y_1})}$$

$[\alpha] = g(t_1, t_2, Y_1, Y_2)$

$$\chi_{(m,n)}(Y_1 = 1, Y_2 = 1) = \dim(m,n) = \frac{(m+1)(n+1)(m+n+2)}{2}$$

[Can use Taylor series expansion here]

g is the generating function of all characters of irreps of SU(3)

$$g(t_1, t_2, 1, 1) = \frac{1 - t_1 t_2}{(1-t_1)^3(1-t_2)^3} = 1 + 3t_1 + 3t_2 + 8t_1t_2 + 6t_1^2 + 6t_2^2$$

[dim of the irred rep as expected] $\left\{ \begin{array}{l} 3t_1 \text{ for fund rep} \\ 3t_2 \text{ for anti-fund rep} \end{array} \right\}$

$t_1 = 0$
[sum is only over n & $m=0$]

$$\sum_{n=0}^{\infty} \chi_{(0,n)}\, t_2^n = \frac{1}{(1-t_2Y_2)(1-t_2\frac{Y_1}{Y_2})(1-t_2\frac{1}{Y_1})} \quad [\beta]$$

[$Y \to$ Keeps track of integer weights - roughly integer of some angle]

[t's are parameters - variables that helps keep track of highest weight]

$$\sum_m : \sum \chi_{(0,n)} t_2^n + \sum \chi_{(1,n)} t_1 t_2^n + \ldots$$ [this is how term goes when $t_1 = 0$]

Analogy to statistical mechanics: $g \longleftrightarrow Z$ partition function

$t \longleftrightarrow e^{-\mu}$ μ is chemical potential, t - fugacity

[Can think of χ as part^{n} f^{n} also - but fixed q-#s] $\to [(0,n)]$

[$t=0$: restricted to ground states]

$g(0,t_2)$ $[\beta]$ is generating function for symmetric products of antifund rep

[i.e. we have all the sym products of all Δs in] [the lattice]

$$\underline{t_2 = 0}: \quad \sum_{m=0}^{\infty} \chi_{(m,0)}(Y_1, Y_2)\, t_1^m = \frac{1}{(1-t_1Y_1)(1-t_1\frac{Y_2}{Y_1})(1-t_1\frac{1}{Y_2})}$$

generating function of symmetric products of the fund rep.
[can measure the order of the sym product]

2 basic representations: fundamental (3) antifundamental ($\underline{3}$)

[For any value Y_1 & Y_2]

$$g\,(\underbrace{t_1 = 0}_{Y_1, Y_2})\; g(\underbrace{t_2 = 0}_{Y_1, Y_2}) \Rightarrow$$ [see next line for correct version]

[the generating function of $3 \otimes \overline{3}$

$$g(t_1, t_2, Y_1, Y_2) = (1 - t_1 t_2)\, g(t_1 = 0, Y_1, Y_2)\, g(t_2 = 0, Y_1, Y_2)$$

$$\chi_{(m,0)}\chi_{(0,n)} = \chi_{(m,n)} + \chi_{(m-1,n-1)}$$

$$\frac{(m+2)(m+1)}{2}\frac{(n+2)(n+1)}{2} = \frac{(m+1)(n+1)(m+n+2)}{2} + \frac{mn(m+n)}{2}$$

$$(m,2)\times(3,n) = \underbrace{(m+3,n+2)}_{\text{[highest weight]}}+\ldots$$

a tensor of SU(3) consists of two types of indices

$$T^{j_1,\cdots,j_n}_{i_1,\cdots,i_m} \qquad (m,n) \qquad T^{j_1 j_2 j_3 \cdots j_m}_{i_1\cdots i_n} = T^{j_2 j_1 j_3\cdots j_m}_{i_1 \cdots i_n}$$

lower : fundamental — all j's symmetrized — [exchange $j_1 \leftrightarrow j_2$ still same tensor]

upper : antifund — i's "

2 actions on indices : contraction δ_i^j invariant tensor of SU(3)

$V_i V'$ [form tensor product - $\bar{3}$ & 3 → invariant that contracts as a singlet]

