Classical Lie gps/Lie algebras: A_n, B_n, C_n, D_n
Exceptional Lie gps/Lie algebras: E_6, E_7, E_8, F_4, G_2
} See summary on p.4 (L21)

A_n

SU(3): $\alpha_1 = (1,0)$, $\alpha_2 = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ [2D]

$$A_{ij} = 2\frac{(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$$ giving a sym $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ [Cartan matrix] $[\alpha_1, \alpha_2]$ since the roots have the same length ($\Rightarrow A_{ij}$ is symmetric)

Let instead $\underset{\text{(vector in } \mathbb{R}^3)}{e_i \in \mathbb{R}^3} = \{(1,0,0),(0,1,0),(0,0,1)\}$

and $\alpha_1 = e_1 - e_2 = (1,-1,0)$, $\alpha_2 = e_2 - e_3 = (0,1,-1)$

Since $(e_i, e_j) = \delta_{ij}$, we have $(\alpha_1,\alpha_1) = (\alpha_2,\alpha_2) = 2$

$(\alpha_1,\alpha_2) = (e_1 - e_2, e_2 - e_3) = -1$

[hence the Cartan matrix above]

All roots: $\alpha_1 + \alpha_2 = e_1 - e_3$, $-\alpha_i$ [i.e. $\alpha_1, \alpha_2, \alpha_1+\alpha_2, -\alpha_1, -\alpha_2, -\alpha_1-\alpha_2$]

and in the end we have $e_i - e_j$, $i \neq j$, i.e. 6 roots, 3 positives
3 negatives

Generalise to SU(n): $\mathbb{R}^3 \longrightarrow \mathbb{R}^n$, $e_i = (0 \ldots 0, 1, 0 \ldots 0)$ s.t. $(e_i, e_j) = \delta_{ij}$
$(e_i \in \mathbb{R}^n)$ $\uparrow$ i^{th} position

$\Rightarrow \alpha_i = e_i - e_{i+1} = (0, \ldots, 0, 1, -1, 0, \ldots, 0)$

$(n-1)$ simple roots: $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2 - e_3$, $\ldots$, $\alpha_{n-1} = e_{n-1} - e_n$ [SU(n) condition: det = +1]

Note that we just embed the 2D lattice into a 3D lattice:

$$\sum_j \alpha_i^j = 0$$ (i.e. $1 - 1 = 0$), i.e. $\mathrm{tr}(\alpha_i) = 0$: 2 dof

[Closed sublattice of the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$ s.t. sum of α's entries is zero]

We recover the Cartan matrix:

$\left\{ \begin{array}{lll} \text{A set of positive roots} & e_i - e_j, & i<j \\ \text{" " " negative "} & e_j - e_i, & i<j \end{array} \right\}$ set of all roots $e_i - e_j$, $i \neq j$

Eg. SU(4): positive roots

$\underbrace{\alpha_1, \alpha_2, \alpha_3}_{\text{the simple roots}}$

$\alpha_1 = e_1 \overset{4}{-} e_2$, $\alpha_2 = e_2 \overset{5}{-} e_3$, $\alpha_3 = e_3 \overset{6}{-} e_4$

$e_1 \overset{1}{-} e_3 = \alpha_1 + \alpha_2$, $e_2 \overset{2}{-} e_4 = \alpha_2 + \alpha_3$

$e_1 \overset{3}{-} e_4 = \alpha_1 + \alpha_2 + \alpha_3$

Number of roots in adj rep $= (n^2 - 1) - (n - 1) = n(n-1)$

Number of positive roots $\binom{n}{2}$, hence there should be $\boxed{6}$ positive roots for SU(4) $n=4$

$$\binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{k(k-1)\ldots 1} = \frac{n!}{k!(n-k)!}$$

Note: $\alpha_1+\alpha_2$, $\alpha_2+\alpha_3$ are roots; $\alpha_1+\alpha_3$ is not

Recall the dim formula for an irrep of SU(4)

$$\dim[n_1,n_2,n_3] = \frac{(n_1+1)(n_2+1)(n_3+1)(n_1+n_2+2)(n_2+n_3+2)(n_1+n_2+n_3+3)}{12}$$

No (n_1+n_3+2) term : linked to $\alpha_1+\alpha_3$ not being a root

Recall the character of funda rep of SU(n): $\boxed{Y_i = \text{fugacities}}$

$$\chi_{\underbrace{[1,0\ldots0]}_{(n-1)\text{ entries}}}(Y_1\ldots Y_{n-1}) = \overset{=Z_1}{Y_1} + \overset{=Z_2}{\frac{Y_2}{Y_1}} + \overset{=Z_3}{\frac{Y_3}{Y_2}} + \ldots + \overset{=Z_{n-1}}{\frac{Y_{n-1}}{Y_{n-2}}} + \overset{=Z_n}{\frac{1}{Y_{n-1}}}$$

weights: $(1,0\ldots0)$, $(-1,1,0\ldots0)$, $(0,-1,1,0\ldots0)$, ...

