Imperial College London

MSc TEST Jan 2018

PARTICLE SYMMETRIES AND UNIFICATION

For Students in Quantum Fields and Fundamental Forces

Thursday 11 Jan 2018: 10:00 to 12:00

Answer all questions in Part A and Part B
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the 2 answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in 2 answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

Go to the next page for questions

SECTION A: Particle Symmetries

1. This question is about the Heisenberg group.

(i) Consider the set of matrices

$$H = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}.$$

Write down the conditions that define a group. Show that H forms a group under matrix multiplication. [4 marks]

(ii) Give the conditions that define an abstract Lie algebra. Find the form of the Lie algebra $\mathfrak{h}$ corresponding to the Lie group H.

Now consider the operators $\hat{q}$, $\hat{p}$ satisfying the Heisenberg commutation relations (with $\hbar = 1$)

$$\left[\hat{q}, \hat{p}\right] = \mathrm{i}\hat{e},$$

where $\hat{e}$ is the identity operator. Show that the Lie algebra generated by the anti-Hermitian operators $\{\mathrm{i}\hat{q}, \mathrm{i}\hat{p}, \mathrm{i}\hat{e}\}$ is isomorphic to $\mathfrak{h}$. [5 marks]

(iii) Consider the set of matrices of the form

$$\mathrm{e}^X = \sum_{n=0}^{\infty} \frac{1}{n!} X^n \qquad \text{where} \qquad X \in \mathfrak{h}.$$

Show that the series e^X terminates in a finite number of terms and hence calculate e^X explicitly. Show that $\left\{\mathrm{e}^X : X \in \mathfrak{h}\right\} = H$. [4 marks]

(iv) Define the adjoint representation of a matrix Lie algebra. Is the adjoint representation of H reducible? Is it decomposible? [3 marks]

(v) Consider the set $N \subset H$ given by

$$N = \left\{ \begin{pmatrix} 1 & 0 & 2\pi n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}.$$

Show that N forms an abelian normal subgroup of H. Define the quotient group H/N. Discuss whether H and H/N are simply connected. [4 marks]

[Total 20 marks]

SECTION B: Unification – the Standard Model

2. Consider the Lagrangian

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi - \frac{1}{2}\lambda \left(\phi^\dagger \phi\right)^2 + \bar{\psi}_L i\partial\!\!\!/ \psi_L + \bar{\psi}_R i\partial\!\!\!/ \psi_R - y\left(\bar{\psi}_L \phi \psi_R + \bar{\psi}_R \phi^\dagger \psi_L\right),$$

where ϕ is a two-component complex Lorentz scalar, ψ_L is a two-component left-handed Weyl spinor, and ψ_R is a single-component right-handed Weyl spinor.

(i) What is the particle spectrum (i.e., masses and numbers of degrees of freedom) if $m^2 > 0$? [5 marks]

(ii) Show that the theory has internal symmetry group SU(2) × U(1) × U(1), and how the fields ϕ, ψ_L and ψ_R transform under these symmetry transformations. [5 marks]

(iii) Find the vacuum state of theory if $m^2 = -\mu^2 < 0$. Expand the Lagrangian to quadratic order in fields about the vacuum state, and use it to determine the particle spectrum. [6 marks]

(iv) Find the residual symmetry group, and discuss how it is related to the differences in the particle spectrum between parts (i) and (iii). [4 marks]

[Total 20 marks]

End of examination paper