$(V'^*)^j V_i \delta^i_j$

$$T^{a,b,j_1\cdots j_n}_{i_1\cdots i_m} = -T^{b,a,j_1\cdots j_n}_{i_1\cdots i_m} = \varepsilon^{abk} T'^{\,j_1\cdots j_n}_{k\, i_1 \cdots i_m}$$

ε_{abk} [can do same]

Using the ε symbol can turn 2 indices in antisymmetrization to a single index with symmetrization

[Simple example]

$W^k = V_i V'_j \varepsilon^{ijk}$ — [can use 2 ε's to do two antisym's]

$\bar{3}$ $\quad 3\times 3$

$V_i V'^j \delta^i_j$

$3\times \underline{3}$ $\quad 1$

[get singlet]

[δ & ε only 2 invariant tensors]

[contracts] [antisymmetrises]

$$\frac{1}{1-t} = e^{\log \frac{1}{1-t}} = e^{-\log(1-t)} = e^{\sum_{k=1}^{\infty} \frac{t^k}{k}} = e^{\sum_{k=1}^{\infty} \frac{f(t^k)}{k}}, \quad f(t)=t, \; f(t^k)=t^k$$

$$\frac{1}{(1-t_1)(1-t_2)} = e^{\sum_{k=1}^{\infty} \frac{t_1^k + t_2^k}{k}} \qquad f(t_1,t_2) = t_1 + t_2$$

def: for a function $f(t_1, t_2, \ldots)$ that vanishes at origin $f(0,0,\ldots,0)=0$ Plethystic exponential is $PE[f] = e^{\sum_{k=1}^{\infty} \frac{f(t_1^k, t_2^k, \ldots)}{k}}$

$$\frac{1}{1-t} = PE[t]$$

$$\frac{1}{(1-t_1)(1-t_2)} = PE[t_1 + t_2]$$

Note

$$V_i W^j = V_i W^j - \frac{1}{3}\delta_i^j V_k W^k + \frac{1}{3}\delta_i^j V_k W^k$$

• What is the nth symmetric product of the fundamental rep of $SU(2)$?

• 2nd rank symmetric product of 2 of $SU(2)$ $\quad 2 \otimes 2 = 3_S \oplus 1_A$

- using highest notation $[1] \otimes [1] = [2] + [0]$

- a representation of highest weight n, denoted $[n]$ has dimension $n+1$

- introduce the notation $\mathrm{sym}^n[1] = [n]$

$T_{\alpha\beta\sigma}$ $\quad \alpha, \beta, \sigma = 1, 2$

For $T_{\alpha\beta\sigma\delta}$ here 5

$T_{111}, T_{122}, T_{112}, T_{222}$

$T_{\alpha\beta\gamma} = T_{\beta\alpha\gamma} = T_{\beta\gamma\alpha} = \ldots$

Plethystic Exponential

$$PE[f(t_1, t_2, \ldots, t_n)] = e^{\sum_{K=1}^{\infty} f(t_1^K, t_2^K, \ldots, t_n^K)/K}$$

$$f(t_1 = 0, t_2 = 0, \ldots, t_n = 0) = 0$$

Examples

$$f(t) = t \qquad PE[t] = e^{\sum_{K=1}^{\infty} \frac{t^K}{K}} = e^{-\log(1-t)} = \frac{1}{1-t}$$

$$PE[f_1 + f_2] = PE[f_1]\, PE[f_2]$$

$$f(t_1, t_2) = t_1 + t_2, \quad PE[t_1 + t_2] = \frac{1}{(1-t_1)(1-t_2)}$$

$$PE\left[\sum_n a_n t^n\right] = \prod_n (1 - t^n)^{-a_n}$$

$$\boxed{PE[at^n] = \frac{1}{(1-t^n)^a} \quad \downarrow \text{because} \quad = e^{\sum_{K=1}^{\infty} \frac{at^{Kn}}{K}} = e^{-a\log(1-t^n)}}$$

Already derived:

$$\sum_{n=0}^{\infty} \chi_n(x) t^n = \frac{1}{(1-tx)(1-\frac{t}{x})}$$

$$= PE\left[tx + \frac{t}{x}\right]$$

$$= PE[\chi_1(x) t]$$

$$PE[\chi_1(x)t] = \sum_{n=0}^{\infty} \chi_n(x) t^n = \sum_{n=0}^{\infty} \chi_{\text{sym}^n[1]} \, t^n$$