[centred around origin]

They sum up to $0 \longrightarrow$ linked to the fact we have a triangle and the sum of weights $=0$

Change the fugacities: [See note 1, p.3]

$$(Y_i)_{1\leq i\leq n-1} \longrightarrow (Z_i)_{1\leq i\leq n} \quad \text{with } Z_i = \frac{Y_i}{Y_{i-1}} \qquad Y_0 = Y_n = 1$$

We have then

$$\prod_{i=1}^{n} Z_i = 1$$

$$\begin{aligned}
\chi_{[1,0\ldots0,1]} &= \text{character of adjoint} = \text{funda}\times\text{antifunda} - \text{singlet} \\
&= \chi_{[1,0\ldots0]}\,\chi_{[0\ldots0,1]} - \underbrace{\chi_{[0\ldots0]}}_{1} \\
&= \sum_i Z_i \cdot \sum_j \frac{1}{Z_j} - 1 \\
&= \sum_{ij} Z_i Z_j^{-1} - 1 \\
&= \Big(\sum_{i=j} + \sum_{i\neq j}\Big)\big(Z_i Z_j^{-1}\big) - 1 \\
&= n-1 + \sum_{i\neq j} Z_i Z_j^{-1}
\end{aligned}$$

The weights are $e_i - e_j$, $i\neq j$ (i.e. the roots)

roots = weights of the adjoint

For SU(2), $\chi_{[1]}(X) = X + \frac{1}{X} = Z_1 + Z_2$ s.t. $Z_1 Z_2 = 1$

$$\mathrm{Tr}\, e^{iJ_3\theta} = \begin{pmatrix} e^{\frac{i\theta}{2}} & 0 \\ 0 & e^{-\frac{i\theta}{2}} \end{pmatrix}, \quad e^{\frac{i\theta}{2}} = X \qquad \det = 1 = \prod_i Z_i$$

So $\prod_i Z_i = 1 \iff \det = 1$

Using a multi index notation: $Z^\alpha = \prod_{i=1}^n Z_i^{\alpha_i}$

So $\sum_{i \neq j} Z_i Z_j^{-1} = \sum_{\alpha \in \Phi} Z^\alpha$ $\left(= \sum_{i\neq j} Z_i/Z_j\right)$

[Z's are convenient but one loses track of highest weights: the Y's keep track]

$B_n \sim SO(2n+1)$

Vectors $e_i \in \mathbb{R}^n$ (to describe roots) $\alpha_1 = e_1 - e_2,\ \alpha_2 = e_2 - e_3, \dots, \alpha_{n-1} = e_{n-1} - e_n$

then length $= \sqrt{2}$

and the last one is shorter: $\alpha_n = e_n$: length $= 1$

Compute the Cartan matrix:

$\{\alpha_i\}$ the set of simple roots

All roots: $\pm(e_i \pm e_j), \pm e_n$: $2n^2$ roots

(without counting $e_i + e_j$ twice)

$\pm e_i \pm e_j$, $i<j$, $\pm e_i$

$2(n-1)n$ $\quad 2n \quad \Rightarrow 2n^2$ roots

Positive roots: $e_i \pm e_j$, $i<j$, e_n

Negative roots: $-(e_i \pm e_j)$, $i<j$, $-e_n$

For $\underline{B_2}$:

$\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2 - e_3$ the positive roots

$\alpha_1 + \alpha_2 = e_1$, $\alpha_1 + 2\alpha_2 = e_1 + e_2$

(Roots, positive roots, simple roots, Cartan matrix, properties of Lie Algebra) (represented by Dynkin diag)

Note 1: Why do we use Y rather than Z?

Y's give the highest weights (h.w.'s)

Y_1 is the h.w. of funda

Y_2 " " " " 2^{nd} (rank) A-S

Y_3 " " " " 3^{rd} (rank) A-S

Summary of Dynkin Diagrams For Groups

Positive roots, simple roots, Cartan matrix
From these, proceeded to classification of Lie algebras

Classical:

- $A_n \sim SU(n+1)$
- $B_n \sim SO(2n+1)$
- $C_n \sim Sp(n)$
- $D_n \sim SU(2n)$

