$$SO(4) = \{ M \text{ s.t. } \forall\, V' = MV \Rightarrow V'^T V' = V^T V = V^T M^T M V \}$$
$$= \{ M \text{ s.t. } M^T M = \mathbb{1},\ \det M = 1 \}$$

where $V^T = (X_1 \dots X_4)$, $V^T V = \sum_{i=1}^{4} X_i^2$

$SO(3,1) = \{$ s.t. $X_1^2 + X_2^2 + X_3^2 - X_4^2$ is preserved $\}$

i.e. one of the comp. is $-X_i^2$ instead of $+X_i^2$

We can define $\eta = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$, the $V^T \eta V = X_1^2 + X_2^2 + X_3^2 - X_4^2$

If we demand this to be constant, and $V' = MV$, then gives the condition

$$M^T \eta M = \eta$$

General set of matrices that preserve the length given by the matrix $\eta = (3,1)$

$$= \{ M \in GL(4,\mathbb{R}) \text{ st } M^T \eta M = \eta,\ \det M \overset{?}{=} 1 \}$$

cf Poincaré Sym

and $SO(2,2) = \{ M \in GL(4,\mathbb{R}) \text{ st } M^T \rho M = \rho \text{ where } \rho = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \det M \overset{?}{=} 1 \}$

Simplectic group $Sp(n)$?

Recall: $SO(n) = \{ M \in GL(n,\mathbb{R})$ which preserves a length, i.e. $M^T M = \mathbb{1},\ \det M = 1 \}$

$$SU(n) = \{ M \in GL(n,\mathbb{C}) \text{ st } M^+ M = \mathbb{1},\ \det M = 1 \}$$

and $$Sp(n) = \{ M \in GL(2n,\mathbb{R}) \text{ st } M^T J M = J \}$$

where $J = \begin{pmatrix} 0_m & \mathbb{1}_m \\ -\mathbb{1}_m & 0 \end{pmatrix}$

Sp(n) appears in Hamiltonian mechanics, in analytical mechanics, with Poisson Brackets.

Note that η is symmetric, but J is antisymmetric.

SU(4): Cartan subalgebra $U(1)^3$

analog: for SU(3) = cartan subalgebra is $U(1)^2$

for SU(2) = " " " $U(1)$

i.e. the rep are characterized by 2 and 1 non-negative integers

or more generally for SU(n), Cartan subalgebra is $U(1)^{n-1}$

and the rep is characterized by (n-1) non-negative integers

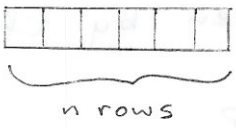
These integers are n giving dim = n+1

n, m — " — dim = $(n+1)(m+1)(n+m+2)/2$

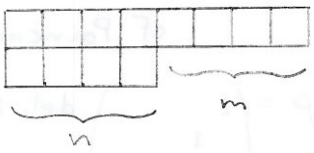
and n_1, n_2, n_3 — " — dim = ??

Non-negative integers lead to Young diagrams:

there is a one-to-one rel^n between integers and a diagram.

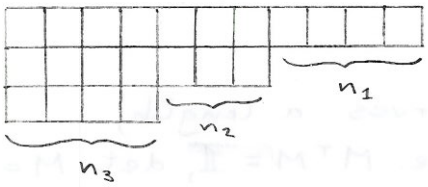


rep of SU(2)



rep of SU(3) where m = the diff between 1st and 2nd row

n = " " 2nd and 3rd "



rep of SU(4) ?

Example

- □ (1) i.e. SU(2) in the funda rep, i.e. n = 1

or SU(n) with integers $\underbrace{(1,0\ldots0)}_{n-1 \text{ integers}}$

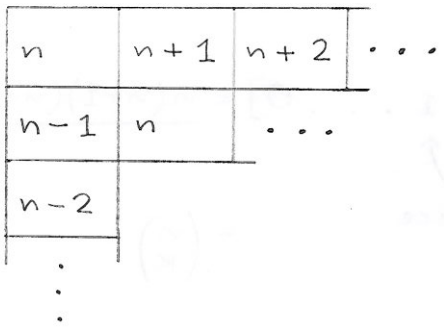
because { First row = one box
{ others = none

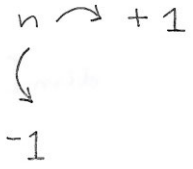
- $(2,0\ldots0)$
  $(0,1,0\ldots)$ } 2 options for 2 boxes

- $(3,0\ldots0)$
  $(1,1,0\ldots0)$
  $(0,0,1,0\ldots0)$

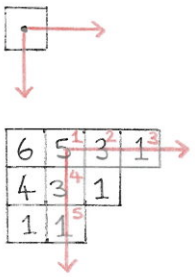
Formula for dim of irreps of SU(n) using Young diagrams?

For a given Young diagram for SU(n)





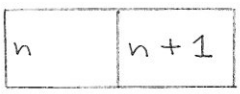
Hook lengths: count the number of boxes along the hook



e.g.

The ratio of these numbers is the dimension of the rep of SU(n) with this Young diagram.