$*$: $\text{sym}^n[1] = [n]$

→ PE generates symmeterisation of irreps [of SU(3)]
t keeps track of the order of symmetrization

SU(3),
$$\frac{1}{(1-tY_1)(1-t\frac{Y_2}{Y_1})(1-t\frac{1}{Y_1})} = \sum_{n=0}^{\infty} \chi_{(n,0)}(Y_1 Y_2) t^n$$

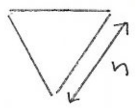
nth rank symmetric representation of SU(3)

$$\| $$

$$PE[\chi_{(1,0)}(Y_1 Y_2)t] = \sum_{n=0}^{\infty} \chi_{(n,0)} t^n$$

For $Y_1 = Y_2 = 0$,
$\chi_{(1,0)} = \dim(1,0)$
$= \dim$ Fund$^{\underline{m}}$
$= 3$

$$PE[3t] = \frac{1}{(1-t)^3} = \sum_{n=0}^{\infty} \frac{(n+2)(n+1)\, t^n}{2}$$

Geometrically
- recall the nth rank sym rep of SU(3) looks like a lattice triangle of size n 

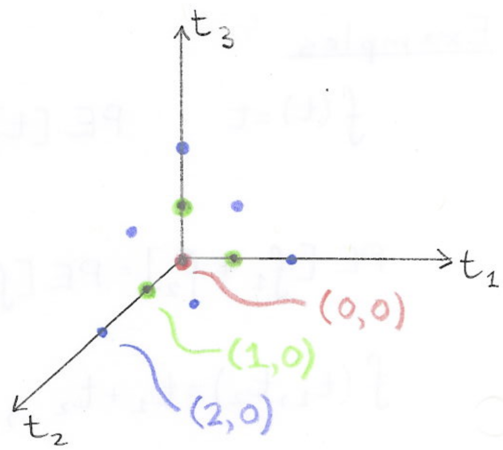
(0, 0)

(1, 0)

(2, 0)

(3, 0)

Put on a 3D lattice



$$\frac{1}{(1-t_1)(1-t_2)(1-t_3)}$$
$$= \sum_{\substack{n_1=0\\n_2=0\\n_3=0}}^{\infty} t_1^{n_1} t_2^{n_2} t_3^{n_3}$$

$t_1 = tY_1 \qquad t_2 = t\frac{Y_2}{Y_1} \qquad t_3 = \frac{t}{Y_2}$

[this gives us what we wanted]

Similarly
$$PE[\chi_{(0,1)} t] = \sum_{n=0}^{\infty} \chi_{(0,n)} t^n$$

The weights are the opposite ones to the fund$^{\underline{m}}$ and the sym product of Fund$^{\underline{m}}$

So $\chi_{(0,1)} = Y_2 + \frac{Y_1}{Y_2} + \frac{1}{Y_1}$ = opposite powers of $\chi_{(1,0)}$

$$PE[\chi_2(x)t] = \frac{1}{(1-tx^2)(1-t)(1-\frac{t}{x^2})} = \sum_{n=0}^{\infty} a_n t^n$$ $a_n = ?$ Exercise

For the adjoint of SU(2), i.e. for the triplet

$a_n = 2$ $\quad a_n = \mathrm{sym}^n([2])$

a_n is the symmetric product of SU(2)

- antisymmetrisation of [3] of SU(2), weights: 3, 1, −1, −3

number of weights $\binom{4}{2}$

$([3] \otimes [3])_A = [4] + ?$ $\quad \uparrow [n]$ [see note below]

how many elements? 6 because taking two weights over four i.e. $\binom{4}{2} = 6$

(Note: antisym is picking (3,1) ~ picking (1,3))

$$([3] \otimes [3])_A = [4] + [0]$$

$\underline{4}, \underline{2}, \underline{0}, 0, \underline{-2}, \underline{-4}$ Get by adding

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Note

What is left?