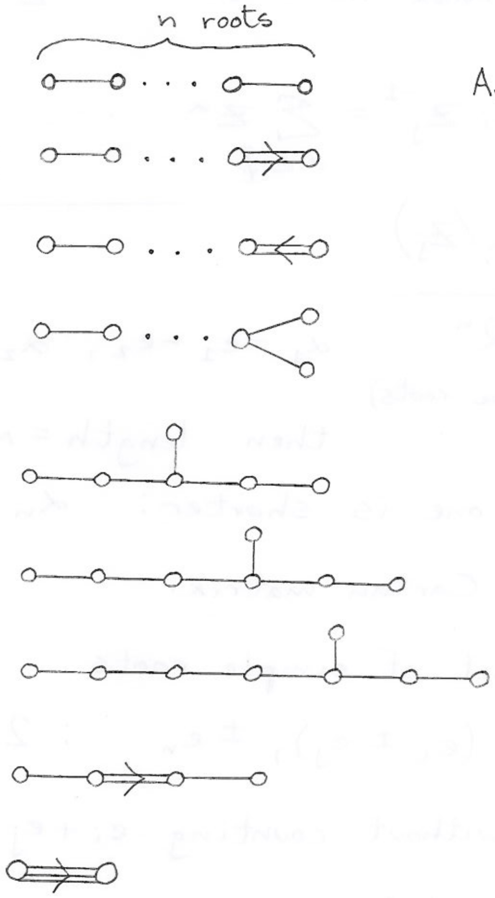
E_6

E_7

E_8

F_4

G_2



$$A_n: \begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix}$$

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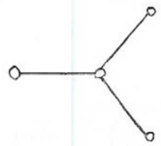
Specific Cases: (of root systems)

- D_4:  So D_3:  So $SO(6) = D_3 \equiv A_3 = SU(4)$: 

$$SO(6) \equiv SU(4)$$

- B_2:  $B_2 \equiv C_2$ i.e. $Sp(2) \equiv SO(5)$

- D_2:  i.e. $D_2 = A_1 \times A_1$  so $SO(4) \cong SU(2) \times SU(2)$

(cf the matrix)

- D_5:  $\equiv E_5$

In general:

A_n: $n+1$ dim vector space, basis vector $= \{e_i\} = \{(0 \dots 0, \underset{\substack{\uparrow \\ i^{th}\ \text{posn}}}{1}, 0, \dots 0)\}$

Set of roots $(e_i - e_j),\ i \neq j$ $\quad n(n+1)$ roots

$(e_i - e_j),\ i < j$ $\quad \dfrac{n(n+1)}{2}$ positive roots

$$\begin{aligned} \dim(\text{adjoint}) &= n(n+1) + n \\ &= n^2 + 2n \\ &= (n+1)^2 - 1 = \dim(\text{adj of } SU(n+1)) \end{aligned}$$

Formulas for dim of irreps of A_n:

recall $\underline{A_1 \simeq SU(2)}, \dim[n] = n+1$ $\quad$ one simple root, positive $= \alpha$, negative $= -\alpha$

For $\underline{A_2 \simeq SU(3)}$

$$\dim[n_1, n_2] = \frac{(n_1+1)(n_2+1)(n_1+n_2+2)}{2},$$

2 simple roots α, β

3 positive roots $\alpha, \beta, \alpha+\beta$ [or: $\alpha_1, \alpha_2, \alpha_1+\alpha_2$]

For $\underline{A_3 \simeq SU(4) \simeq SO(6)}$

$$\dim[n_1, n_2, n_3] = \underbrace{\frac{1}{12}}_{2\times2\times3}\Big((n_1+1)(n_2+1)(n_3+1)(n_1+n_2+2)(n_2+n_3+2)(n_1+n_2+n_3+3)\Big)$$

3 s. roots: $\alpha_1, \alpha_2, \alpha_3 = e_1 - e_2, e_2 - e_3, e_3 - e_4$

positive roots: same and $\alpha_1 + \alpha_2 = e_1 - e_3$, $\alpha_2 + \alpha_3 = e_2 - e_4$

$$\alpha_1 + \alpha_2 + \alpha_3 = e_1 - e_4$$

For $SU(N)$ $\boxed{\dim[n_1 \ldots n_{N-1}] = \prod_{1 \leq i < j \leq N} \frac{n_i + \ldots + n_{j-1} + (j-i)}{(j-i)}}$ by convention, $n_N = 0$

check: for $N=2$: $i \in \{1\}$, $\dim[n_1] = \frac{n_1 + 1}{1} = n_1 + 1$ (OK)

for $N=3$: $(i,j) \in \{(1,2), (1,3), (2,3)\}$

$$(1,2) \to n_1 + 1, \quad (1,3) \to \frac{n_1 + n_2 + 2}{2}, \quad (2,3) \to n_2 + 1$$ (OK)

$\underline{B_n}$: root sys

basis $\{e_i\}$ in n dim vector space

$$\text{roots} = \left\{ \begin{array}{lll} \pm e_i \pm e_j & = 2n(n-1) & \text{long roots } (i<j) \\ \pm e_i & = 2n & \text{short roots} \end{array} \right\} 2n^2 \text{ roots}$$

