roots $\longrightarrow$ positive roots, simple roots α_i, Fundamental weights $\rightarrow$ [give rise to basic rep of group] [see later] [β]

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} \quad , \quad \frac{2(\Lambda_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij} \qquad \Lambda = (A^{-1})_{ij}\alpha_j$$

[α][see below]

The highest weight of irrep

$$\Lambda = \sum_{i=1}^{K} n_i \Lambda_i \qquad \text{if } \Lambda = \Lambda_i \text{ each of } i\text{'s basic irrep}$$

[see later] [γ]

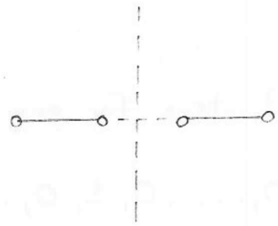
A_n: $\dim[0, \ldots, 0, \underset{K\text{-th}}{1}, 0, \ldots 0] = \binom{n+1}{K}$ K-th rank antisymmetric representation of $SU(n+1)$

$$\binom{n+1}{K} = \binom{n+1}{n+1-K}$$

[Prop binomial coeff^n have symmetry]

Set $N = n+1$

$$\dim[n_1, \ldots, n_{N-1}] = \dim[n_{N-1}, \ldots, n_1]$$



Example

$A_3 = D_3 = SU(4), SO(6)$

$\dim[n_1, n_2, n_3] = \dim[n_3, n_2, n_1]$

$\dim[1,0,0] = 4 = \dim[0,0,1] \qquad \dim[0,1,0] = 6$

[antisym product]

$[n_1, n_2, n_3]$ is in the tensor product of $Sym^{n_1}[1,0,0] \otimes Sym^{n_2}(\underbrace{\Lambda^2[1,0,0]}_{[0,1,0]})$

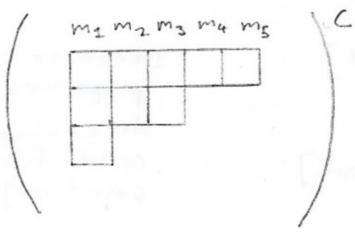
$\otimes Sym^{n_3}(\underbrace{\Lambda^3[1,0,0]}_{[0,0,1]})$

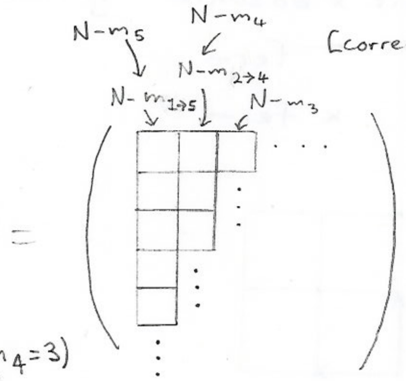
[when take K-th asym product $\rightarrow$ move 1 along rows of [0,0,0s]]

[just deal with algebra not group $\rightarrow$ so only δ_{ij}, not η_{ij} at [α]]

For a representation of $SU(N)$ of Dynkin labels

$[n_1, \ldots, n_{N-1}]$ the complex conj rep is $[n_{N-1}, \ldots, n_1]$

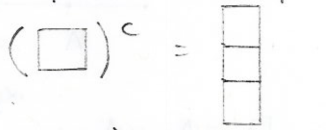


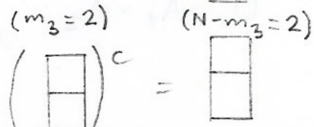


[corrected]

[Originally ...m_1,...m_2, etc. changed to ...m_5,...m_4, etc.]

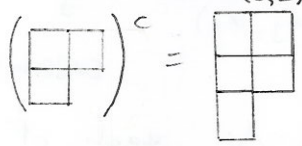
For $SU(4)$





All irreps $[0, n, 0]$ are real, symmetric traceless n-th rank tensors of $SO(6)$

$(2,1)$ $\quad$ $(4-1, 4-2) = (3,2)$



i.e. $[1,1,0]^c = [0,1,1]$

For $SU(6)$ $\quad [0,0,n,0,0]$ is real

$[n_1, n_2, n_3 \ldots n_3, n_2, n_1]$ in general

[All asym reps are real]
[All this A_n]
[Fund weights correspond to K-th rank asym rep]

$[[\beta] \to [\gamma]$ true for any group] [now apply to B_n]

$\underline{B_n}$ $\quad [0, \ldots, 0, 1, 0, \ldots, 0]$

$\dim [0, \ldots, 0, 1] = 2^n$ spinor of $SO(2n+1)$

Examples

$B_1 = C_1 = A_1 \cong SU(2)$ $\quad 2^1 = 2$ fund. rep

$SO(3)$

$\det(A_{B_n}) = 2$

$Sp(1)$

[Stern-Gerlach experiment: spin relate to ang. momenta + Fermi stats]

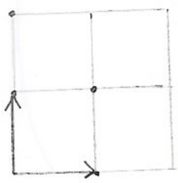
2 types of irreps $[n_1, \ldots, n_n]$, if n_n is even bosonic, odd fermionic

tensor product of

bosonic $\times$ bosonic gives bosonic [stay in bosonic lattice]

" $\times$ ferm gives ferm [stay in fermionic lattice]

ferm $\times$ ferm gives bosonic [move from ferm to bosonic]

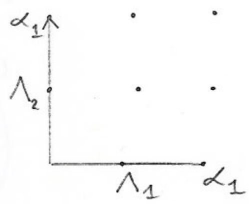


[can use determinant to find area]
[see note 1, p.4]