How many weights in [4]? 5

Where? 4, 2, 0, −2, 4

Left? 0 ⇒ singlet i.e. 1, i.e. [0]

(anti) Symmetric $\underset{\text{tensor}}{\wedge}$ products

Many particle physics: bosons, n particles, wave functions symmetric
fermions " antisymmetric

plethystic exponential: generates symmetric products

$$PE[t_1+t_2+t_3] = \underbrace{\frac{1}{(1-t_1)(1-t_2)(1-t_3)}}_{[\alpha]} = \sum_{\substack{n_1=0\\ n_2=0\\ n_3=0}} t_1^{n_1} t_2^{n_2} t_3^{n_3}$$

antisymmetrization?

$$= 1+(t_1+t_2+t_3)+(t_1^2+t_1t_2+t_2^2+t_3^2+t_1t_3+t_2t_3)+ \\ + \ldots$$

[PE →]
[takes weights - takes all possible combinations of weights]

[cf partition function for a bosonic operator → a SHO looks like part$^{\underline{n}}$ f$^{\underline{n}}$ for bos. op. → the [α] term]

$$PE[t_1+t_2+t_3] = (1+t_1)(1+t_2)(1+t_3)$$
$$= e^{\log(1+t_1)+\log(1+t_2)} = [\beta]$$

[like Pauli exclusion principle, weight can either be there or not]

$$(1+t) = e^{\log(1+t)} = e^{\sum_{K=1}^{\infty} \frac{t^K(-1)^{K+1}}{K}}$$

$$[\beta] = e^{\sum_{K=1}^{\infty} \frac{(-1)^{K+1}(t_1^K+t_2^K+t_3^K)}{K}}$$

Def$^{\underline{n}}$

For a function $f(t_1,\ldots,t_n)$ s.t. $f(t_1=0,\ldots,t_n=0)=0$,

$$PE_F[f] = e^{\sum_{K=1}^{\infty} \frac{(-1)^{K+1}}{K} f(t_1^K,\ldots,t_n^K)}$$

PE_F generates antisymmetrizations

Example: take fundamental rep of $SU(2)$, $\chi_1(x) = x+\frac{1}{x}$

$$PE_F\left[\left(x+\tfrac{1}{x}\right)t\right] = (1+tx)\left(1+\tfrac{t}{x}\right) = 1+t\left(x+\tfrac{1}{x}\right)+t^2 = \chi_0(x)+\chi_1(x)t \\ +\chi_0(x)t^2$$

$([1]\times[1])_A = 0$ $\qquad \Lambda^2[1]$

x Keeps track of the SU(2) weight

t Keeps track of the order of antisymmetrization

$$PE_F[f(x)t] = \sum_{n=0}^{\infty} a_n t^n \qquad a_n = \Lambda^n[f(x)] \;\; [\gamma]$$

Example:

[Polynomial order 3 in t - 4^{th} order & above identically vanish]

$$PE_F[\chi_{(1,0)}(Y_1,Y_2)t] = (1+tY_1)\left(1+t\frac{Y_2}{Y_1}\right)\left(1+t\frac{1}{Y_2}\right)$$

$$= 1 + t\chi_{(1,0)}(Y_1,Y_2) + \left(Y_2 + \frac{Y_1}{Y_2} + \frac{1}{Y_1}\right)t^2 + t^3$$

$$\chi_{1,0}(Y_1,Y_2) = Y_1 + \frac{Y_2}{Y_1} + \frac{1}{Y_2}$$

$$= \chi_0(Y_1,Y_2) + \chi_{1,0}(Y_1,Y_2)t + \underbrace{\chi_{(0,1)}(Y_1,Y_2)t^2}_{} + \chi_{0,0}(Y_1,Y_2)t^3$$

[From [γ] can read antisymmetrizat$^{\underline{n}}$]

$$\Lambda^2[1,0] = [0,1]$$

$$\Lambda^3[1,0] = [0,0]$$

$3 \otimes 3 = 6_S + \underline{3}_A$

$\underline{3} \otimes \underline{3}$ [another example]

Example:

