Particle Symmetries Problem Set 2 (2014)

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Eigenvalues and Casimir Invariants

1. For each of the algebras in the Cartan classification compute the eigenvalues of the Cartan matrix. Show that each eigenvalue a_i satisfies the relation

$$a_i = 4\sin^2\left[\frac{c_i - 1}{2h}\pi\right],$$

where c_i is an integer and h is the Coxeter number. Compute the integers c_i and evaluate their product. Show that the order of the Weyl group is equal to this product.

2. For a Lie algebra of dimension d and rank r set α to be the highest root, and α_i to be the simple roots. The Coxeter number is defined to be

$$h = 1 + \sum_{i=1}^{r} m_i; \qquad \alpha = \sum_{i=1}^{r} m_i \alpha_i \tag{0.1}$$

Find its value for any Lie algebra, and show that $r(h+1) = d$, and $h = c_r$.

3. For a given Lie algebra define ρ (Weyl vector) to be the half sum of positive roots,

$$\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha. \tag{0.2}$$

Find ρ for each Lie algebra in Cartan's classification and show that the following identity holds

$$\rho = \sum_i \Lambda_i, \tag{0.3}$$

where $\{\Lambda_i\}$ is the set of all fundamental weights of the Lie algebra.