Example



$\frac{n(n+1)}{1.2}$ = dim [2,0...0]: 2nd rank sym of SU(n)

e.g. n = 2, dim [2] = 3: dim of the 2nd rank sym of SU(2)

i.e. spin 1 rep of SU(2): dim = 3

n = 3, dim [2,0] = 6: dim of sym $3 \otimes 3$:

$3 \otimes 3 = \underbrace{6_S} \oplus \underline{3}_A$

n
n-1

$\dim [0,1,0\ldots 0] = \frac{n(n-1)}{1.2} = \binom{n}{2}$ = dim of 2nd rank antisym rep of SU(n)

2
1

e.g. $n=2$, $\dim [0] = 1$: singlet

$n=3$, $\dim [0,1] = 3$: anti-Funda of SU(3)

$n=4$, $\dim [0,1,0] = 6$

□ i.e. $[0,0,1]$ for $n=4$, $\dim [0,0,1] = 4$

□ because $\frac{n(n-1)(n-2)}{3\times 2\times 1} = 4$ ↑ For $n=4$

□

or more generally:

K { n | n-1 | ⋮ | n-K+1 }

$\dim [0,0\ldots \underset{\uparrow}{1} \ldots 0] = \frac{n(n-1)(n-2)\ldots(n-K+1)}{K!}$

K^{th} place

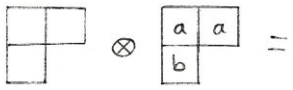
$= \binom{n}{K}$

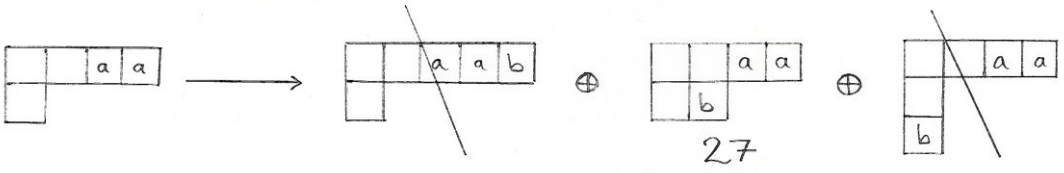
= dim K^{th} rank antisym rep of SU(n)

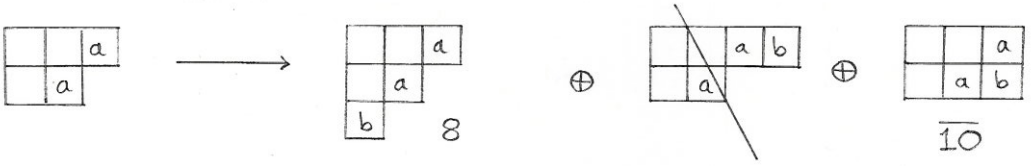
and $\dim [K,0\ldots 0] = \binom{n+K-1}{K}$

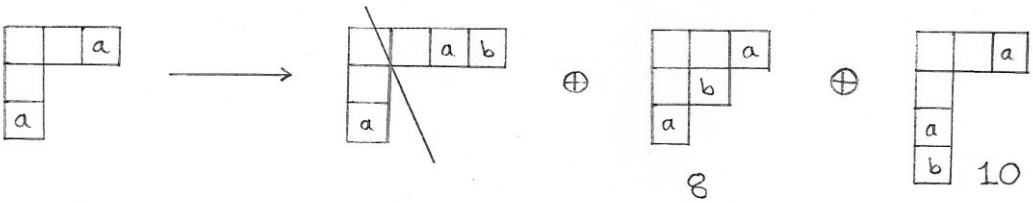
[The Following was already on the board when the lecture began → it gave rise to the exercise in the notes for this lecture]