A_1 : $\Lambda = \frac{1}{2}\alpha$ [Fund weight] [+ve root]

[because of properties of determinant could find solution]

$D_2 = A_1 \times A_1$



[Area 4 times as large for D_2 as area picked out by roots]

$[0,\dots,0,1,0,\dots,0]$ (1 in the K-th place)

K-th rank antisymmetric rep. of $SO(2n+1)$ [vector space - always real by construction]

of dimension $\binom{2n+1}{K}$ real rep

[referring back to {bottom p2} root diagrams]

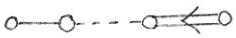
$[1,0]$ spinor

$[0,1]$ "

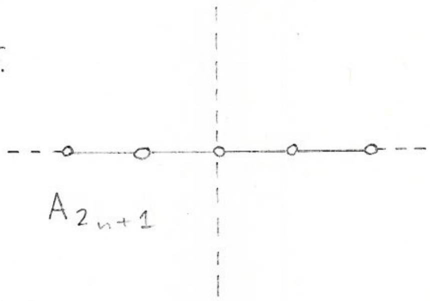
$[2,0]$ self dual 2^{nd} rank antisym rep of $SO(4)$

a representation that appears in physics $[1,0\dots0,1]$ — 1 = vector, 1 = spinor ~ gravitino [appears in $d \geq 4$ dim-context supgrav]

		dim	
B_2:	$[0,1]$	4, pseudoreal	spinor reps
B_3:	$[0,0,1]$	8, real	
B_4:	$[0,0,0,1]$	16, real	

C_n 

cf.



$A_5 \simeq SU(6)$ $[1,0,0,0,0]$

$$\chi_{[1,0,0,0,0]} = Y_1 + \frac{Y_2}{Y_1} + \frac{Y_3}{Y_2} + \frac{Y_4}{Y_3} + \frac{Y_5}{Y_4} + \frac{1}{Y_5}$$

Set the fugacities of C_3 to be Z_i

$$Z_1 = Y_1 = Y_5 \qquad Z_2 = Y_2 = Y_4 \qquad Z_3 = Y_3$$

$$\chi_{[1,0,0]_{C_3}} = Z_1 + \frac{Z_2}{Z_1} + \frac{Z_3}{Z_2} + \frac{Z_2}{Z_3} + \frac{Z_1}{Z_2} + \frac{1}{Z_1} \quad \text{pseudoreal}$$

$[0,\dots,0,1,0,\dots,0]_{C_n}$ (1 in the K-th place) traceless K-th rank antisymmetric rep of $Sp(n)$

[singlet derived from sym or asym - determine pseudoreal or not]

[How know traceless?]

[not simple dim to write down - not binomial since traceless]

Recall for $SO(n)$ 2 vectors V_i, W_i $i = 1,\dots,n$

$V_i W_j + V_j W_i$ second rank symmetric

$V_i W_i$ is a singlet

$Sp(n)$ V_α, W_β antisymmetric $V_\alpha W_\beta - V_\beta W_\alpha$ reducible

$J^{\alpha\beta} V_\alpha W_\beta$

[see traceless by analogy]

Note 1 Relation Between Determinant And Area (general formulation)

In 2D: area of a parallelogram spanned by

$\underline{a} = a_1 \underline{i} + a_2 \underline{j}$

$\underline{b} = b_1 \underline{i} + b_2 \underline{j}$

is given by

$$\|\underline{a} \times \underline{b}\| = \left| \det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \right|$$

Physics: A_μ, $\mu = 0 \dots 3$ so we are in $SO(4)$
i.e. $D_2 = A_1 \times A_1$

4D rep: $[3,0]$, $[0,3]$ or $[1,1]$, or $[1,0]+[0,1]$

$[3,0]$: spin $\frac{3}{2}$ rep for one $SU(2)$ do not want this for general vectors

$[0,3]$: idem

$[1,1]$: the vector rep we want

$[1,0]+[0,1]$: 2 spinors, which do not work properly as well

For the $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

∂_μ transf in $[1,1]$ (i.e. as a vector A_μ)

$F_{\mu\nu}$ —"— —"— $\Lambda^2[1,1]$ antisym product of two vectors

$$\dim \Lambda^2[1,1] = \binom{4}{2} = 6$$

Compute $\Lambda^2[1,1]$ in irrep with PE_F

or $f(X_1,X_2) = (X_1 + X_1^{-1})(X_2 + X_2^{-1}) = \chi_{[1,1]}(X_1,X_2)$

then $\frac{1}{2}f^2(X_1,X_2) - \frac{1}{2}f(X_1^2,X_2^2) = (2,0)(0,2)(0,0)(0,0)(0,-2)(-2,0)$

from $(1,1)(1,-1)(-1,1)(-1,-1)$

Highest weight: $[2,0]$

so $\Lambda^2(1,1) = [2,0]+[0,2]$ (sym in the exchange of $[\cdot,\cdot]$)

So $F_{\mu\nu}$ is reducible

It's a reducible rep of $SO(4)$: the adjoint rep of $SO(4)$

Can then be written

$F_{\mu\nu} = F_{\mu\nu}^{1} + F_{\mu\nu}^{2}$ which transforms without mixing

Using $\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} \equiv \tilde{F}_{\mu\nu}$, we can write $F^{\pm}_{\mu\nu} \equiv \frac{1}{2}(F_{\mu\nu} \pm \tilde{F}_{\mu\nu})$

and the $F_{\mu\nu} = F^+_{\mu\nu} + F^-_{\mu\nu}$

Does this work?