Λ symmetric product of 3 of SU(3)

n-th

[i.e. the "6" in $3\otimes 3$ above]

$$PE[\chi_{(1,0)}(Y_1,Y_2)t] = \sum_{n=0}^{\infty} a_n t^n$$

[Highest weight - top LHS of lattice]

$$a_n = \chi_{\mathrm{sym}^n}[1,0] = [n,0]$$

$$PE = \frac{1}{(1-tY_1)\left(1-t\frac{Y_2}{Y_1}\right)\left(1-t\frac{1}{Y_2}\right)} = 1 + \chi_{(1,0)}t + \overbrace{\left(Y_1^2 + \frac{Y_2^2}{Y_1^2} + \frac{1}{Y_2^2} + Y_2 + \frac{Y_1}{Y_2} + \frac{1}{Y_1}\right)}^{\chi_{(2,0)}(Y_1,Y_2)}t^2 + \dots$$

$$\left(1 + tY_1 + t^2Y_1^2 + \dots\right)\left(1 + t\frac{Y_2}{Y_1} + t^2\frac{Y_2^2}{Y_1^2} + \dots\right)\left(1 + t\frac{1}{Y_2} + t^2\frac{1}{Y_2^2} + \dots\right)$$

$$3 \times 3 \times 3 = 10 + 8 + 8 + 1$$

$$[1,0]\times[1,0]\times[1,0] = [3,0]_S + 2[1,1] + [0,0]_A$$

$$\chi_{1,0}\,\chi_{0,1} = \left(Y_1 + \frac{Y_2}{Y_1} + \frac{1}{Y_2}\right)\left(Y_2 + \frac{Y_1}{Y_2} + \frac{1}{Y_1}\right)$$

$$= \underbrace{Y_1 Y_2 + \frac{Y_1^2}{Y_2} + 1} + \underbrace{\frac{Y_2^2}{Y_1} + 1 + \frac{Y_2}{Y_1^2}} + \underbrace{1 + \frac{Y_1}{Y_2^2} + \frac{1}{Y_1 Y_2}}$$

$$= \chi_{(1,1)} + \chi_{(0,0)}$$

$$\dim[n_1, n_2] = \frac{(n_1+1)(n_2+1)(n_1+n_2+2)}{2}$$

$$PE[F(Y_1 \dots Y_n)t] = e^{\sum_{k=1}^{\infty} \frac{F(Y_1^k, \dots, Y_n^k)t^k}{k}}$$

$$= 1 + F(Y_1, \dots, Y_n)t + \left[\tfrac{1}{2}F(Y_1^2, \dots, Y_n^2) + \tfrac{1}{2}F(Y_1, \dots, Y_n)^2\right]t^2 + \dots$$

$$\boxed{Sym^2[F(Y_1 \dots Y_n)] = \tfrac{1}{2}F(Y_1^2, \dots, Y_n^2) + \tfrac{1}{2}F(Y_1, \dots, Y_n)^2}$$

[Generic formula for symmetrising 2nd rank tensor]

Examples: (i) 2D rep of SU(2) noted $\mathrm{sym}^2[1] = [2]$

$$\left(\text{i.e. } \tfrac{1}{2} \otimes \tfrac{1}{2} = \underline{\underline{1_S}} \oplus 0_A\right)$$

[Character of the Fundamental rep:]

$$\mathrm{Sym}^2[\chi_n(x)] = \mathrm{Sym}^2\left[x + \frac{1}{x}\right]$$

$$= \frac{1}{2}\left(x^2 + \frac{1}{x^2}\right) + \frac{1}{2}\left(x + \frac{1}{x}\right)^2$$

$$= \frac{1}{2}x^2 + \frac{1}{2}\frac{1}{x^2} + \frac{1}{2}\left(x^2 + 2 + \frac{1}{x^2}\right)$$

$$= x^2 + 1 + \frac{1}{x^2}$$

$$\boxed{\underline{\text{NB}}\ \chi_1(x) = x + \frac{1}{x}}$$

$$= \chi_2(x)$$

$$(\text{i.e. } 3 \otimes 3 = \underline{\underline{6_S}} \oplus \underline{3}_A)$$

[Let's compute the symmetric tensor product of the fundamental rep of SU(3)]

(ii) $$\mathrm{Sym}^2[\chi_{(1,0)}(Y_1, Y_2)] = \mathrm{Sym}^2\left[Y_1 + \frac{Y_2}{Y_1} + \frac{1}{Y_2}\right]$$