$\dim(\text{adj of } B_n) = 2n^2 + n$

$\dim(\text{adj of } SO(2n+1)) = \binom{2n+1}{2} = (2n+1)n$

<u>recall</u>:
number of root + rank
= dim adjoint

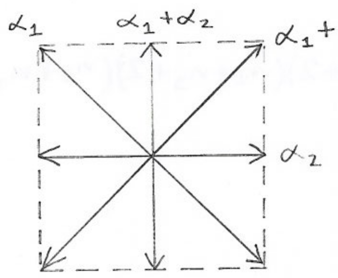
$$\text{positive roots} = \left\{ \begin{array}{ll} e_i \pm e_j & i<j \\ e_i & \end{array} \right.$$

simple roots: $e_1 - e_2 = \alpha_1$, $e_2 - e_3 = \alpha_2$ $\ldots$ $e_{n-1} - e_n = \alpha_{n-1}$, $e_n = \alpha_n$

$\underline{B_2}$

positive roots: $\underbrace{e_1 - e_2}_{=\alpha_1}$, $\underbrace{e_1 + e_2}_{\alpha_1 + 2\alpha_2}$, $\underbrace{e_1}_{\alpha_1 + \alpha_2}$, $\underbrace{e_2}_{=\alpha_2}$ (4 pos. roots, 8 roots)

Lattice:



sum of coefficients

$$\dim[n_1, n_2] = \frac{(n_1+1)(n_2+1)(n_1+n_2+2)(n_1+2n_2+3)}{2.3}$$

So $\dim[1,0] = \frac{2\times 1\times 3\times 4}{2\times 3} = 4$, $\dim[0,1] = \frac{1\times 2\times 3\times 5}{2\times 3} = 5$

spin rep of $SO(5)$ defining rep of $SO(5)$

For C_2: the 4D represⁿ is the defining rep

" 5D " " " 2nd rank traceless antisym rep of $Sp(2)$

Root Systems

dimension Formulas

$$\dim[n_1, n_2]_{C_2} = \frac{(n_1+1)(n_2+1)(n_1+n_2+2)(n_1+2n_2+3)}{6}$$

[note this is c_2 not b_2]

[n_1, n_2 - integers: Dynkin labels]

$\dim[1,0]_{C_2} = 4$

$\dim[0,1]_{C_2} = 5$

[Can be done only for simply laced algebras]

[$\alpha \in \Phi^+$]

[or could say $\alpha > 0$ [for all +ve roots]]

Weyl dimension formula, For a representation of highest weight Λ the dim at this rep is

$$\boxed{\prod_{\alpha \in \Phi^+} \frac{(\Lambda + \rho, \alpha)}{(\rho, \alpha)}}$$

[works for any algebra]

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$$

[$G_2 \to$ a simple example]

Cartan matrix

Example G_2: the roots convenient to work in a 3d space, e_1, e_2, e_3

[$G_2 \to$ 12 roots, 6 short, 6 long $\to$ hexagon graph]

12 roots

$\pm(e_1 - e_2)$, $\pm(e_1 - e_3)$, $\pm(e_2 - e_3)$ short

$\pm(2e_1 - e_2 - e_3)$ $\pm(2e_2 - e_1 - e_3)$ $\pm(2e_3 - e_1 - e_2)$ long

$\alpha_1 = e_1 - e_2$ simple roots $(e_i, e_j) = \delta_{ij}$

$\alpha_2 = 2e_2 - e_1 - e_3$

Cartan matrix $A_{ij} = \dfrac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$ $A = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$ [α]

[originally (α_j, α_j) $\to$ check convention]

[unitary, orthogonal, symplectic } can rep as matrices BUT 5 exceptional algebras - not easy construct as matrices - so "groups" from them are reliant on the algebra]

$(\alpha_1, \alpha_1) = 2$

$(\alpha_1, \alpha_2) = -1 - 2 = -3$

$(\alpha_2, \alpha_2) = 4 + 1 + 1 = 6$

$\Lambda = n_1 \Lambda_1 + n_2 \Lambda_2$

Λ_1, Λ_2 fundamental weights

$$\frac{2(\Lambda_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij} = \frac{2(\alpha_i, \Lambda_j)}{(\alpha_i, \alpha_i)}$$

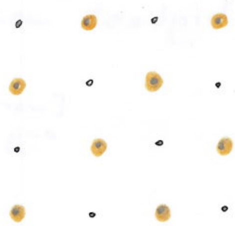
Λ's are basis vectors for the dual lattice to the root lattice

Weight lattice has a sublattice which is the root lattice

α_i generate the root lattice

Λ_i " " weight lattice

root lattice is a sublattice of the weight lattice of index equal to $\det A$



a sublattice of index 2

$\Lambda_i = m_{ij}\alpha_j$

$$\delta_{ij} = \frac{2(\Lambda_i, \alpha_j)}{(\alpha_i,\alpha_i)} = 2m_{jK}\frac{(\alpha_i, \alpha_K)}{(\alpha_i,\alpha_i)} = m_{jK}A_{iK}$$

[Note]

$\underline{SU(2)}$: one simple root, α

[adjoint rep]

[but if a highest weight $\to$ move]

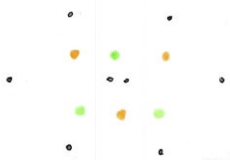
[lattice of index 2]