$$\varepsilon^{\mu\nu\rho\sigma}F^+_{\mu\nu} = \varepsilon^{\mu\nu\rho\sigma}\left(\tfrac{1}{2}F_{\mu\nu} + \tfrac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}\right) = \frac{1}{2}\tilde{F}^{\rho\sigma} + \frac{c}{2}\varepsilon^{\rho\sigma}{}_{\alpha\beta}F^{\alpha\beta}$$

$= F^{\rho\sigma +}$

shows that

i.e. $F^+_{\mu\nu}$ has an eigenvalue $+1$ under $\varepsilon^{\mu\nu\rho\sigma}$
as well as

$F^-_{\mu\nu}$ " " " -1 " "

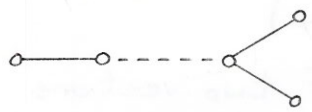
We can identify these self dual and anti-self dual tensors $F^+_{\mu\nu}$ and $F^-_{\mu\nu}$ as the elements transforming in $[2,0]$ and $[0,2]$

The middle dimensional antisym tensor:

$\frac{2n}{2} = n$ SO(2n): the n-th rank antisym rep of SO(2n) is reducible into SD and ASD tensors

Self dual $\Leftrightarrow F^+ = 0$, antiself dual $\Leftrightarrow F^- = 0$

D_n:



SO(2n): basic rep are $[0, \ldots, \underset{\uparrow K\text{th}}{1}, \ldots, 0]$

For $K = n-1$ or n: spinor rep of SO(2n), dim 2^{n-1}
distinguished by calling spinor and spinor (!)

Dirac spinor (on SO(4)) is reducible in $[1,0]+[0,1]$ (because massless part)
called two Weyl spinors, the L and R ones
(whereas Dirac spinor is L-R sym)

Let $A_{D_n} = 4$ so set the last two indices to be n_{n-1}, n_n being even or odd (each)

giving $[n_1 \ldots n_{n-1}, n_n] \otimes [m_1 \ldots m_{n-1}, m_n] = [\ldots n_{n-1}+m_{n-1}, n_n+m_n]$

Selection rules:

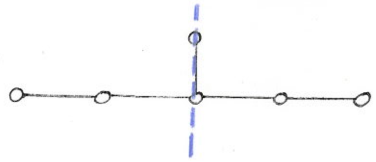
$[\ldots \text{odd}, \text{odd}] \otimes [\ldots \text{odd}, \text{odd}] = [\ldots \text{even}, \text{even}]$

odd, even   odd, odd   even, odd

⋮   ⋮   ⋮

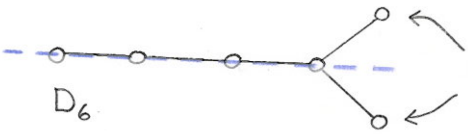
Decomposition of exceptional groups E_i $(i = 6, 7, 8)$

E_6:



diag symmetric under reflection, so under complex conjugation

(cf.



D_6 — spinor complex rep, complex conj of each other)

$\dim[1, 0 \ldots 0]_{E_6} = 27 = \dim[0 \ldots 0, 1]_{E_6}$

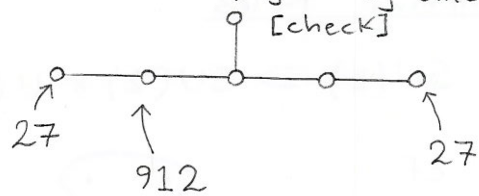
traceless antisym

[i.e. adjoint]

$\dim \mathrm{Adj}(E_6) = 78$

$\dim[0, 1, 0 \ldots 0] = 912$

highest weight given by this node [check]



E_7: dim funda $E_7 = 56$ pseudoreal; dim Adj $E_7 = 133$

E_8: dim Adj $E_8 = 248$

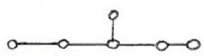
Recall: every dot in Dynkin diag is simple root and funda weight (one α_i corresponds to one Λ_j:

$$\Lambda_i = (A^{-1})_{ij}\, \alpha_i \quad \text{with} \quad A_{ij} = 2\frac{(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$$

[i.e. roots allow us to form the Cartan matrix]

i.e. $$2\frac{(\Lambda_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}$$)

each node is represented by n and we can write:



$\dim\begin{pmatrix} & & n_4 & & \\ n_1 & n_2 & n_3 & n_5 & n_6 \end{pmatrix}$

every rep is represented by a highest weight Λ

decomposition in Λ_i:

$$\Lambda = \sum_i n_i \Lambda_i, \quad \text{with } \Lambda_i \text{ the funda weights}$$

Dynkin diagram contains all possible roots

GUT:

$SU(5) \supset SU(3) \times SU(2) \times U(1)$

$SO(10)$

E_6

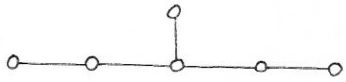
{How to couple a scalar field to a gauge field? Use covariant derivative}

Gauge Field Coupling — R = representation

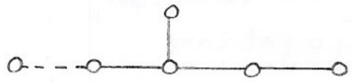
$D_\mu \phi_R = \partial_\mu \phi_R + i g A_\mu \phi_R$ $\qquad \frac{1}{2}|D_\mu \phi_R|^2$