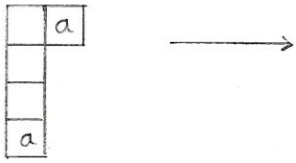
$8 \otimes 8$ $SU(3)$

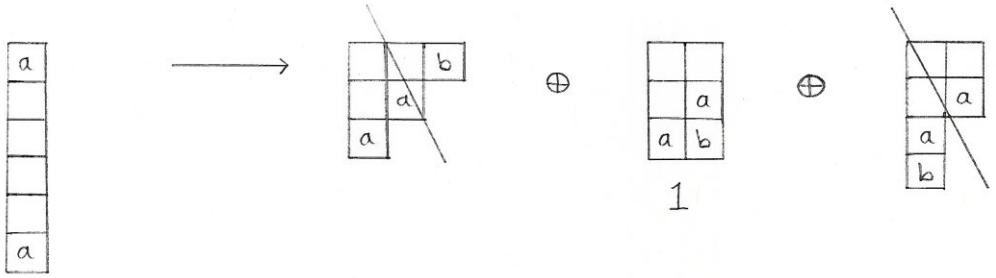


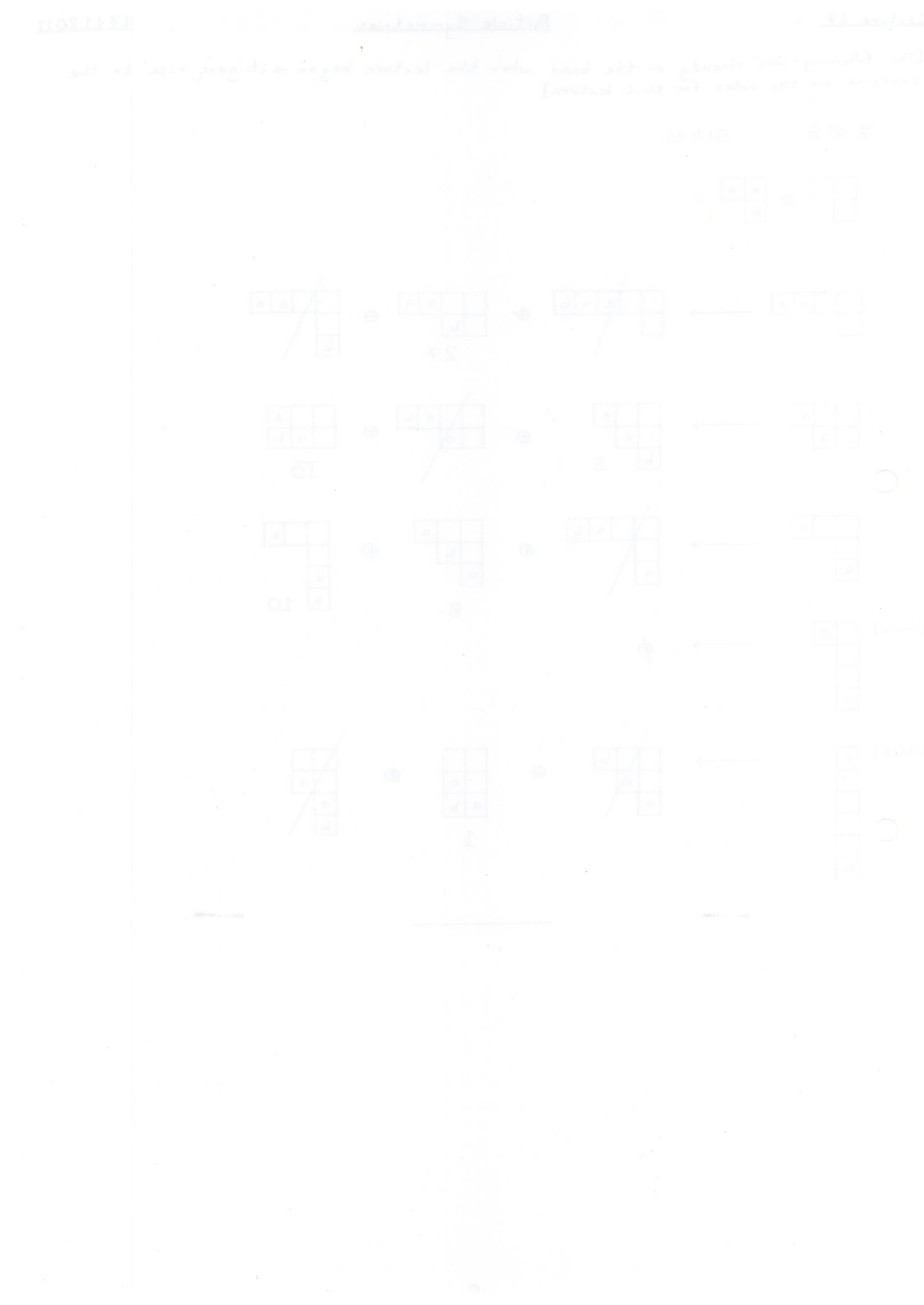






[check] 

[check] 



Exercise: $[1,1]^2 = [2,2] + [3,0] + [0,3] + 2[1,1] + [0,0]$ Show for SU(3)

How many terms in symmetrical product of 2 "8"s?

$\dim \mathrm{Sym}^2(8) = \frac{8\cdot 9}{2} = 36$

$\mathrm{Sym}^2[1,1] = [2,2] + [1,1] + [0,0]$

dim: 36 — 27 — 8 — 1

$\dim \Lambda^2(8) = \frac{8\cdot 7}{2} = 28$

$\Lambda^2[1,1] = [3,0] + [0,3] + [1,1]$

dim: 28 — 10 — 10 — 8

$8 \otimes 8 = 27 + 10 + \overline{10} + 8 + 8 + 1$

We have in general:

$[n_1,0] \times [0,n_2] = [n_1,n_2] + [n_1-1,0] \times [0,n_2-1]$

$[2,0][0,2] = [2,2] + [1,0] \times [0,1]$ for instance

$$PE[\chi_{[1,1]}\, t] = \sum_{n=0}^{\infty} a_n t^n \qquad a_n = 2$$

Mix Sym:

$$[1,0]^3 = \underset{\mathrm{Sym}^3[1,0]}{[3,0]^{10}} + 2[1,1]^{8} + \underset{\Lambda^3[1,0]}{[0,0]^{1}}$$

Last time: reps of SU(n)

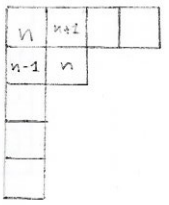
hook Formulas for dimensions

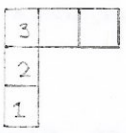
Characters of such representations: how to compute them?