$1 = (\Lambda, \alpha) = (m\alpha, \alpha) = 2m$

$\Lambda = \frac{1}{2}\alpha$

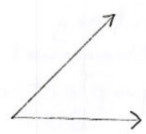
[weight lattice contains $\frac{1}{2}$ integral points but adjoint rep contains integral points]

SU(3)



[adjoint - grey
fund$^{\underline{m}}$ - orange
anti-fund$^{\underline{m}}$ - green]

[fund$^{\underline{m}}$ doesn't sit in adjoint rep$^{\underline{n}}$]



[vectors are the fund$^{\underline{m}}$ weights]

the matrix m_{ij} is A^{-1}

$$A^{-1} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$$ [cf [α]]

$\Lambda_1 = 2\alpha_1 + 3\alpha_2 = 2\alpha_1 + \alpha_2$

$\Lambda_2 = \alpha_1 + 2\alpha_2 = 3\alpha_1 + 2\alpha_2$

[Precisely did the transpose]

$\Lambda = n_1\Lambda_1 + n_2\Lambda_2 = (2n_1 + 3n_2)\alpha_1 + (1n_1 + 2n_2)\alpha_2$

positive roots : $e_1 - e_2 = \alpha_1$, $e_2 - e_3 = \alpha_1 + \alpha_2$, $e_1 - e_3 = 2\alpha_1 + \alpha_2$

$$2e_2 - e_1 - e_3 = \alpha_2 \,,\quad 2e_1 - e_2 - e_3 = 3\alpha_1 + \alpha_2 \,,\quad -2e_3 + e_1 + e_2 = 3\alpha_1 + 2\alpha_2$$

[6 +ve roots → allows to compute ρ (rho)]

[+ve linear combinations of simple roots give +ve roots]

$$\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha = 5\alpha_1 + 3\alpha_2$$

[now compute scalar product]

$$(\Lambda, \alpha_1) = (2n_1 + n_2)\underbrace{(\alpha_1, \alpha_1)}_{=2} + (3n_1 + 2n_2)\underbrace{(\alpha_1, \alpha_2)}_{=+3} = n_1$$

$$(\Lambda, \alpha_2) = (2n_1 + 3n_2)\underbrace{(\alpha_1, \alpha_2)}_{=-3} + (n_1 + 2n_2)\underbrace{(\alpha_2, \alpha_2)}_{=6} = 3n_2$$

Recall: $\Lambda_i = (A^{-1})_{ij}\,\alpha_j$ Fundamental weights (basis vectors for weight lattice)

$$2\frac{(\Lambda_i,\alpha_j)}{(\alpha_j,\alpha_j)} = (A^{-1})_{iK}\, 2\frac{(\alpha_K,\alpha_j)}{(\alpha_j,\alpha_j)}$$

$$= (A^{-1})_{iK}\,(A)_{Kj}$$

$$= \delta_{ij}$$

[root lattice is always a sublattice of the weight lattice]

$$\boxed{\begin{array}{l}\text{Cartan Matrix}\\ \text{Fulton Harris}\\ A_{ij} = 2\dfrac{(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)}\end{array}}$$

For all groups: $\Lambda = \sum_{i=1}^{K} n_i \Lambda_i$ (i = the number of simple roots)

$$2\frac{(\Lambda,\alpha_i)}{(\alpha_i,\alpha_i)} = 2\sum_j n_j \frac{(\Lambda_j,\alpha_i)}{(\alpha_i,\alpha_i)}$$

$$= \sum_j n_j \delta_{ij}$$

$$= n_i$$

So $\boxed{n_i = 2\dfrac{(\Lambda,\alpha_i)}{(\alpha_i,\alpha_i)}}$

So for G_2 again:

$$\boxed{\begin{array}{lll} \alpha_1 = e_1 - e_2 & A = \begin{pmatrix} 2 & -1 \\ -3 & 2\end{pmatrix}, & A^{-1} = \begin{pmatrix} 2 & 1 \\ 3 & 2\end{pmatrix} \\ \alpha_2 = 2e_2 - e_1 - e_3 & & \\ (\alpha_1,\alpha_1) = 2 & \Lambda_1 = 2\alpha_1 + \alpha_2 & \Lambda = n_1\Lambda_1 + n_2\Lambda_2 \\ (\alpha_1,\alpha_2) = -3 & \Lambda_2 = 3\alpha_1 + 2\alpha_2 & (\Lambda,\alpha_1) = n_1 \\ (\alpha_2,\alpha_2) = 6 & & (\Lambda,\alpha_2) = 3n_2 \end{array}}$$

$$\Lambda = (2n_1 + 3n_2)\alpha_1 + (n_1 + 2n_2)\alpha_2\ ;\ (\Lambda,\alpha_1) = n_1\ ,\ (\Lambda,\alpha_2) = 3n_2$$

$$\rho = 5\alpha_1 + 3\alpha_2 \implies (\rho,\alpha_1) = 5.2 + 3.(-3) = 1$$

$$(\rho,\alpha_2) = 5.(-3) + 3.6 = 3$$

Compute now the dimension:

[Dimension Formula, for a representation of highest weight Λ]

$$\dim[n_1,n_2]_{G_2} = \prod_{\alpha>0} \frac{(\Lambda+\rho,\alpha)}{(\rho,\alpha)} \qquad \boxed{\rho = \frac{1}{2}\sum_{\alpha>0}\alpha}$$

$$\overbrace{3(\Lambda,\alpha_1)}^{n_1} + \overbrace{2(\Lambda,\alpha_2)}^{3n_2}$$

$$+3\underbrace{(\rho,\alpha_1)}_{1} + 2\underbrace{(\rho,\alpha_2)}_{3}$$

$$= \frac{n_1+1}{1} \times \frac{3n_2+3}{3} \times \frac{n_1+3n_2+4}{4} \times \frac{2n_1+3n_2+5}{5} \times \frac{3n_1+3n_2+6}{6} \times \frac{3n_1+6n_2+9}{9}$$

for: (+ve roots) α_1, α_2, $\alpha_1+\alpha_2$, $2\alpha_1+\alpha_2$, $3\alpha_1+\alpha_2$, $3\alpha_1+2\alpha_2$

$$= \frac{(n_1+1)(n_2+1)(n_1+3n_2+4)(2n_1+3n_2+5)(n_1+n_2+2)(n_1+2n_2+3)}{2.3.4.5}$$

So

$$\begin{cases} \dim[1,0]_{G_2} = \dim \text{ funda} = \dfrac{\cancel{2}.\cancel{5}.7.\cancel{3}.\cancel{4}}{\cancel{2}.\cancel{3}.\cancel{4}.\cancel{5}} = 7 \\ \dim[0,1]_{G_2} = \dim \text{ adjoint} = \dfrac{\cancel{2}.7.{}^{2}\cancel{8}.\cancel{3}.\cancel{5}}{\underset{1}{\cancel{2}.\cancel{3}.\cancel{4}.\cancel{5}}} = 14 \end{cases}$$

(this is only antifundamental for SU(3))

These are the two basic reps of G_2:

any other rep $[n_1, n_2]$ is in the tensor product

$[n_1, n_2] \in \mathrm{Sym}^{n_1}[1,0] \otimes \mathrm{Sym}^{n_2}[0,1]$

For a rep with a set of weights $\{w_i\}$, the complex conjugate rep has the set of weights $\{-w_i\}$

If the two sets are equal, it is called a <u>real</u> rep
$\{w_i\} = \{-w_i\}$

• For instance, for $SU(2)$:

$$\chi_{[n]}(X) = \sum_{m=-\frac{n}{2}}^{\frac{n}{2}} X^{2m} \qquad \text{sym on } n \quad \left(\text{i.e. } \sum_{-\frac{n}{2}}^{\frac{n}{2}} = \sum_{\frac{n}{2}}^{-\frac{n}{2}}\right)$$

So all $SU(2)$ irreps are real

(Spin j rep (of $SU(2)$) has weight $-j \leq m \leq j$ so sym)

<u>Real rep</u>: funda for instance V_α, W_α $(\alpha = 1,2)$

Contraction lives in $2 \otimes 2 = 1_A \oplus 3_S$

so invariant contraction is AS

i.e. it is $V_\alpha W_\beta \varepsilon^{\alpha\beta}$ (singlet)

But adjoint rep: A_i, B_i in (3) of $SU(2)$,

we have $3 \otimes 3 = 1_S \oplus 3_A \oplus 5_S$

(i.e. $[2] \otimes [2] = [0] \oplus [2] \oplus [4]$)

The invariant contraction is now sym:

$A_i B_j \delta^{ij}$ is a singlet

Then the real irreps are either real (contraction with δ)
or pseudoreal (—— " —— —"— ε)
[Levi-Civita]

• For <u>$SU(3)$</u>: complex conj of $[m,n]$ is $[n,m]$

so real reps of $SU(3)$ are $[n,n]$ (not pseudoreal in fact)
i.e. $[n,n]_{SU(3)}$ is real

• For <u>$SO(4) \simeq SU(2) \times SU(2)$</u>

reps are $[n_1, n_2]_{SO(4)}$ (one n for each rep of each $SU(2)$)

$\dim [n_1, n_2]_{SO(4)} = (n_1+1)(n_2+1)$

and $\chi_{[n_1,n_2]} = \chi_{[n_1]}(X_1)\chi_{[n_2]}(X_2)$ (two different fugacities)

Lecture 24 — Particle Symmetries — 02.12.2011

The product of two pseudo real reps is real (1.) [See below] [P×P→R]

— " — — " — — " — pseudo real with real — " — pseudo real [P×R→P]

— " — — " — — " — two real reps — " — real [R×R→R]

(1.) (anti × anti = sym, $\varepsilon\varepsilon = \delta$, fermions × fermions = bosons ...)