ϕ is a vector space; A_μ transforming it, rep gives the interaction, i.e. the action of A_μ on ϕ, and the mass of fields

Indeed, consider Dynkin diagram for E_6



taking a subset

 $\equiv D_5 = SO(10)$

then

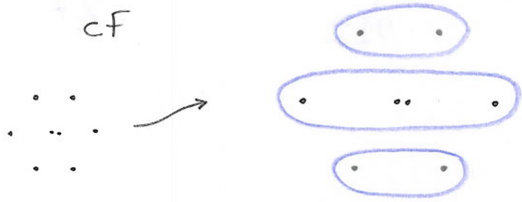
 $\equiv A_4 = SU(5)$

but need to understand how to extend the mechanisms: branching

Branching Rules

$SU(3) \supset SU(2) \times U(1)$ $\longleftarrow$ $\left\{ \begin{array}{l} \text{Suppose we have a group } SU(3) \\ \text{and we want a subgroup decomposition} \\ SU(2)\times U(1) \end{array} \right\}$

cf



in $SU(3)$ rep

decomposed into $SU(2)$ different rep

or 3×3 matrix:

$$\left(\begin{array}{c|c} 2\times2 & 1\times2 \\ \hline 2\times1 & 1\times1 \end{array}\right)$$

Hence $3 \longrightarrow 2+1$

but charged under the other gp: $2_1 + 1_{-2}$

charges

2^{nd} rank sym of $SU(3) = 6 \longrightarrow ?$

Use the Fugacities:

$$\chi_{[1,0]}(Y_1, Y_2) = Y_1 + \frac{Y_2}{Y_1} + \frac{1}{Y_2} = \chi_3$$

$3 =$  $\equiv \left.\begin{array}{l} \text{doublet} \\ \text{singlet} \end{array}\right\}$ with different charges $\left\{ \begin{array}{l} \chi_3 = q^1 \chi_2 + q^{-2}\chi_1 \\ 3 = 2_1 + 1_{-2} \end{array} \right.$

choosing this mapping:

$$\chi_3 = q\left(\frac{1}{x} + x\right) + \frac{1}{q^2}$$

Note: $SU(2) \left\{ \begin{array}{l} \text{Fugacity} \\ \text{highest weight} \end{array} \right\} = x$, $U(1) \left\{ \begin{array}{l} \text{Fugacity} \\ \text{charge} \end{array} \right\} = q$,

set • $SU(2)$ fugacity to x

• $U(1)$ — " — — " — q

giving:

$$Y_1 = qX, \quad \frac{Y_2}{Y_1} = \frac{q}{X}, \quad \frac{1}{Y_2} = \frac{1}{q^2} = \left(qX \cdot \frac{q}{X}\right)^{-1} = (q^2)^{-1} \quad \text{OK}$$

Fugacity map:

$$\boxed{Y_1 = qX, \ Y_2 = q^2}$$ for the branching rules

not choosing:

X and $\frac{1}{X}$ symmetry for SU(2) properties $\left(qX \Rightarrow \frac{q}{X}\right)$

product of terms with X or $\frac{1}{X}$ = terms without it $\Rightarrow \frac{1}{q^2}$

Fugacity map not unique: 3 possible maps:

(1) $Y_1 = qX, \ Y_2 = q^2$

(2) $Y_1 = qX, \ Y_2 = \frac{X}{q}$

(3) $Y_1 = \frac{q}{X}, \ Y_2 = q^2$

Any of these will solve

$$Y_1 + \frac{Y_2}{Y_1} + \frac{1}{Y_2} = q\left(X + \frac{1}{X}\right) + \frac{1}{q^2}$$

For 6:

= triplet
doublet
singlet

and $$\chi_{[2,0]} = Y_1^2 + \frac{Y_2^2}{Y_1^2} + \frac{1}{Y_2^2} + Y_2 + \frac{Y_1}{Y_2} + \frac{1}{Y_1}$$

Dynkin labels for E_6

with the first map: $(Y_1 = qX, \ Y_2 = q^2)$

$$\chi_{[2,0]} = q^2X^2 + \frac{q^4}{q^2X^2} + \frac{1}{q^4} + q^2 + \frac{qX}{q^2} + \frac{1}{qX}$$

$$= q^2\left(X^2 + 1 + \frac{1}{X^2}\right) + \frac{1}{q}\left(X + \frac{1}{X}\right) + \frac{1}{q^4}$$

i.e. $$\chi_{[2,0]} = q^2\chi_{[2]} + q^{-1}\chi_{[1]} + q^{-4}\chi_{[0]}$$

$$\Rightarrow \quad 6 = 3_2 + 2_{-1} + 1_{-4}$$

Once you have chosen a map there are 3 ways of implementing this map (in this case).

For X_n there are n ways of choosing this map.

