Particle Symmetries Problem Set 2 (2012)

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Length of Highest weights

Given a root system $\{\alpha_i\}_{i=1}^r$, for an algebra of rank r, define the Cartan matrix to be

$$C_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}, \tag{0.1}$$

where $(.,.)$ is the usual norm in a Euclidean vector space. The fundamental weights $\{\Lambda_i\}_{i=1}^r$ are given by

$$\Lambda_i = (C^{-1})_{ij}\alpha_j$$

The highest weight of a representation with Dynkin labels $[n_1, n_2, \ldots, n_r]$ is given by

$$\Lambda = \sum_{i=1}^{r} n_i \Lambda_i.$$

1. For $SU(2)$ the Cartan matrix is equal to 2.
   (a) Find a root α_1 that satisfies relation (1.1) with the Cartan matrix (note that this is not unique).
   (b) Find an expression for the fundamental weight Λ_1 and express it in terms of the root α_1.
   (c) Compute the length square of the highest weight Λ as a function of the Dynkin labels n_i.
2. For $SU(3)$ the Cartan Matrix of $SU(3)$ is
   $$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$
   Repeat steps (a) to (c) for $SU(3)$.
3. Find in books, or online the Cartan matrix for each of the classical algebras, $SO(2n+1)$, $Sp(n)$, $SO(2n)$. Repeat these steps for each of these algebras. Find the simple roots, the fundamental weights, and the length square of the highest weight Λ.
4. Find the Cartan matrix and Repeat these steps for each of the exceptional algebras, E_6, E_7, E_8, F_4, G_2.