[Only use Y.D.s for SU(n)]

[Y.D. = Young Diagram]

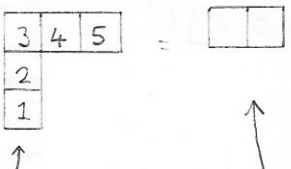
SU(4)





$[n_1,n_2]$

For SU(3)



$= \square\square$

dim: $\frac{3\cdot 4\cdot 5\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 2\cdot 1} = 6$ | dim: $\frac{3\cdot 4}{2\cdot 1} = 6$

i.e. 3 variables

[This is the Young diagram in this case] [Y.D]

SU(4) $\dim [1,0,0] = 4$, $\chi_{[1,0,0]}(Y_1,Y_2,Y_3) = Y_1 + \frac{Y_2}{Y_1} + \frac{Y_3}{Y_2} + \frac{1}{Y_3}$ $\square \equiv [1,0,0]$

SU(3) $\chi_{[1,0]}(Y_1,Y_2) = Y_1 + \frac{Y_2}{Y_1} + \frac{1}{Y_2}$

SU(2) $\chi_{[1]}(X) = X + \frac{1}{X}$

Note: [on selection of this form] ↳ [could have put] [instead]

$Y_1 = Z_1 + Z_2 + Z_3 + Z_4$

$[1,0,0]: Z_1 + Z_2 + Z_3 + Z_4$

$[0,1,0]: Z_1Z_2 + Z_1Z_3 + Z_1Z_4 + Z_2Z_3 + Z_2Z_4 + Z_3Z_4 = Y_2 \Rightarrow Z_2 = \frac{Y_2}{Y_1}$ [turns out same]

$$PE_F[\chi_{[1,0,0]}t] = e^{\sum_{K=1}^{\infty}\frac{(-1)^{K+1}(Y_1^K+\frac{Y_2^K}{Y_1^K}+\frac{Y_3^K}{Y_2^K}+\frac{1}{Y_3^K})t^K}{K}}$$

[Note: PE_F → fermionic]

[because antisym]

Dynkin labels

[Cartan sub-algebra]

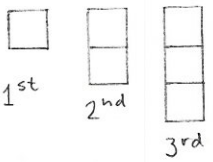
$$= (1+tY_1)^{[a]}(1+t\tfrac{Y_2}{Y_1})^{[b]}(1+t\tfrac{Y_3}{Y_2})^{[c]}(1+t\tfrac{1}{Y_3})^{[d]}$$

$[\chi_{[0,1,0]}:]$

$\left[\binom{4}{2}=6 \text{ \#terms}\right]$

$$= 1+\chi_{[1,0,0]}t+\chi_{[0,1,0]}t^2+\chi_{[0,0,1]}t^3+t^4$$

$$\binom{4}{2}=\frac{4!}{(4-2)!\,2!}=\frac{4\cdot3\cdot\cancel{2}}{\cancel{2}\cdot2}=6$$



[these are the K^{th} antisym product of [1,0,0]]

1st 2nd 3rd ...

[fundamental rep^n] [1st rank antisym] dim = 4

$$\chi_{[1,0,0]}(Y_1,Y_2,Y_3) = Y_1+\frac{Y_2}{Y_1}+\frac{Y_3}{Y_2}+\frac{1}{Y_3}$$

$$\chi_{[0,1,0]} = Y_2+\frac{Y_3Y_1}{Y_2}+\frac{Y_1}{Y_3}+\frac{Y_3}{Y_1}+\frac{Y_2}{Y_1Y_3}+\frac{1}{Y_2}$$

[2nd rank antisymmetric (product) tensor of SU(4)] dim = 6

$$\chi_{[0,0,1]} = Y_3+\frac{Y_2}{Y_3}+\frac{Y_1}{Y_2}+\frac{1}{Y_1}$$

[Anti-fund. rep^n] [3rd rank antisym] dim = 4

[abc] [abd] [acd] [bcd] [see abcd above]

Weights = −weights (Funda so directly Known

highest weights are indeed [1,0,0], [0,1,0], [0,0,1]

$[n_1,n_2,n_3]$ is contained in the tensor product

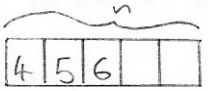
$$Sym^{n_1}[1,0,0]\times Sym^{n_2}[0,1,0]\times Sym^{n_3}[0,0,1]$$

[SU(n), SO(n), SP(n) → 3 types of groups → any antisym rep^n that is reducible?]
[Start by considering any sym. rep^n that is reducible.]

$$PE[\chi_{[1,0,0]}t] = \sum_{n=0}^{\infty} Sym^n[1,0,0]t^n = \sum_{n=0}^{\infty}[n,0,0]t^n$$

[Question: did we have to use the form of variables above? If choose alternatives, the form above still unfolds -see bottom p.1]

Notation: $\chi_{Sym^n[1,0,0]} := Sym^n[1,0,0]$

Young diagram $[n,0,0]$ = 

$$\dim = \frac{4\cdot5\cdots(n+3)}{n!} = \binom{n+3}{3} = \frac{(n+3)(n+2)(n+1)}{6}$$

$Y_1=Y_2=Y_3=1$: $PE[4t] = \frac{1}{(1-t^4)} = \sum_{n=0}^{\infty} a_n t^n$, [Taylor expansion]

($PE[4t] \leftarrow \chi_{[1,0,0]}$)

a_n is a polynomial of order 3 in n

$$= \sum_{n=0}^{\infty}\underbrace{\binom{n+3}{3}}_{\text{[dim of the sym product]}}t^n$$

enough to evaluate first 4 numbers

$$a_n = \alpha n^3 + \beta n^2 + \gamma n + \delta$$

[sym power same dim^n as sym product] [confirm]

dim = 4.