Branching rules $SU(3) \supset SU(2) \times U(1)$

Fugacity map $3 \longrightarrow 2_1 + 1_{-2}$ } recap of lecture 28

e) Now: $E_6 \supset SO(10) \supset SU(5) \supset SU(3) \times SU(2) \times U(1)$ [Used to be popular for SM representn]

[see note p2]

Decompose 5D vector: $\rightarrow$ 3 + 2D vector (i.e. decomp of funda of SU(5))

$$5 \longrightarrow (3,1)_2 + (1,2)_{-3}$$

(a) triplet of SU(3) = singlet under SU(2)

(b) doublet under SU(2) = singlet under SU(3)

$$\begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \hline \cdot \\ \cdot \end{pmatrix} \begin{matrix} \}\ 3\text{-d} \\ \}\ 2\text{-d} \end{matrix}$$

(b) [singlet SU(3) doublet SU(2)]

Cartan generators

$$0 = \mathrm{Tr}\left(\begin{array}{ccc|cc} a & & & & \\ & a & & & \\ & & a & & \\ \hline & & & b & \\ & & & & b \end{array}\right)$$

generator $\Rightarrow$ traceless

$3a + 2b = 0$

$a = 2,\ b = -3$ [choose these] [since normalizn is a choice] [then write 2, −3 @ (a) & (b)]

chosen to be proportional to $\left(\begin{array}{c|c} I_{3\times3} & 0 \\ \hline 0 & I_{2\times2} \end{array}\right)$

[Cartan gen's − trace zero]

[singlet carries U(1) charges − arbitrary normalization − only care e-charge some integer # times some value $\Rightarrow$ deal w lattice for non-Abelian groups, though this is Abelian]

[Fugacity map tells decomp for any repn]

Characters $\chi_{[1,0,0,0]SU(5)} = \chi_{[1,0]SU(3)}\chi_{[0]SU(2)}q^2 + \chi_{[0,0]SU(3)}\chi_{[1]SU(2)}q^{-3}$

$$[1,0,0,0]_{SU(5)} = Y_1 + \frac{Y_2}{Y_1} + \frac{Y_3}{Y_2} + \frac{Y_4}{Y_3} + \frac{1}{Y_4}$$

Note: using $[n_1 \dots n_n]$ for $\chi_{[n_1 \dots n_n]}$

[decomp want equal to this repn − intro fugacities]

$$= q^2\left(Z_1 + \frac{Z_2}{Z_1} + \frac{1}{Z_2}\right) + q^{-3}\left(X + \frac{1}{X}\right)$$

SU(3) fugacities Z_1, Z_2

SU(2) " X

U(1) " q

Pick a fugacity rep:

$$\underline{\underline{Y_1 = q^2 Z_1}}$$

(choice) (1st monomial) [funda of SU(5) = funda of SU(3) $\times q^2$ (charge)]

$\Rightarrow Y_2 = Y_1 \cdot q^2 \dfrac{Z_2}{Z_1} = q^4 Z_2$ (2nd monomial) [do fugacity map then find interpretn]

[2nd rank antisym SU(5) = 2nd rank antisym SU(3) $\times (q^2)^2$]

[character fundamental × character singlet]

$Y_3 = Y_2 \cdot \dfrac{q^2}{Z_2} = q^6$ (3rd monomial)

[3rd rank antisym SU(5) = $(q^2)^3$: singlet so no dependence on SU(3) fugacities (Z_i)]

[monomial] [ternomial]

$Y_4 = q^3 X$ (5th monomial)

[4th rank antisym SU(5) = map to SU(2)]

Check: $\dfrac{Y_4}{Y_3} = \dfrac{q^3 X}{q^6} = \dfrac{X}{q^3}$ (4th monomial) [i.e. OK]

[Y_1: SU(5) − funda]

[Y_2: SU(5) − 2nd rank anti-funda]

[Z_2 SU(3) − 2nd rank anti-funda]

[Y_3 − 3rd rank anti-sym rep SU(5) $\Rightarrow$ 3 × charge U(1) charge]

[3rd rank rep $\rightarrow$ singlet]

$\bar{5} \longrightarrow (\bar{3},1)_{-2} + (1,2)_3$ (because $\bar{2}=2$)

[cf Y_4 - possible to decompose to SU(2) repⁿ] [i.e. $X_{q^3} = Y_4$ as above $\Rightarrow$ coherent]

[Can tell how SU(5) decompose to repⁿs of SU(3)]

[4d lattice SU(5) - different ways to choose sublattices also choose fugacities → natural basis, Y_1 correspond to fund. weight etc. decomp doesn't depend on fugacity maps]

$$10: [0,1,0,0]_{SU(5)} = \left(Y_2 + \frac{Y_3Y_1}{Y_2} + \frac{Y_4Y_1}{Y_3} + \frac{Y_1}{Y_4}\right) + \left(\frac{Y_3}{Y_1} + \frac{Y_4Y_2}{Y_3Y_1} + \frac{Y_2}{Y_1Y_4}\right) + \left(\frac{Y_4}{Y_2} + \frac{Y_3}{Y_2Y_4}\right) + \left(\frac{1}{Y_3}\right)$$

[2nd rank ASym rep SU(5)]
[" " " & Sym simple - add squares on the sym]

[10 monomials]

(By taking AS terms of product $\chi_{[1,0,0,0]}\chi_{[1,0,0,0]}$)

[using $Y_3^{-1} = q^{-6}$]

$$\underbrace{Y_2}_{q^4 Z_2\ (c)} + \underbrace{\frac{Y_3Y_1}{Y_2}}_{q^4 \frac{Z_1}{Z_2}\ (d)} \quad \underbrace{\frac{Y_4Y_1}{Y_3}}_{q^3 \times \frac{q^{-2}}{q^6} Z_1} \quad \underbrace{Y_3}_{q^{-6}}$$