[degree of symmetrization]

$$= \frac{1}{2}\left(Y_1^2 + \frac{Y_2^2}{Y_1^2} + \frac{1}{Y_2^2}\right) + \frac{1}{2}\left(Y_1^2 + \frac{Y_2}{Y_1} + \frac{1}{Y_2}\right)^2$$

$$= Y_1^2 + \frac{Y_2^2}{Y_1^2} + \frac{1}{Y_2^2} + Y_1 + \frac{Y_2}{Y_1} + \frac{1}{Y_2}$$

[Let's do a similar formula for antisym: almost same but we need to keep track of the signs]

$$= \chi_{(2,0)}(Y_1, Y_2)$$

$$\boxed{\underline{\text{NB}}\ \chi_{(1,0)}(Y_1, Y_2) = Y_1 + \frac{Y_2}{Y_1} + \frac{1}{Y_2}}$$

For antisymmetrization:

$$\Lambda^2[f(Y_1 \dots Y_m)] = \frac{1}{2} f(Y_1 \dots Y_m)^2 - \frac{1}{2} f(Y_1^2 \dots Y_m^2)$$

and indeed:

$$\Lambda^2[f(Y_1 \dots Y_m)] + \mathrm{sym}^2[f(Y_1 \dots Y_m)] = f(Y_1 \dots Y_m)^2$$

Look now at:

$$PE[f(Y_1 \dots Y_m)t] = \sum_{N=0}^{\infty} \mathrm{sym}^N[f(Y_1 \dots Y_m)]\, t^N$$

$$= \exp\left(\sum_{K=1}^{\infty} f(Y_1^K \dots Y_m^K)\frac{t^K}{K}\right)$$

$$= \prod_{K=1}^{\infty} \exp\left(f(Y_1^K \dots Y_m^K)\frac{t^K}{K}\right)$$

$$= \prod_{K=1}^{\infty} \sum_{m_K=0}^{\infty} \left(\frac{f(Y_1^K \dots Y_m^K)\, t^K}{K} \right) \frac{1}{m_K!}$$

[using [α]] (below) [many such products, indexed by K] [now looks like a power series in t]

$$\Rightarrow \mathrm{sym}^N [f(Y_1 \dots Y_m)]\, t^N = \sum_{\{m_K\}\,\text{s.t.}} \prod_{K=1}^{\infty} \frac{f(Y_1^K \dots Y_m^K)^{n_K}}{(K)^{m_K}\, m_K!}$$

[OK for 0 to ∞] | [Solution is:] $\sum K m_K = N$

[α] $\boxed{e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!}}$

$\sum K m_K = N$:?? Explanation:

N=1 : $K = 1, \quad m_1 = 1, \quad m_j = 0$ for $j > 1$: 1 solution

N=2 : $m_2 = 1, \quad m_j = 0 \quad j \geqslant 3 \quad (K=2)$ i.e. $2 = 2.1 = 2$ (I)

and $m_1 = 2, \quad m_j = 0 \quad j \geqslant 2 \quad (K=1)$ $\quad 2 = 1.2 = 1+1$ (II) } : 2 solution

[cf the counting of the degeneracy of harmonic oscillator → partition problem]

N=3 : $m_1 = 3, \quad m_j = 0 \quad j \geqslant 2$ i.e. $3 = 1.3 = 1+1+1$

$m_1 = 1, \quad m_2 = 1, \quad m_j = 0 \quad j \geqslant 3$ $\quad 3 = 1.1 + 2.1 = 1+2$

$m_3 = 1, \quad m_j = 0 \quad j \geqslant 2$ $\quad 3 = 3.1 = 3$ } : 3 solutions

[Note exchange of K and m_K]

Probability of partitions of N!

N=4 : 4, 3+1, 2+2, 2+1+1, 1+1+1+1 : 5 solutions

[How to count the partitions:
→ Young diagram
→ partition function for the harmonic oscillator]

Consider the Function (of partition of SHO)

$$f(t) = \prod_{m=1}^{\infty} \frac{1}{1-t^m} = \sum_{N=0}^{\infty} P_N t^m$$

P_N is the number of partitions of N in same pb!