[polynomial determined by 4 coefficients]

Recall

- $SU(4)$ rep by $[n_1, n_2, n_3]$, $n_i \in \mathbb{N}$ [i.e. rep characterised by 3 #'s]
  is in tensor product of $Sym^{n_1}[1,0,0] \times Sym^{n_2}[0,1,0] \times Sym^{n_3}[0,0,1]$

(1) $$PE[[1,0,0]t] \underset{(1)}{=} \sum_{n=0}^{\infty} sym^n[1,0,0]t^n \underset{(2)}{=} \sum_{n=0}^{\infty} [n,0,0]t^n$$

$\chi_{[1,0,0]}(\gamma_1, \gamma_2, \gamma_3)$ $\qquad$ $\chi_{Sym^n[1,0,0]}$

via $Sym^n[1,0,0] \overset{(b)}{=} [n,0,0]$ $\overset{(a)}{}$

[i.e. (a) is reducible to (b)]
[can only do this for $SU(n)$]

and similarly

(2) $$PE[[0,0,1]t] \underset{(1)}{=} \sum_{n=0}^{\infty} Sym^n[0,0,1]t^n \underset{(2)}{=} \sum_{n=0}^{\infty} [0,0,n]t^n$$

id.

Note

(1) Combinatorial, so "easy"

(2) More complicated

then $PE[[0,1,0]t] = \sum_{n=0}^{\infty} Sym^n[0,1,0]t^n = ?$

More interesting: cannot claim $Sym^n[0,1,0] \overset{?}{=} [0,n,0]$ [See (*) below]

Example

$\dim Sym^2[0,1,0] = ?$

$\dim [0,1,0] = 6 \Rightarrow \dim Sym^2[0,1,0] = \frac{6 \cdot 7}{2} = 21$

Note: we have used

$$PE[nt] = \frac{1}{(1-t)^n} = \sum_{K=0}^{\infty} \binom{n+K-1}{K} t^K$$

For a rep of dim n, the K^{th} sym tensor power has dim $\binom{n+K-1}{K}$

and the K^{th} antisym tensor power has dim $\binom{n}{K}$

We know the beginning $= Sym^2[0,1,0] = [0,2,0] + \ldots$

because the highest weights are additive

(*) i.e. $Sym^n[0,1,0] \neq [0,n,0]$

End of it?

Young diagram: $[0,2,0]$ =

$$\dim[0,2,0] = \frac{4\times5\times3\times4}{1\times2\times2\times3} = 20$$

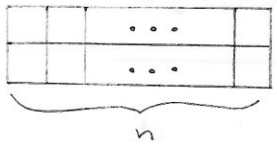
Thus not enough and find:

$$\mathrm{Sym}^2[0,1,0] = [0,2,0] + [0,0,0]$$

← { i.e. only one possible $[.,.,.]$ we can use to make up the 1 dimension

↖ dim = 1

For $[0,n,0]$, dim = ?

Young diag = $2\{$ 

$$\dim = \frac{(n+1)(n+2)^2(n+3)}{12}$$

using

<u>or in general</u>: $\dim[n_1,n_2,n_3] = ?$

recall: $SU(2)$, $\dim[n] = n+1$ (1st order) ; $SU(3)$, $\dim[n_1,n_2] = \frac{(n_1+1)(n_2+1)(n_1+n_2+2)}{2}$ (3rd order)

polynomial of deg 1 and deg 3

- for $SU(4)$, $\dim[n_1,n_2,n_3] = \frac{(n_1+1)(n_2+1)(n_3+1)(n_1+n_2+2)(n_2+n_3+2)(n_1+n_2+n_3+3)}{12}$ (6th order)

(deg = 6; For SU(5) deg = 10 ...) i.e. order is $\binom{n}{2}$

Let's compute:

$$\sum_{n=0}^{\infty}\dim[0,n,0]t^n = \sum_{n=0}^{\infty}\frac{(n+1)(n+2)^2(n+3)t^n}{12} = \frac{(1+t)}{(1-t)^5}$$

long computation (not checked!)

$$\text{i.e. } \dim[0,n,0] = \frac{(n+1)(n+2)^2(n+3)}{12}$$

via $= \binom{n+4}{4} + \binom{n+3}{4} \cdots$

Now try to restore Y dependence and replace dimensions with characters:

and $\sum_{n=0}^{\infty}\chi_{[0,n,0]}(Y_1,Y_2,Y_3)t^n = ?$ since $\sum\dim[0,n,0]t^n = \frac{1-t^2}{(1-t)^6}$