[dim = 3 + (3×2) + 1 = 10]

$$10 \longrightarrow (\bar{3},1)_4 + (3,2)_{-1} + (1,1)_{-6}$$

[just by writing down $\underbrace{1^{st}\ \text{term}}_{\text{i.e.}\,(c)}$]

$\frac{1}{q} \times Z_1$ [this gives the $(3,2)_{-1}$ term]

[(d) part of anti-Fund rep - only need pick some to get repⁿ]

trace:		
3 weights with $q=4 \Rightarrow +12$		
3×2 — " — — " — $q=-1 \Rightarrow -6$		traceless
1 — " — — " — $q=-6 \Rightarrow -6$		

Interpretation

[$(3,2)_{-1}$: Fit naturally with LH quarks]

[$(\bar{3},1)_4$: — " — — " — — " — RH quarks]

[$(1,1)_{-6}$: take to predict massive particle not yet found] [nature not LR symmetric]
[or RH electron → singlet under both]

[adjoint SU(5) contains adjoint SU(3) and SU(2)]

[leptons - singlet under SU(3) and doublet under SU(2)]

Note: can go to larger groups
SO(10) ⊃ SU(5) } see (e)
E_6 ⊃ SO(10)... } above

We have:

$$U(N) \supset U(n_1) \times U(n_2) \qquad n_1 + n_2 = N$$

$$SO(N) \supset SO(n_1) \times SO(n_2)$$

$$Sp(N) \supset Sp(n_1) \times Sp(n_2)$$

[natural embedding
U - complex vector space
SO - real vector space
Sp - 2n dim real vector space]

[Rank:] $N-1$

$$SU(N) \supset S(U(n_1) \times U(n_2)) \simeq \overbrace{SU(n_1)}^{n_1 - 1} \times \overbrace{SU(n_2)}^{n_2 - 1} \times \overbrace{U(1)}^{1}$$

[Removing Cartan element prop to id (non-zero trace) on both sides]

[removing traceless subgroup in Cartan algebra]

$G \supset H$

[How find such subgroup?]

$$\underbrace{SO(8)}_{\text{rank: } 4} \supset SO(4) \times SO(1) = \underbrace{SU(2)^4}_{4 \cdot 1}$$

[rank]
[preserved in this decomposition]

[rule: given G, what are all possible subgroups?
e.g. can I embed E_6 inside E_7?]

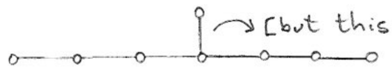
$E_7 \supset E_6 \times SU(2)$ ← [correct?] [No]

$E_7 \supset E_6 \times U(1)$

$E_7 \supset SO(12) \times SU(2)$

$E_7 \supset ? \times ?$
e.g. $E_7 \supset E_6 \times ?$
rank: $7 = 6 + ?$

Consider the Dynkin diagram:

E_8: 

[but this should be here]

Branching Rules:

$G \supset H$ and write $R_G = \sum_i R_{H,i}$ with fugacity maps

We have:

$$U(N) \supset \prod_i U(n_i), \text{ with } \sum_i n_i = N$$

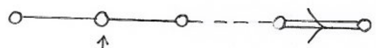
$$SU(N) \supset U(1)^{K-1} \prod_{i=1}^{K} SU(n_i), \text{ idem}$$

$$SO(N) \supset \prod_i SO(n_i), \text{ idem}$$

$$Sp(N) \supset \prod_i Sp(n_i), \text{ idem}$$

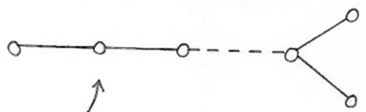
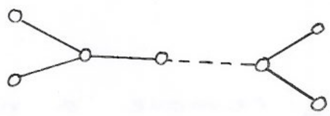
Dynkin method: to compute the decomposition (for classical and exceptional groups) use extended Dynkin diagram (for affine Lie algebra): add one node that has a scalar product with the node of the adjoint rep

Examples

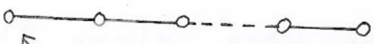
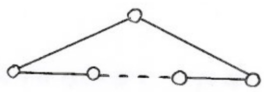
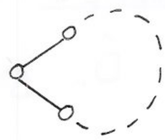
B_n:  $\Rightarrow$ $\hat{B}_n$: 

adj rep
(because dim adj = n+2)

extended diagram
(has n+1 nodes)

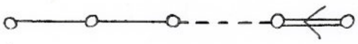
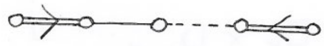
D_n:  $\Rightarrow$ $\hat{D}_n$: 

2^{nd} rank AS again is the adj

A_n:  $\Rightarrow$ $\hat{A}_n$:  $\equiv$ 

adj

(more symmetrically)

C_n:  $\Rightarrow$ $\hat{C}_n$: 

$\sim$adj = [2,0...0]

because for $SO(n)$, adj $\equiv 2^{nd}$ rank AS

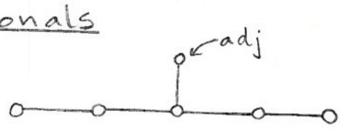
for $Sp(n)$, adj $\equiv 2^{nd}$ rank Sym

and because it is $[2,0\ldots0]$, we will put 2 lines from the node corresponding to $[1,0\ldots0]$

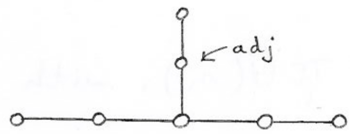
<u>Exceptionals</u>

[notes copied from board p.5 → begin here]