Here, the sum $n_1 + n_2 =$ if odd, rep pseudoreal; if even, rep real

Examples:

$[0,1]_{SO(4)}$ has dim 2, spinor rep

$[1,0]_{SO(4)}$ as well (or spinor)

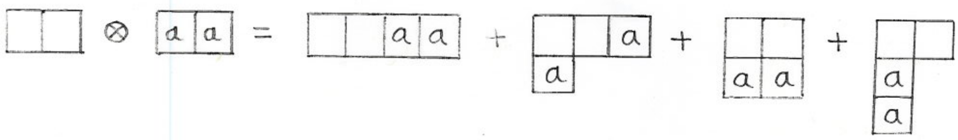
} pseudoreal

$[1,1]_{SO(4)}$ has dim 4, vector rep, real rep

$[1,2]_{SO(4)}$ has dim 6, ?

For product of $n_1 n_2$:
$n_1 + n_2$ odd $\Rightarrow n_1 n_2$ pseudoreal
$n_1 + n_2$ even $\Rightarrow n_1 n_2$ real

Note on Young Diagrams



no two a's in the same column

Root Systems [remove all brackets][with ° °]

A_n: roots $\pm(e_i - e_j)$ $\quad i<j \quad i,j = 1,\ldots,n+1 \quad$ $n(n+1)$ roots, positive roots $e_i - e_j$,

simple roots $\alpha_i = e_i - e_{i+1}$

[simply laced]
[all roots have same length]

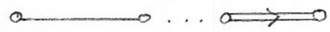
[Dynkin diag:] 

B_n: $\pm(e_i \pm e_j), \pm e_i \quad \underbrace{i<j \quad i,j = 1,\ldots,n}$ $\quad \underbrace{2n(n-1)}_{\text{long}} + \underbrace{2n}_{\text{short}} = 2n^2$, $(e_i \pm e_j), e_i$

[same condition applies to C_n, D_n]

$\alpha_i = e_i - e_{i+1} \quad i = 1,\ldots,n-1$

$\alpha_n = e_n$

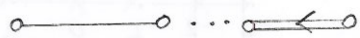


[very similar to B_n]

C_n: $\pm(e_i \pm e_j), \pm 2e_i \qquad \underbrace{2n(n-1)}_{\text{short}} + \underbrace{2n}_{\text{long}} \qquad (e_i \pm e_j), 2e_i$

$\alpha_i = e_i - e_{i+1} \quad i = 1,\ldots,n-1$

$\alpha_n = 2e_n$



D_n: $\pm e_i \pm e_j \qquad 2n(n-1) \qquad e_i \pm e_j$

$\alpha_i = e_i - e_{i+1} \quad i = 1,\ldots,n-1$

$\alpha_n = e_{n-1} + e_n$

[last two roots orthogonal if compute Cartan matrix]

[Choice - pick any set of roots you wish]

E_8: $\alpha_i = e_i - e_{i+1} \qquad i = 1,\ldots,6$

$\alpha_7 = e_6 + e_7$

$\alpha_8 = \frac{1}{2}\sum_{i=1}^{8} e_i \qquad$ [1st case of non-integer entry]

Example [of choosing roots]

A_3: $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)}$

[i.e. the elements of A are given by the RHS]

[Any set of vectors that satisfy this product - once know Cartan matrix can extract all info for it - & can read from Dynkin diagrams]

Det A:

the root lattice is a sublattice of the weight lattice of index equal to the Det A

[lattice of roots as sub-lattice of lattice of weights]

	Det
A_n:	$n+1$
B_n:	2
C_n:	2
D_n:	4

A_n [tells us $n+1$ types rep$^{\underline{n}}$s]

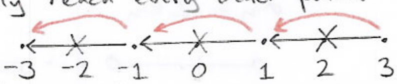
$\left.\begin{array}{l}B_n\\C_n\end{array}\right\}$ [2 types rep$^{\underline{n}}$s: ½ & whole integral spin: saw previously
½-fundamental rep$^{\underline{n}}$
whole - not fundamental but adjoint SU(2)]

D_n [4 types rep$^{\underline{n}}$s:
saw for $D2 = SO(4) \rightarrow$ copy of 2 SU(2)s]

[Why does det tell us # types rep$^{\underline{n}}$?]