$= (1-t)^2\,\mathrm{PE}[\chi_{[0,n,0]}t]$

$= (1-t^2)\,\mathrm{PE}[6t]$

[should be [0,1,0]?] → dim Funda [1,0,0]

giving $$\boxed{PE[\chi_{[0,1,0]}\, t] = \frac{1}{1-t^2} \sum_{n=0}^{\infty} \chi_{[0,n,0]}\, t^n}$$

$$= PE[[0,1,0]t] = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} [0, n_1, 0]\, t^{n_1+2n_2}$$ by expanding $\frac{1}{1-t^2}$

and should be by definition $= \sum_{n=0}^{\infty} Sym^n[0,1,0]t^n$

Let $n = n_1 + 2n_2$, where $0 \le n_2 \le \left[\frac{n_1}{2}\right]$

↖ integer from n_1

giving $$= \sum_{n=0}^{\infty} \sum_{n_2=0}^{[n/2]} [0, n-2n_2, 0]\, t^n$$

$$\Rightarrow \boxed{Sym^n[0,1,0] = \sum_{n_2=0}^{[n/2]} [0, n-2n_2, 0]}$$

In the same way, $SU(2)$: $Sym^n[2] = ?$ (exercise)

Now: $Sym^3[0,1,0] = ?$

starts with $[0,3,0]$

dim $Sym^3[0,1,0] = \frac{6.7.8}{2.3} = 56$ $\left(= \binom{8}{3}\right)$ ← $\binom{n+K-1}{K}$ $[n=6, K=3]$

and dim $[0,3,0] = \frac{4.5.6.3.4.5}{2.3.4.2.3} = 50$

missing dim 6: dim $[0,1,0] = 6$ [this is only rep with dim 6]

$\Rightarrow Sym^3[0,1,0] = [0,3,0] + [0,1,0]$ coherent with the previous formula

Now consider

SU(n): the fundamental rep is $[\underbrace{1,0\ldots0}_{n-1}]$

$$PE_f[[1,0\ldots0]t] = \sum_{K=0}^{n} \wedge^K [1,0\ldots0]t^K = \sum_{K=0}^{n} [0,\ldots,\underset{\uparrow K^{th}\text{ position}}{1},\ldots,0]t^K$$

Character of the fundamental rep is:

$$\chi_{[1,0,\ldots,0]}(Y_1\ldots Y_{n-1}) = Y_1 + \frac{Y_2}{Y_1} + \frac{Y_3}{Y_2} + \ldots + \frac{Y_{n-1}}{Y_{n-2}} + \frac{1}{Y_{n-1}}$$

The basic irrep of SU(n) are the set of K^{th} antisym products of the funda rep of dim $\binom{n}{K}$ for $K = 1,\ldots,n-1$.

Any other rep with highest weights $[n_1\ldots n_{n-1}]$

appears in

$$\prod_{K=1}^{n-1} Sym^{n_K}[0,\ldots,\underset{\uparrow K^{th}}{1},\ldots,0]$$

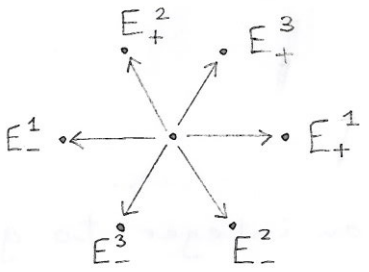
Finite dimensional Lie Algebras

Classification Cartan $\underbrace{A, B, C, D}_{\text{classical gps}}, E, F, G$

[To understand classn we need:] $\equiv SU(n), SO(n), Sp(n)$

Root system:

Adjoint rep of $SU(3)$



also called the roots

$\equiv$ weights of adjoint rep of $SU(3)$

= annihilation & creation

def: roots = weights of the adjoint rep.

if there is a E_+^i, there is E_-^i: raising op & lowering op

as vectors, $E_+^1 + E_+^2 = E_+^3$ e.g. $(1,0) + \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

Recall $E_+^1 (0,0) = (1,0)$

$E_+^2 (0,0) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$E_+^3 (0,0) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

simple roots = basis of the root system

Basis of roots such that all roots can be expressed as $\mathbb{Z}$ (integer) linear combinations of the basis = simple roots

$\underbrace{\{E_+^1, E_+^2\}}_{\text{[pick for basis]}}$, with $E_+^3 = E_+^1 + E_+^2$, $E_-^i = -E_+^i$ $\left\{\begin{array}{l} \text{i.e. } E_-^1 = -E_+^1 \\ E_-^2 = -E_+^2 \\ \Rightarrow E_-^3 = -E_+^1 - E_+^2 \end{array}\right\}$

dim 2 = dim of the lattice = rank of the Lie algebra

A root sys of dim r, where r is the rank of the Lie algebra, called Φ and $\alpha \in \Phi$, α a root, Φ the root system, i.e. a vector subspace with a scalar product,

s.t. (1) roots span Φ

(2) $\alpha \in \Phi \Rightarrow -\alpha \in \Phi$

(3) reflections: the root system is invariant under reflections i.e. for $\alpha, \beta \in \Phi$:

$$\sigma_\alpha(\beta) = \beta - 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}\alpha \in \Phi$$

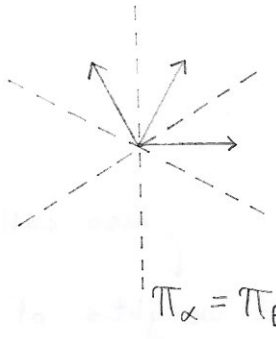
[projection in plane orth to α]