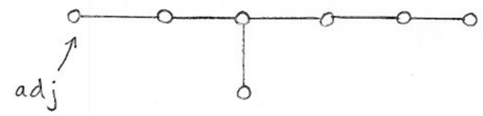
E_6:



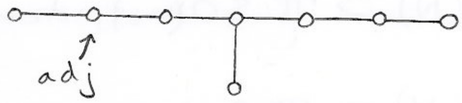
$\Rightarrow$ $\hat{E}_6$:



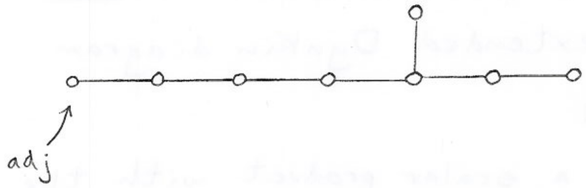
E_7:



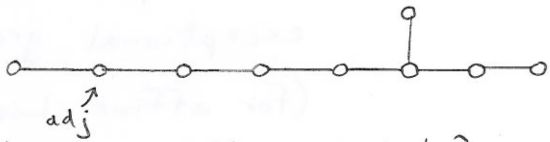
$\Rightarrow$ $\hat{E}_7$:



E_8:

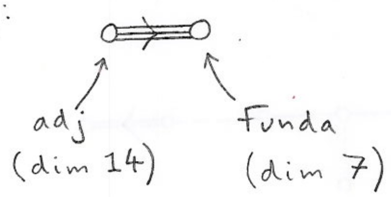


$\Rightarrow$ $\hat{E}_8$

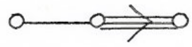


(where is the adjoint? can compute the dim)

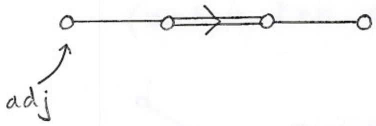
G_2:



$\Rightarrow$ $\hat{G}_2$:



F_4:



$\Rightarrow$ $\hat{F}_4$:

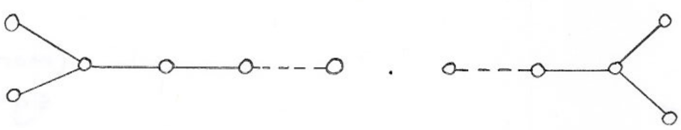
<u>Prescription</u>: remove a node from the extended Dynkin diag $\hat{G}$

the remaining diag is a subgp of the original gp G

<u>Example</u>

$\hat{D}_n$ gives by removing a node somewhere along the line

[node removal indicated by "."]

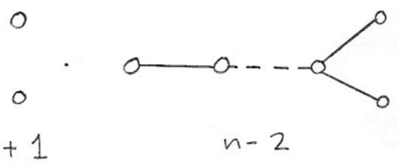


K nodes n−K nodes

$\Rightarrow D_n \supset D_K \times D_{n-K}$

i.e. $SO(2n) \supset SO(2K) \times SO(2n-2K)$

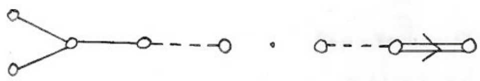
Specific cases:



$\Rightarrow D_n \supset SO(2n-4) \times \underbrace{SU(2) \times SU(2)}_{SO(4)}$

Note that this method is not complete and useful mainly for exceptional cases

cf $\hat{A}_n$: gives A_n only $\Rightarrow$ useless

- $\hat{B}_n$ gives  $= D_K$ and B_{n-K}

i.e. $SO(2n+1) \supset SO(2K) \times SO(2n-2K+1)$

for the last node, gives $SO(2n+1) \supset SO(2n)$

- $\hat{G}_2$ gives G_2 the trivial answer (via )

$A_1 \times A_1 = SU(2) \times SU(2)$ (via o · o)

$A_2 = SU(3)$ (via o—o ·)

So $SU(2) \times SU(2) \subset G_2$, $SU(3) \subset G_2$
(or $SO(4)$)

[notes copied from board p.5 $\rightarrow$ end here]

[notes copied from board p.6 $\rightarrow$ begin here]

- $\hat{F}_4$ gives F_4

$A_1 \times C_3 \equiv SU(2) \times Sp(\ \ )$

$A_2 \times A_2 \equiv SU(3) \times SU(3)$

$A_3 \times A_1 \equiv SU(4) \times SU(2)$

$B_4 \equiv SO(9)$

- $\hat{E}_8$ (also called E_9) gives E_8

$A_1 \times E_7$

$A_2 \times E_6$

$A_3 \times D_5 \equiv SU(4) \times SO(10) \equiv SO(6) \times SO(10)$

(because $A_3 = D_3$)

$A_4 \times A_4 = SU(5) \times SU(5)$ [α] [see p.4]

$A_1 \times A_2 \times A_5$

$A_1 \times A_7$

$D_8 \equiv SO(16)$

$A_8 \equiv SU(9)$

So $A_3 \times D_5 = SO(6) \times SO(10)$
$\subset SO(16) = D_8$

So $A_3 \times D_5$ is not maximal
(can embed it in another subgp)

[α] $E_8 \supset SU(5) \times SU(5)$

$$248 \longrightarrow \underbrace{(24,1) + (1,24)}_{\text{not enough}} + \underbrace{(5,\bar{5}) + (\bar{5},5)}_{(\text{funda, adj}) + \overline{(\quad)}}$$

still not enough

or $+ (5,\overline{10}) + (10,\bar{5})$

$+ (5,10) + (\bar{5},\overline{10})$ OK?