Indeed $f(t) = \frac{1}{(1-t)(1-t^2)(1-t^3)\dots}$

$$= \underbrace{(1 + t + t^2 + t^3 \dots)}_{K=1} \underbrace{(1 + t^2 + t^4 \dots)}_{K=2} \underbrace{(1 + t^3 + \dots)}_{K=3}$$

$$= (1) + (t) + (t^2 + t^2) + (t t^2 + t^3 + t^3)$$

From $\frac{1}{1-t}$ | from $\frac{1}{1-t}$ | from $\frac{1}{1-t^2}$ | from $\frac{1}{1-t}$ and $\frac{1}{1-t^2}$ | From $\frac{1}{1-t}$ | From $\frac{1}{1-t^3}$

$$= 1 + t + 2t^2 + 3t^3$$

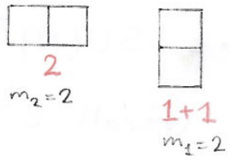
Young diagrams:

N boxes pushed to the top left corner without spaces
How many different cases?

$N=1$

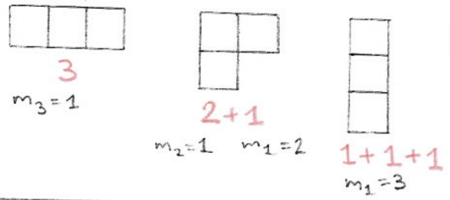
1 sol$^{\underline{n}}$

$N=2$



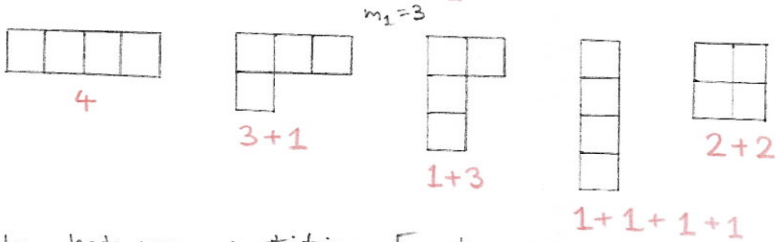
2 sol$^{\underline{n}}$

$N=3$



3 sol$^{\underline{n}}$

$N=4$



5 sol$^{\underline{n}}$

equivalent problem:
m_K = the number of rows with K boxes
and
$N = \sum K m_K$

$1 \leftrightarrow 1$ rel$\underline{n}$ between partition function and Young diagram
m_K is the number of rows with K boxes

$\{m_K\}$: For $N < \infty$, infinite number of them is 0.
finite number of them is >0.

$$\mathrm{Sym}^N[f(Y_1 \cdots Y_m)] = \sum_{\substack{\{m_K\} \\ \sum_K K m_K = N}} \prod_K \frac{f(Y_1^K \cdots Y_m^K)^{m_K}}{K^{m_K} \cdot m_K!}$$

For $N=2$:

$$\left.\begin{aligned} m_2 = 1 &\longrightarrow \frac{1}{2^1 . 1!} f(Y_1^2 \ldots Y_m^2)^1 \\ m_1 = 2 &\longrightarrow \frac{1}{1^2 . 2!} f(Y_1 \ldots Y_m)^2 \end{aligned}\right\} \frac{1}{2} f(Y^2) + \frac{1}{2} f(Y)^2 \qquad \underline{\text{OK}}$$

For $N=3$:

$$m_3 = 1 \longrightarrow \frac{1}{3^1 . 1!} f(Y_1^3 \ldots Y_m^3)^1$$

$$m_2 = 1,\ m_1 = 2 \longrightarrow \frac{1}{2^1 . 1!} f(Y_1^2 \ldots Y_m^2) \times \frac{1}{1^2 . 1!} f(Y_1^1 \ldots Y_m^1)^2$$

$$m_1 = 3 \longrightarrow \frac{1}{1^3 .\ 3!} f(Y_1^1 \dots Y_m^1)^3$$

$$\Rightarrow \mathrm{Sym}^3[f] = \frac{1}{3} f(Y^3) + \frac{f(Y) f(Y^2)}{2} + \frac{1}{6} f(Y)^3$$

So far, developed tools of sym & antisym products of characters, in particular over $SU(2)$ and $SU(3)$.

What about higher dimensional algebras, such as $SU(N)$? or $SO(N)$?

(recall $SO(3) \sim SU(2)$)

$SO(4)$ extended to $SO(3,1)$, $SO(2,2)$