(4) $2\frac{(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$

$\sigma_\alpha(\beta)$ = sym of β under a reflection associated with α

(a reflection associated with α is the reflection with respect to the axis perpendicular to α, i.e. $\sigma_\alpha(\alpha) = -\alpha$)

Example



$\sigma_{E_+^1}(E_+^3) = E_+^2$

3 axis of reflection in adj of SU(3)

We want $\frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ to be an integer to get only integers of $E_\pm^i$

(This classification leads to finite dim Lie Alg)

Angle between α and β: denoted $\theta_{\alpha\beta}$

$4\frac{(\alpha,\beta)}{(\alpha,\alpha)}\frac{(\beta,\alpha)}{(\beta,\beta)} \in \mathbb{Z}$ and $= (2\cos\theta_{\alpha\beta})^2 \in \mathbb{Z}$

↑ only a few angles satisfy this condition

$$(2\cos\theta)^2 \in \mathbb{Z} \Rightarrow \cos\theta \in \left\{0, \pm\frac{1}{2}, \pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{3}}{2}, \pm 1\right\}$$

$$\Rightarrow \quad \theta \in \{90°, 60° \text{ or } 120°, 45° \text{ or } 135°, 30° \text{ or } 150°, 0° \text{ or } 180°\}$$

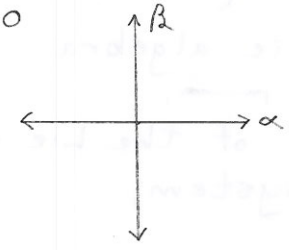
Example $r = 2$: 4 options/cases, 2 roots

$\sigma_\alpha(\beta) = \beta + n.\alpha$ where $n = 0, 1, 2, 3$ (4 root sys)

simple roots are the basis of the root system

(α, β) in our case

1) $n = 0$

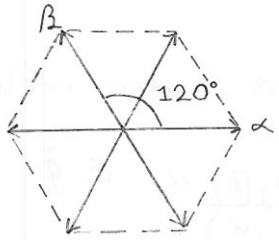


Algebra: $SU(2) \times SU(2) = A_1 \times A_1$

because in Cartan's classification,

$SU(n)$ has the algebra of A_{n-1}

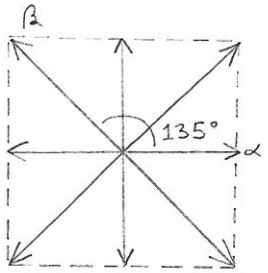
2) $n = 1$



Algebra: $SU(3) = A_2$

s.root = simple root
rank = dim of the Cartan subalgebra

3) n = 2



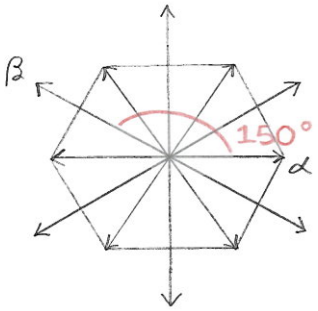
1 short s.root $\alpha \Rightarrow$ 4 short roots

1 long s.root $\beta \Rightarrow$ 4 long roots

dim Adjoint = number of roots = 8 + 2 = 10

i.e. $SO(5) = B_2$ in Cartan classification
↑only one with dim 10

4) n = 3



1 short s.root $\alpha \Rightarrow$ 6 short roots

1 long s.root $\beta \Rightarrow$ 6 long roots

group lattice called G_2

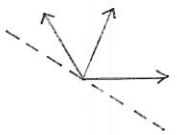
dim Adjoint = 14 (12 + 2)

Choose some to be positive = raising op^n, i.e. the half of them

Divide a group system into two halves = + and − subsets

There is a subset of roots called positive/negative roots ↴ (raising operators)

cf



Root Systems

Schweigert & Fuchs (R)

set of roots, positive roots, simple roots = basis

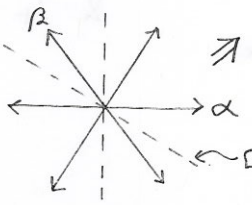
positive roots:

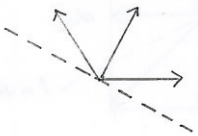
recall that if $\alpha \in \Phi$ only $\pm\alpha \in \Phi$ out of $\alpha, -\alpha$ only one is positive root, the other is a negative root

if $\alpha, \beta \in \Phi^+$ & $(\alpha+\beta) \in \Phi$ then $(\alpha+\beta) \in \Phi^+$

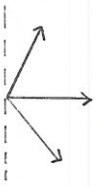
[positive roots span a part of the space ⇔ positive roots are neighbouring]

[6 roots]





or



[we took +ve to RHS, but only a convention, could take to RHS of e.g. vertical dotted line]

[i.e. only $\pm\alpha \in \Phi$ i.e. $n\alpha \notin \Phi$ for $n \in \mathbb{Z} \setminus \{\pm 1\}$]

[subset of positive roots $\equiv \Phi^+$]

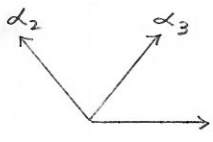
An element in Φ^+ is a simple root if it can not be written as a sum of two positive roots