· $\hat{E}_6$ gives E_6

$A_1 \times A_5$ so $A_1 \times A_2 \times A_5$ in $\hat{E}_8$

$A_2 \times A_2 \times A_2$ not maximal because $(A_1 \times A_5) \times A_2 \subset A_2 \times E_6 \subset E_8$

(and again the same)

For real rep of G of dim n, we can embed it in $SO(n)$

$$G \supset SO(n)$$
$$n \longleftarrow n$$

Example

$SU(n) \subset SO(n^2-1)$

adj $\longleftarrow$ (n^2-1) rep

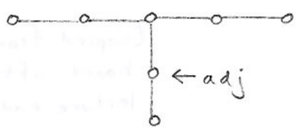
from the dim of the adj to the rep of $SO(\text{dim adj})$

[Notes copied from board p.6 → end here]

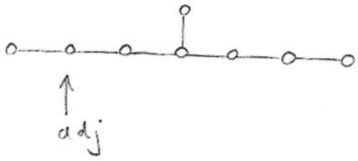
[Lct pp. 2-3]

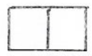
[notes copied from board after lecture ended]

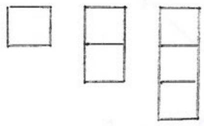
$\hat{E}_6$:



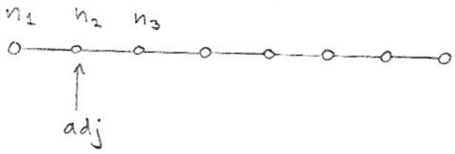
$\hat{E}_7$:



 $[2,0,0,0]$

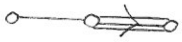
 $[0,1,0,0]$

$\hat{E}_8$:

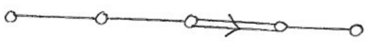


 $[0,0,1,0]$

$\hat{G}_2$:

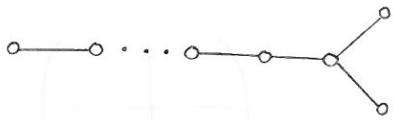


$\hat{F}_4$:



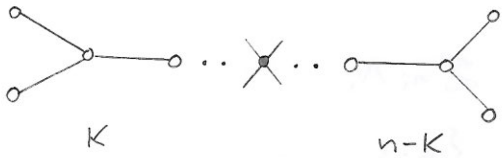
A_n: $[1,0\ldots 0,1]$

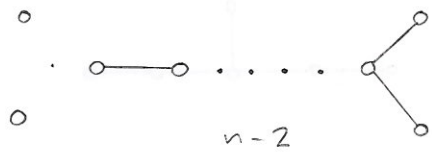
B_n: $[0,1,0\ldots 0]$



remove a node From from the extended Dynkin diagram $\hat{G}$ remaining diagram is a subgroup of G

$\hat{D}_n$:





$SO(2n-4)\times SU(2)\times SU(2)$

$D_n \supset D_K \times D_{n-K}$ $\qquad SO(2n)\supset SO(2K)\times SO(2n-2K)$

$\hat{B}_n$:

$SO(2n+1)\supset SO(2K)\times SO(2n-2K+1)$

$\hat{G}_2 \supset G_2$ $\qquad SO(2n+1)\supset SO(2n)$

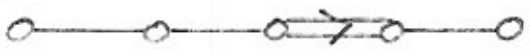
o · o $\supset A_1\times A_1$ $\qquad SU(2)\times SU(2)\subset G_2$

$\supset A_2$ $\qquad SU(3)\subset G_2$

[copied from board after lecture ended]

Branching Rules

[cF pp. 3-4]



$F_4,\ A_1 \times C_3,\ A_2 \times A_2,\ A_3 \times A_1,\ B_4$

$\hat{E}_8 = E_9$

$SO(6)$

$SU(4) \times SO(10)$

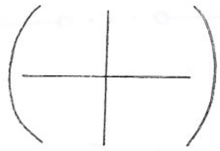
$E_8,\ A_1 \times E_7,\ A_2 \times E_6,\ \underbrace{A_3 \times D_5}_{\text{not maximal}},\ A_4 \times A_4$

$\underbrace{A_1 \times A_2 \times A_5}_{\text{not maximal}},\ A_1 \times A_7,\ D_8,\ A_8$

$SO(16) \subset E_8$

$A_1 \times A_2 \times A_5 \subset A_2 \times E_6 \subset E_8$

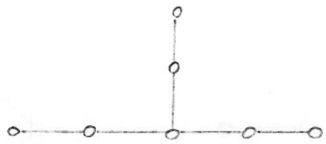
Dynkin



$E_8 \supset SU(5) \times SU(5)$

$248 \longrightarrow (24,1) + (1,24) + (5,\overline{10}) + (10,\bar{5}) + (5,10) + (\bar{5},\overline{10})$

$\hat{E}_6$



$\Lambda = \sum n_i \Lambda_i$

$\{n_i\}$

$E_6,\ A_1 \times A_5,\ A_2 \times A_2 \times A_2$

$SU(3)^3 \subset E_6$

For a real representation of an group G of dimension n

$G \subset SO(n)$ <u>Example</u> $SU(n) \subset SO(n^2-1)$

$n \longleftarrow n$ $\qquad$ $\text{adj} \longleftarrow n^2-1$