SU(2)

If highest weight is 3,
α^{-1} is jump of length 2,
can only reach every other point



So need 2 diff$^{\underline{t}}$ reps to reach all points

[Can only jump by units determined by roots - each step is a root]

[jump 2 - cannot get to points × in middle once have highest weight: can divide lattice into that many - is given by det]

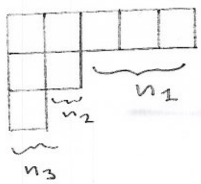
[SU(2) ⊗ SU(2) generic rep$^{\underline{n}}$]

$[n_1, n_2]$ $\quad n_1 <$ even / odd, $\quad n_2 <$ even / odd $\Big\}$ [i.e. 4 types rep$^{\underline{n}}$: can see this from root diagram]

Example:

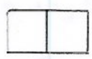
an irrep of SU(N) is denoted by N-1 non-negative integers $[n_1, \dots, n_{N-1}]$

[Young's diagram]



number of boxes, (mod N), = 0, ..., N-1

$A_2 \cong SU(3)$  $= [1,0]$ [fund. rep$^{\underline{n}}$],  $= [0,1]$ [anti-fund rep$^{\underline{n}}$],  $= [1,1]$

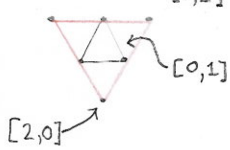
 $= [2,0]$

[sit on same lattice] [check]

Example:

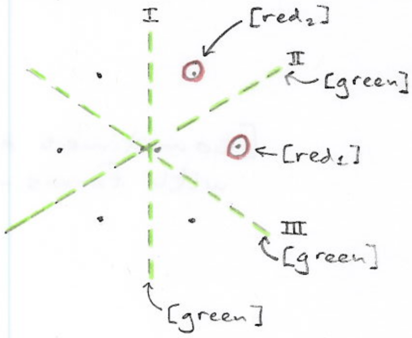
For $[1,0] \Rightarrow \overbrace{Y_1 + \frac{Y_2}{Y_1} + \frac{1}{Y_2}}^{X_{[1,0]}}$, $[0,1] \Rightarrow \overbrace{Y_2 + \frac{Y_1}{Y_2} + \frac{1}{Y_1}}^{X_{[0,1]}}$, $[2,0] \Rightarrow \overbrace{Y_1^2 + \frac{Y_2^2}{Y_1^2} + \frac{1}{Y_1^2} + \underbrace{Y_2 + \frac{Y_1}{Y_2} + \frac{1}{Y_1}}_{X_{[0,1]}}}^{X_{[2,0]}}$

[sit on same sublattice]



Group of reflections

S_3 Weyl Group W



[How get from red$_1$ to red$_2$? By a reflection]

Weyl group of $SU(N)$ is S_N

$$W_{A_n} = S_{n+1}$$

[reln N to n is by shift of 1]

$$|W_{B_n}| = 2^n n! \qquad |W_{G_2}| = 12$$

$$|W_{C_n}| = 2^n n!$$

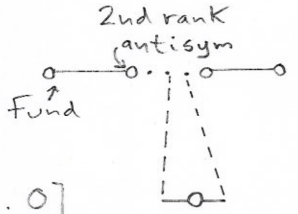
$$|W_{D_n}| = 2^{n-1} n!$$

[each root corresponds to element of Weyl group
-doesn't have to be case
→Weyl grows factorial
→# roots grows quadratic]

Recall for each simple root α_i there is a corresponding

Fundamental weight Λ_i

$$\frac{2(\Lambda_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij} \qquad \Lambda = \sum_{i=1}^{n} n_i \Lambda_i$$

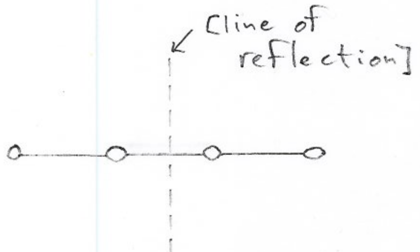


[each node corresponds to
- simple root
- fund weight]

Basic rep $[0 \ldots 0, \underset{i\text{-th}}{1}, 0 \ldots 0]$

[@ i^{th} place:] i-th rank antisym rep $\binom{n+1}{i}$ [= dim]

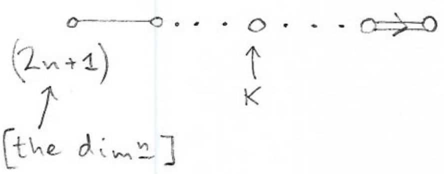
A_n:



[under this refl^n [of complex conjugation] the dim^n is preserved]

$$\dim[n_1, n_2 \ldots, n_{N-1}] = \dim[n_{N-1}, n_{N-2}, \ldots, n_1]$$

B_n:



$$\dim[0, \ldots, 0, \underset{K}{1}, 0 \ldots 0]$$

[last node corr to short root → dif^t fund weight]

K-th rank anti-symmetric rep of B_n

$K = 1, \ldots, n-1$

[have $n-1$ basic rep^{n}s & then dif^t for last node]

$K = n$: [last node:] $\dim[0, \ldots, 0, 1] = 2^n$ Spinor rep

Examples:

$\underline{B_2}$ $[1,0]$ of dim 5 vector $[=2.2+1]$
$[0,1]$ of dim 4 spinor $[=2^2]$

$\underline{B_3}$ $[1,0,0]$ of dim 7 vector $[=2.3+1]$
$[0,1,0]$ " " 21 2nd rank antisym [sometimes associate with forms - in 7D space]
$[0,0,1]$ " " 8 spinor $[=2^3]$