Ex:

α, β simple ⇒ γ not

Using the simple roots one can write all the roots as linear combinations of them (integer)

Example: SU(3) two simple roots: α_1 and α_2



$$\alpha_1, \alpha_2, \underbrace{\alpha_1+\alpha_2}_{\alpha_3}, -\alpha_1, -\alpha_2, \underbrace{-\alpha_1-\alpha_2}_{-\alpha_3}$$

Cartan matrix:

[normalized scalar product btwn dif$^{\underline{s}}$ vectors] (i.e. simple roots)

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$$ where $\{\alpha_i\}$ are the set of r simple roots

Example: SU(3) $\alpha_1 = (1, 0)$ $\alpha_2 = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$$A_{11} = 2, \quad A_{22} = 2, \quad A_{12} = A_{21} = \frac{2\left(-\frac{1}{2}\right)}{1} = -1$$

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

graphical notation for the Cartan matrix, off diagonal components are negative integers

called Dynkin diagram of the algebra

- A_n SU(n+1) (algebra for)

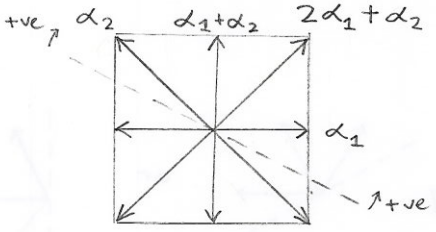
$$A = \begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix}$$

diagram: n nodes

o—o represents a (-1) in the off diagonal entry also corresponds to an angle between the two simple roots equal to 120°

• Recall the root diagram of B_2 for SO(5)

[called B_2 in Cartan class 4]
[equivalent to SO(5)]



[choose simple roots as max span of +ve roots]

α_1 is shorter than α_2

lie algebra contains 4 short roots & 4 long roots

{[spin rep ↔ short root]
[vector rep ↔ long root]}
{for SO, not for SpGp just a special rep in short...}

Cartan matrix $\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$ [not sym]

the roots are (positive)

$\alpha_1, \alpha_2, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2$

[always these whichever simple roots we choose]

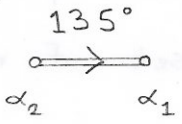
$\alpha_1 = (1,0) \qquad \alpha_2 = (-1,1)$

$\Rightarrow (\alpha_1, \alpha_2) = -1 \Rightarrow A_{12} = \frac{2\cdot(-1)}{1} = -2$

$A_{21} = \frac{2\cdot(-1)}{2} = -1$

Dynkin diagram

one node for each of simple roots

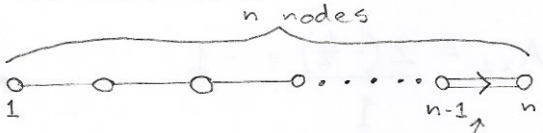


for SO(5) i.e. B_2

longer than because of >

[double line indicates short (length 1)
single line indicates long (length $\sqrt{2}$)]

In general $B_n := SO(2n+1)$



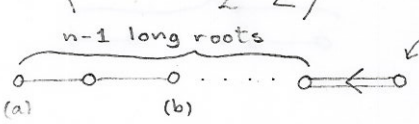
B_n has a single short simple root

$n-1$ longer simple roots

[simple root angle 135°]
[⟹ 135° between roots]

$A_{B_n} \begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -2 & 2 \end{pmatrix}$

C_n :- Sp(n)

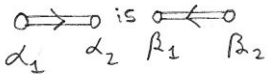


[not connected (a), (b) implies scalar product is 0, at 90°]

A_{C_n} =

$$\begin{pmatrix} 2 & -1 & & & 0 \\ -1 & \ddots & \ddots & & \\ & \ddots & \ddots & -1 & \\ & & -1 & 2 & -2 \\ 0 & & & -1 & 2 \end{pmatrix}$$

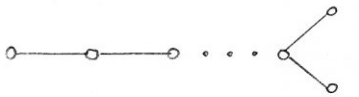
$B_2 = C_2$ $\quad Sp(2) \cong SO(5)$ [spin extension SO(5) isomorphic to Sp(2)]



$\alpha_1 = \beta_2$, $\alpha_2 = \beta_1$

$SO(3,1)$ $\quad A_1 \times A_1$

D_n :- $SO(2n)$



[all 120°]

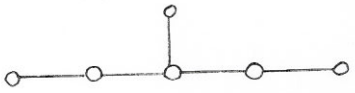
Algebras where all roots have the same length are called "simply laced"
otherwise "non-simply laced"

A, D, E simply laced

B, C, F, G non-simply laced

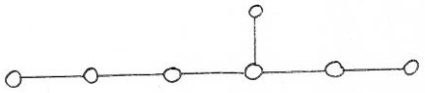
[all are Lie algebras:
A, B, C, D are simplectic unitary LAs]

E_6

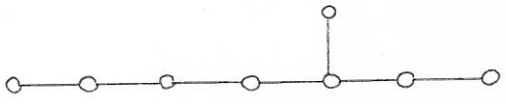


[classification of all the finite dim rep of Lie algebras]

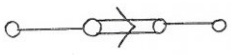
E_7



E_8



F_4



G_2

[3 lines = 150°]